Math. Program., Ser. A 87: 37–56 (2000)

Digital Object Identifier (DOI) 10.1007/s101079900107

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On the separation of maximally violated mod-*k* cuts

Received May 29, 1997 / Revised version received May 10, 1999
Published online November 9, 1999

Abstract. Separation is of fundamental importance in cutting-plane based techniques for *Integer Linear Programming* (ILP). In recent decades, a considerable research effort has been devoted to the definition of effective separation procedures for families of well-structured cuts. In this paper we address the separation of Chvátal rank-1 inequalities in the context of general ILP's of the form $\min\{c^T x : Ax \le b, x \text{ integer}\}$, where A is an $m \times n$ integer matrix and b an m-dimensional integer vector. In particular, for any given integer k we study *mod-k* cuts of the form $\lambda^T A\, x \le \lfloor \lambda^T b \rfloor$ for any $\lambda \in \{0, 1/k, \dots, (k-1)/k\}^m$ such that $\lambda^T A$ is integer. Following the line of research recently proposed for mod-2 cuts by Applegate, Bixby, Chvátal and Cook [1] and Fleischer and Tardos [19], we restrict to *maximally violated* cuts, i.e., to inequalities which are violated by $(k-1)/k$ by the given fractional point. We show that, for any given k, such a separation requires $O(mn \min\{m, n\})$ time. Applications to both the symmetric and asymmetric TSP are discussed. In particular, for any given k, we propose an $O(|V|^2|E^*|)$-time exact separation algorithm for mod-k cuts which are maximally violated by a given fractional (symmetric or asymmetric) TSP solution with support graph $G^* = (V, E^*)$. This implies that we can identify a maximally violated cut for the symmetric TSP whenever a maximally violated *(extended) comb inequality* exists. Finally, facet-defining mod-k cuts for the symmetric and asymmetric TSP are studied.

Key words. integer programming – separation – symmetric/asymmetric traveling salesman problem

1. Introduction

Separation is of fundamental importance in cutting-plane based techniques for *Integer Linear Programming* (ILP). In recent decades, a considerable research effort has been devoted to the definition of effective separation procedures for families of well-structured cuts. This line of research was originated by the pioneering work of Dantzig, Fulkerson and Johnson [12] on the *Traveling Salesman Problem* (TSP) and led to the very successful branch-and-cut approach introduced by Padberg and Rinaldi [28]. Most of the known methods have been originally proposed for the TSP, a prototype in combinatorial optimization and integer programming.

In spite of the large research effort, however, polynomial-time exact separation procedures are known for only a few classes of facet-defining TSP cuts. In particular, no efficient separation procedure is known at present for the famous class of comb inequalities [22]. The only exact method is due to Carr [8], and requires $O(n^{2t+3})$ time for separation of comb inequalities with t teeth on a graph of n nodes.

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Mathematics Subject Classification (1991): Primary 90C10, 90C27

Applegate, Bixby, Chvátal and Cook [1] recently suggested concentrating on maximally violated combs, i.e., on comb inequalities which are violated by $1/2$ by the given fractional point x^* to be separated. This is motivated by the fact that maximally violated combs exhibit a very strong combinatorial structure, which can be exploited for separation. Their approach is heuristic in nature, and is based on the solution of a suitably-defined system of mod-2 congruences. Following this approach, Fleischer and Tardos [19] were able to design an $O(|V|^2 \log |V|)$-time exact separation procedure for maximally violated comb inequalities for the case where the support graph $G^* = (V, E^*)$ of the fractional point x^* is planar.

