Time Series Analysis via Kalman Filtering

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Aims

- To apply Dynamic Linear Models (DLMs) to time series, that do not show a particular trend or seasonal variation.

- To use the local level model to generate observed data.

- To apply the Kalman filter to perform inference on the unknown state vector and forecast.

- To test the performance of the Kalman filter on multiple observation sequences and in the case of missing data.
Theory

Dynamic Linear Models (DLMs)

Observation (1) and state (2) equation:

\[ Y_t = F_t \theta_t + v_t, \quad v_t \sim \mathcal{N}(0, V_t) \]  
\[ \theta_t = G_t \theta_{t-1} + w_t, \quad w_t \sim \mathcal{N}(0, W_t) \]  

**Assumption:** \( \theta_0 \sim \mathcal{N}(m_0, C_0) \)

Kalman filter

1) Prior at \( t \): \( p(\theta_t|D_{t-1}) \)

2) One-step forecast: \( p(Y_t|D_{t-1}) \)

3) Posterior at \( t \): \( p(\theta_t|D_t) \)
Generating Data

Local Level Model

\[ Y_t = \theta_t + v_t, \quad v_t \sim \mathcal{N}(0, V_t) \]  
(3)

\[ \theta_t = \theta_{t-1} + w_t, \quad w_t \sim \mathcal{N}(0, W_t) \]  
(4)

Figure: Generated state vector sequence by the random walk and the observation data sequence by local level model.
Generating Multiple Observation Sequences

(a) 50 filtered state vector sequences together with the true state vector sequence.

(b) The fraction of the true state vector sequence laying within the range of the filtered state vector sequences.
Missing Data - Univariate

Randomly and periodically missing data

(a) Randomly missing data

(b) Periodically missing data
Missing Data - Univariate

Range of missing data

<table>
<thead>
<tr>
<th>Level</th>
<th>4</th>
<th>2</th>
<th>0</th>
<th>-2</th>
<th>-4</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>Time step</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
</tr>
</tbody>
</table>

- **data**
- **forecast for data**
- **forecast for data with missing values**
- **range of missing values**
Missing Data - Multivariate

Range of missing data

Figure: Univariate and multivariate method applied to a bivariate VAR model Smith et al. [2022]
Missing Data - Multivariate

Range of missing data

![Graph showing univariate and multivariate forecasts with missing data points and range of missing values in the first series.](image-url)
Missing Data - Multivariate

Range of missing data - different method

![Graph showing univariate and multivariate forecasts with missing data points and range of missing values in the first series.](image-url)
Missing Data - Multivariate

Comparing the two multivariate methods

<table>
<thead>
<tr>
<th>Correlation</th>
<th>Standard multivariate method (univariate)</th>
<th>Modified multivariate method</th>
</tr>
</thead>
<tbody>
<tr>
<td>90%</td>
<td>$\text{MSE}(Y)/\text{MSE}(\theta)$: 15.873/4.331</td>
<td>$\text{MSE}(Y)/\text{MSE}(\theta)$: 14.259/2.263</td>
</tr>
<tr>
<td>80%</td>
<td>$\text{MSE}(Y)/\text{MSE}(\theta)$: 15.531/3.906</td>
<td>$\text{MSE}(Y)/\text{MSE}(\theta)$: 14.678/2.537</td>
</tr>
<tr>
<td>70%</td>
<td>$\text{MSE}(Y)/\text{MSE}(\theta)$: 15.317/3.626</td>
<td>$\text{MSE}(Y)/\text{MSE}(\theta)$: 15.194/2.937</td>
</tr>
<tr>
<td>60%</td>
<td>$\text{MSE}(Y)/\text{MSE}(\theta)$: 15.171/3.421</td>
<td>$\text{MSE}(Y)/\text{MSE}(\theta)$: 15.761/3.401</td>
</tr>
<tr>
<td>50%</td>
<td>$\text{MSE}(Y)/\text{MSE}(\theta)$: 15.070/3.266</td>
<td>$\text{MSE}(Y)/\text{MSE}(\theta)$: 16.359/3.907</td>
</tr>
<tr>
<td>40%</td>
<td>$\text{MSE}(Y)/\text{MSE}(\theta)$: 15.001/3.148</td>
<td>$\text{MSE}(Y)/\text{MSE}(\theta)$: 16.981/4.443</td>
</tr>
</tbody>
</table>

**Table:** Table comparing the two methods by computing the mean-square error for different correlations.
Conclusion

- Generating multiple observation sequences.

- Missing data (univariate) - random, periodic, range.

- Missing data (multivariate) - standard and modified method
Thank you!
References


Kalman Filter in More Detail

\[ \theta_{t-1}|D_{t-1} \sim \mathcal{N}(m_{t-1}, C_{t-1}), t > 1 \]

Prior at \( t \):

\[ \theta_t|D_{t-1} \sim \mathcal{N}(a_t, R_t), \]

\[ a_t = G_t m_{t-1}, \]

\[ R_t = P_t + W_t, P_t = G_t C_{t-1} G_t'. \]

One-step forecast:

\[ Y_t|D_{t-1} \sim \mathcal{N}(f_t, Q_t), \]

\[ f_t = F_t a_t, \]

\[ Q_t = F_t R_t F_t' + V_t. \]

Posterior at \( t \):

\[ \theta_t|D_t \sim \mathcal{N}(m_t, C_t), \]

\[ m_t = a_t + R_t F_t' Q_t^{-1} e_t, e_t = y_t - f_t, \]

\[ C_t = R_t - R_t F_t' Q_t^{-1} F_t R_t. \]
Missing data

Univariate

\[ m_t = a_t, \]

\[ C_t = R_t. \]

Multivariate

Missing data in the first row,

\[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} \rightarrow M_t = \begin{bmatrix}
0 & 1
\end{bmatrix}.
\]

New observation equation,

\[
\tilde{y}_t = \tilde{F}_t \theta_t + \tilde{v}_t,
\]

\[
\tilde{F}_t = M_t F_t,
\]

\[
\tilde{v}_t = M_t v_t.
\]