Modelling Waves in the Ocean

Jack McGinn
Supervisor: Jake Grainger

September 2020
Time Series

- Wind generated waves treated as stochastic process
- Sampling displacement over time of some point in the ocean is denoted by stochastic variable $X_\Delta = [X_{t\Delta}]_{t \in \mathbb{Z}}$ forming discrete time series
Spectral Density

- Continuous Spectral Density

\[ f(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} c(\tau) \exp(-i\omega \lambda) d\tau \]

- Discrete Spectral Density

\[ f_\Delta(\omega) = \frac{\Delta}{2\pi} \sum_{\tau=-\infty}^{\infty} c(\tau \Delta) \exp(-i\omega \tau \Delta) \]

- Aliasing

\[ f_\Delta(\omega) = \sum_{k=-\infty}^{\infty} f(\omega + \frac{2\pi k}{\Delta}) \]
Modelling Waves in the Ocean

Aliasing
Periodogram estimate for spectral density from wave displacement time series

$$ I(\omega) = \frac{\Delta}{2\pi N} \left| \sum_{t=0}^{N-1} X_t \exp(-it\Delta\omega) \right|^2 $$
JONSWAP

JONSWAP model for spectral density of wind generated waves in the ocean

\[ S_G(\omega|\theta) = \alpha \omega^{-r} \exp \left( \frac{-r}{s} \left( \frac{\omega}{\omega_p} \right)^{-s} \right) \gamma \delta(\omega|\theta) \]
JONSWAP

Plotting JONSWAP for varied alpha

- alpha = 0.6
- alpha = 0.7
- alpha = 0.8

Plotting JONSWAP for varied omega_p

- omega_p = 0.6
- omega_p = 0.7
- omega_p = 0.8

Plotting JONSWAP for varied gamma

- gamma = 2.5
- gamma = 3.3
- gamma = 4.1

Plotting JONSWAP for varied \( r \)

- \( r = 3 \)
- \( r = 4 \)
- \( r = 5 \)
Estimating parameters - Whittle Approximation

- Whittle Likelihood function

\[ \ell_W(\theta|X_{\Delta,N}) = - \sum_{\omega \in \Omega} \log(f(\omega|\theta)) + \frac{I(\omega)}{f(\omega|\theta)} \]

- De-biased Whittle Likelihood function, replace \( f(\omega|\theta) \) in fraction with,

\[
E[I(\omega)] = \frac{1}{2\pi} \text{Re} \left( 2\Delta \sum_{\tau=0}^{N-1} \left( 1 - \frac{\tau}{N} \right) c(\tau|\theta) \exp(-i\omega\tau\Delta) - \Delta c(0|\theta) \right)
\]
Simulated Data

Bartlett's periodogram of simulated wave and spectral density it was simulated from.
Real Data

Significant Wave Height throughout half a year

Day number
Real Data continued
Swell

Periodogram and estimated spectral density

- **Periodogram**
- **De-biased Whittle Predicted JONSWAP**
- **Barletts periodogram**

![Graph showing periodogram and estimated spectral density with Omega and estimated spectral density axes.](image-url)
Modelling Waves in the Ocean

Robustness of Removing swell

Variance of alpha against distance between sea state peaks

Variance of omega_p against distance between sea state peaks

Variance of gamma against distance between sea state peaks

Variance of r against distance between sea state peaks
Modelling Waves in the Ocean

Real Parameters

Average alpha per 2.5 hour period through June

Average omega_p per 2.5 hour period through June

Average gamma per 2.5 hour period through June

Average r per 2.5 hour period through June
Further Work

- Improve method of testing robustness
- Improve method for removing swell