It is well known that comb inequalities can be obtained by adding-up and rounding a convenient set of TSP degree equations and subtour elimination constraints weighed by $1/2$, i.e., they are $\{0, \frac{1}{2}\}$-*cuts* in the terminology of Caprara and Fischetti [7]. These authors studied $\{0, \frac{1}{2}\}$-cuts in the context of general ILP's. They showed that the associated separation problem is equivalent to the problem of finding a minimum-weight member of a binary clutter, i.e., a minimum-weight $\{0, 1\}$-vector satisfying a certain set of mod-2 congruences. This problem is NP-hard in general, as it subsumes the max-cut problem as a special case. A key observation, due to Letchford [25], is that the binary clutter problem is in fact polynomially solvable when all weights are equal to 0, as in this case it reduces to finding a solution of a system of mod-2 congruences. This is precisely the situation arising when one restricts to the separation of maximally violated cuts. As a consequence, one can always separate in polynomial time over the family of $\{0, \frac{1}{2}\}$-cuts which are maximally violated by the given x^*. For the TSP, this implies that one can efficiently separate maximally violated members of a family of cuts that properly contains comb inequalities. Interestingly, this family contains facet-inducing cuts which are not comb inequalities.

In this paper we address the separation of Chvátal rank-1 inequalities in the context of general ILP's of the form $\min\{c^T x : Ax \leq b, x \text{ integer}\}$, where A is an $m \times n$ integer matrix and b an m-dimensional integer vector. In particular, for any given integer k we study *mod-k* cuts of the form $\lambda^T A x \leq \lfloor \lambda^T b \rfloor$ for any $\lambda \in \{0, 1/k, \ldots, (k-1)/k\}^m$ such that $\lambda^T A$ is integer. We show that, for any given k, separation of maximally violated mod-k cuts requires $O(mn \min\{m, n\})$ time. Applications to both the symmetric and asymmetric TSP are discussed.

The paper is organized as follows. Section 2 introduces mod-k cuts and the associated separation problem, called mod-k SEP. We show that mod-k SEP is equivalent to finding a $\{0, 1, \ldots, k-1\}$-vector satisfying a certain set of mod-k congruences. As a consequence, maximally violated mod-k cuts can be separated efficiently, for any given k. Section 3 discusses the separation of maximally violated TSP mod-k cuts. In particular, we show how to reduce from $O(|V|^2)$ to $O(|V|)$ the number of tight constraints to be considered in the mod-k congruence system, where $|V|$ is the number of nodes of the underlying graph. Sections 4 and 5 address mod-k cuts for the symmetric and asymmetric TSP. For both problems we show that the family of these cuts contains several classes of facet-defining inequalities. Furthermore, we investigate some interesting connections between mod-k derivation and clique (0-node) lifting and 2-cycle (edge) cloning. Some conclusions are drawn in Sect. 6.

2. Maximally violated mod-k cuts

Given an $m \times n$ integer matrix A and an m-dimensional integer vector b, let $P := \{x \in R^n : Ax \le b\}$, $P_I := \text{conv}\{x \in Z^n : Ax \le b\}$, and assume $P_I \ne P$. A *Chvátal-Gomory cut* is a valid inequality for P_I of the form $\lambda^T A\, x \le \lfloor \lambda^T b \rfloor$, where the *multiplier vector* $\lambda \in R^m_+$ is such that $\lambda^T A \in Z^n$, and $\lfloor \cdot \rfloor$ denotes lower integer part. In this paper we address cuts which can be obtained through multiplier vectors λ belonging to $\{0, 1/k, \dots, (k-1)/k\}^m$ for any given integer $k \ge 2$. We call them *mod-k cuts*, as their validity relies on mod-k rounding arguments. Note that mod-2 cuts are in fact the $\{0, \frac{1}{2}\}$-cuts studied in Caprara and Fischetti [7].

Any Chvátal-Gomory cut is a mod-k cut for some integer $k > 0$, as it is well known that undominated Chvátal-Gomory cuts only arise for $\lambda \in [0,1)^m$, since replacing any λ_i by its fractional part $\lambda_i - \lfloor \lambda_i \rfloor$ always leads to an equivalent or stronger cut. Moreover, λ can always be assumed to be rational, i.e., an integer $k > 0$ exists such that $k\lambda$ is integer. Indeed, for any given $\lambda \in R^m_+$ with $\alpha^T := \lambda^T A \in Z^n$ one can obtain an equivalent (or better) multiplier vector $\tilde{\lambda}$ by solving the linear program $\min\{\tilde{\lambda}^T b : \tilde{\lambda}^T A = \alpha^T, \tilde{\lambda} \ge 0\}$, whose basic solutions are of the form $\tilde{\lambda} = \left[B^{-1}\alpha, 0\right]$ for some basis B of A^T. Hence $\det(B) \cdot \tilde{\lambda}$ is integer, as claimed.

We are interested in the following separation problem, in its optimization version:

mod-k SEP: *Given $x^* \in P$, find $\lambda \in \{0, 1/k, \dots, (k-1)/k\}^m$ such that $\lambda^T A \in Z^n$, and $\lfloor \lambda^T b \rfloor - \lambda^T A\, x^*$ is a minimum.*

Following [7], this problem can equivalently be restated in terms of the integer multiplier vector $\mu := k\lambda \in \{0, 1, \dots, k-1\}^m$. For any given $z \in Z$ and $k \in Z_+$, let $z \bmod k := z - \lfloor z/k \rfloor k$. As is customary, notation $a \equiv b \pmod{k}$ stands for $a \bmod k = b \bmod k$. Given an integer matrix $Q = (q_{ij})$ and $k \in Z_+$, let $\overline{Q} = (\overline{q}_{ij}) := Q \bmod k$ denote the *mod-k support* of Q, where $\overline{q}_{ij} := q_{ij} \bmod k$ for all i, j. Then, mod-k SEP is equivalent to the following optimization problem.

mod-k SEP: *Given $x^* \in P$ and the associated slack vector $s^* := b - Ax^* \ge 0$, solve*

$$\delta^* := \min\, (s^{*\,T} \mu - \theta) \tag{1}$$

subject to

$$\overline{A}^T \mu \equiv 0 \pmod{k} \tag{2}$$

$$\overline{b}^T \mu \equiv \theta \pmod{k} \tag{3}$$

$$\mu \in \{0, 1, \dots, k-1\}^m \tag{4}$$

$$\theta \in \{1, \dots, k-1\}. \tag{5}$$

By construction, $(s^{*\,T} \mu - \theta)/k$ gives the slack of the mod-k cut $\lambda^T A\, x \le \lfloor \lambda^T b \rfloor$ for $\lambda := \mu/k$, computed with respect to the given point x^*. Hence, there exists a mod-k cut violated by x^* if and only if the minimum δ^* in (1) is strictly less than 0. Observe that

$s^* \geq 0$ and $\theta \leq k-1$ imply $\delta^* \geq 1-k$, i.e., no mod-k cut can be violated by more than $(k-1)/k$. This bound is attained for $\theta = k-1$, when the mod-k congruence system (2)–(4) has a solution μ with $\mu_i = 0$ whenever $s_i^* > 0$. In this case, the resulting mod-k cut is said to be *maximally violated*.

Even for $k = 2$, mod-k SEP is NP-hard as it is equivalent to finding a minimum-weight member of a binary clutter [7]. However, finding a *maximally violated* mod-k cut amounts to finding any feasible solution of the congruence system (2)–(4) after having fixed $\theta = k-1$ and having removed all the rows of $(\overline{A}, \overline{b})$ associated with a strictly positive slack s_i^*. For any k *prime* this solution, if any exists, can be found in $O(mn \min\{m, n\})$ time by standard Gaussian elimination in $GF(k)$, which works as follows.

Consider a general congruence system $Qy \equiv d \pmod{k}$, in the unknowns $y_j \in \{0, 1, \ldots, k-1\}$, where the entries of (Q, d) are assumed to be in $\{0, 1, \ldots, k-1\}$. By a series of modulo k row combinations, one can always bring the system to its equivalent form $\tilde{Q}y \equiv \tilde{d} \pmod{k}$, where after a convenient row and column permutation $\tilde{Q}$ and $\tilde{d}$ read

$$\tilde{Q} := \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}, \; \tilde{d} := \begin{bmatrix} \tilde{d}_1 \\ \tilde{d}_2 \end{bmatrix}.$$

Therefore, $Qy \equiv d \pmod{k}$ has a solution if and only if $\tilde{d}_2 = 0$. In this case, a basic solution is obtained as $y = (\tilde{d}_1, 0)$. Note that basic solutions have the property of being *minimal*, in the sense that no other solution $y' \leq y$ exists.

For k nonprime, $GF(k)$ is not a field, hence Gaussian elimination cannot be performed. On the other hand, there exists an $O(mn \min\{m, n\})$-time algorithm to find, if any, a solution of the mod-k congruence system (2)–(4) even for k nonprime, provided a prime factorization of k is known; see, e.g., Cohen [11].

The above considerations lead to the following result.

Theorem 1. *For any given k, maximally violated mod-k cuts can be found efficiently, in $O(mn \min\{m, n\})$ time, provided a prime factorization of k is known.*

It is worth noting that mod-k SEP with $\mu_i = 0$ whenever $s_i^* > 0$ can be solved efficiently by fixing θ to any value in $\{1, \ldots, k-1\}$. We call the corresponding solutions of (2)–(5) *totally tight* mod-k cuts. The following theorem shows that, for k prime, the existence of a totally tight mod-k cut implies the existence of a maximally violated mod-k cut.

Theorem 2. *For any k prime, a maximally violated mod-k cut exists if and only if a totally tight mod-k cut exists.*

Proof. One direction is trivial, as a maximally violated mod-k cut is also a totally tight mod-k cut. Assume now that a totally tight mod-k cut exists, associated with a vector (μ, θ) satisfying (2)–(5) and such that $\mu_i = 0$ for all $s_i^* > 0$. If $\theta \neq k-1$ and k is prime, μ can always be scaled by a factor $w \in \{2, \ldots, k-1\}$ such that $\overline{A}^T w\mu \equiv 0 \pmod{k}$ and $\overline{b}^T w\mu \equiv k-1 \pmod{k}$.

□

Note that Theorem 2 cannot be extended to the case of k nonprime.

The following considerations allow for a typically substantial reduction of the number of variables and congruences in (1)–(5). Suppose decision variables x_j have an associated lower bound L_j, i.e., system $Ax \leq b$ contains a number of inequalities of the form $-x_j \leq -L_j$. Then, it is not hard to see that all lower bound constraints which are tight at the given point x^* need not be taken into account explicitly as long as mod-k separation is concerned. The same holds for tight upper bound constraints of the form $x_j \leq U_j$, as one can always complement x_j to transform this inequality into a tight lower bound constraint. Moreover, any inequality in $Ax \leq b$ with slack $s_i^* \geq \theta$ (where $\theta \leq k-1$ is supposed to be fixed) can be removed, as $\delta^* < 0$ in (1) implies $\mu_i = 0$.

Of course, not all maximally violated mod-k cuts are guaranteed to be facet defining for P_I. In particular, a cut is not facet defining whenever it is associated with a non-minimal solution μ of the congruence system (2)–(4), where θ has been fixed to $k-1$ (barring the case of equivalent formulations of the same facet-defining cut). Indeed, the inequality associated with any solution $\tilde{\mu} \leq \mu$ is violated whenever the one associated with μ is. Hence one is motivated in finding maximally violated mod-k cuts which are associated with minimal solutions. This can be done with no extra computational effort for k prime since, as already observed, for any fixed θ all basic solutions to (2)–(4) are minimal by construction. Unfortunately, the algorithm for k nonprime does not guarantee finding a minimal solution. On the other hand, the following result holds.

Theorem 3. *If there exists a maximally violated mod-k cut for some k nonprime, a maximally violated mod-ℓ cut exists also for every ℓ which is a prime factor of k.*

Proof. First of all, observe that $Qy \equiv d \pmod{k}$ implies $Qy \equiv d \pmod{\ell}$ for each prime factor ℓ of k. Hence, given a solution (μ, θ) of (2)–(5) with $\theta = k-1$, the vector $(\mu, \theta) \bmod \ell$ yields a totally tight mod-ℓ cut, as $\theta \bmod \ell = k-1 \bmod \ell \neq 0$. The claim then follows from Theorem 2.

□

It is then natural to concentrate on the separation of maximally violated mod-k cuts for some k prime.

For several important problems, maximally violated mod-k cuts associated with minimal solutions contain families of inequalities defining facets of P_I, as shown in Sects. 4 and 5 for the TSP.

Moreover, for k prime one can find, again in $O(mn \min\{m, n\})$ time, the minimum threshold $T \geq 0$ such that system (2)–(5) has a solution μ with $\mu_i = 0$ for all $s_i^* > T$. This is achieved by first considering the subsystem defined by the variables μ_i such that $s_i^* = 0$ (with θ fixed w.l.o.g. to $k-1$), and then by iteratively introducing the other variables μ_i according to increasing s_i^*, stopping as soon as the current subsystem has a solution. This may be a useful heuristic for the separation of mod-k cuts which are not necessarily maximally violated.

3. Separation of maximally violated mod-k cuts for the TSP

In this section we discuss the separation of maximally violated mod-k cuts for the symmetric and asymmetric TSP.

The *Symmetric TSP* (STSP) polytope is defined as the convex hull of the characteristic vectors of all the Hamiltonian cycles of a given complete undirected graph $G = (V, E)$. For any $S \subseteq V$, let $\delta(S)$ denote the set of the edges with exactly one end node in S, and $E(S)$ denote the set of the edges with both end nodes in S. Moreover, for any $A, B \in V$ we write $E(A : B)$ for $\delta(A) \cap \delta(B)$. As is customary, for singleton node sets we write v instead of $\{v\}$. For any real function $x : E \to R$ and for any $Q \subseteq E$, let $x(Q) := \sum_{e \in Q} x_e$.

A widely-used STSP formulation is based on the following constraints, called *degree equations*, *subtour elimination constraints (SEC's)*, and *nonnegativity constraints*, respectively:

$$x(\delta(v)) = 2, \quad \text{for all } v \in V \tag{6}$$

$$x(E(S)) \leq |S| - 1, \quad \text{for all } S \subset V, |S| \geq 2 \tag{7}$$

$$-x_e \leq 0, \quad \text{for all } e \in E. \tag{8}$$

We next address the separation of maximally violated mod-k cuts that can be obtained from (6)–(8). Given a point $x^* \in R^E$ satisfying (6)–(8), we call *tight* any node set S with $x^*(E(S)) = |S| - 1$. It is well known that only $O(|V|^2)$ tight sets exist, which can be represented by an $O(|V|)$-sized data structure called *cactus tree* [13]. A cactus tree associated with x^* can be found efficiently in $O(|E^*||V| \log(|V|^2/|E^*|))$ time, where $E^* := \{e \in E : x^*_e > 0\}$ is the support of x^*; see [18] and also [24]. Moreover, we next show that only $O(|V|)$ tight sets need be considered explicitly in the separation of maximally violated mod-k cuts.

Applegate, Bixby, Chvátal and Cook [1] and Fleischer and Tardos [19] showed that tight sets can be arranged in *necklaces*. A necklace of size $q \geq 3$ is a partition of V into a cyclic sequence of tight sets $B_1, \ldots, B_q$ called *beads*; see Fig. 1 for an illustration. To simplify notation, the subscripts in $B_1, \ldots, B_q$ are intended modulo q, i.e., $B_i = B_{i+hq}$ for all integer h. Beads in a necklace satisfy:

(i) $B_i \cup B_{i+1} \cup \ldots \cup B_{i+t}$ is a tight set for all $i = 1, \ldots, q$ and $t = 0, \ldots, q-2$,
(ii) $x^*(E(B_i : B_j))$ is equal to 1 if $j \in \{i+1, i-1\}$, and 0 otherwise.

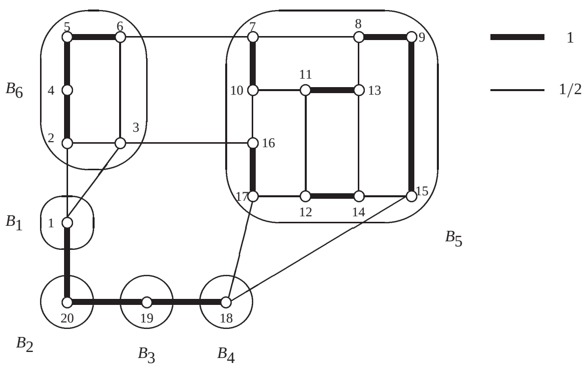

Fig. 1. A fractional point x^* and one of its necklaces

A pair (B_i, B_{i+1}) of consecutive beads in a necklace is called a *domino*. We allow for degenerate necklaces with $q = 2$ beads, in which $x^*(E(B_1 : B_2)) = 2$. Degenerate necklaces have no dominoes.

Given x^* satisfying (6)–(8), one can find in time $O(|E^*||V|\log(|V|^2/|E^*|))$ a family $\mathcal{F}(x^*)$ of $O(|V|)$ necklaces with the property that every tight set is the union of consecutive beads in a necklace of the family. The next theorem shows that the columns in the congruence system (2)–(4) corresponding to tight SEC's are linearly dependent, in $GF(k)$, on a set of columns associated with degree equations, tight nonnegativity constraints, and tight SEC's corresponding to beads and dominoes in $\mathcal{F}(x^*)$.

Theorem 4. *If any STSP mod-k cut is maximally violated by x^*, then there exists a maximally violated mod-k cut whose Chvátal-Gomory derivation uses SEC's associated with beads and dominoes (only) of necklaces of $\mathcal{F}(x^*)$.*

Proof. Let S be any tight set whose SEC is used in the Chvátal-Gomory derivation of some maximally violated mod-k cut. By the properties of $\mathcal{F}(x^*)$, S is the union of consecutive beads $B_1, \ldots, B_t$ of a certain necklace $B_1, \ldots, B_q$ in $\mathcal{F}(x^*)$, $1 \le t \le q-1$. If $t \le 2$, then S is either a bead or a domino, and there is nothing to prove. Assume then $t \ge 3$, as in Fig. 2, and add together:

- the SEC on $B_1 \cup B_2 \cup \ldots \cup B_{t-2}$,
- the SEC on B_{t-1} multiplied by $k-1$,
- the SEC on B_t,
- the degree equations on every $v \in B_{t-1}$,
- the nonnegativity inequalities $-x_e \le 0$ for every $e \in E(B_{t-1} : B_{t+1} \cup \ldots \cup B_q)$,
- the nonnegativity inequalities $-x_e \le 0$ multiplied by $k-1$ for every $e \in E(B_t : B_1 \cup \ldots \cup B_{t-2})$.

This gives the following inequality:

$$\begin{aligned}\alpha^T x &:= x(E(S)) + kx(E(B_{t-1})) - kx(E(B_t : B_1 \cup \ldots \cup B_{t-2}))\\ &\le \alpha_0 := |S| + k|B_{t-1}| - k - 1.\end{aligned}$$

All the inequalities used in the combination are tight at x^*. Moreover, all the coefficients in $\alpha^T x \le \alpha_0$ are identical, modulo k, to the coefficients of the SEC $x(E(S)) \le |S| - 1$. So we can use the inequalities in the derivation of $\alpha^T x \le \alpha_0$ in place of the original SEC to obtain a (different) maximally violated mod-k cut. Applying this procedure recursively yields the result.

□

As an immediate consequence, one has

Theorem 5. *For any given k, maximally violated mod-k cuts for the STSP can be found in $O(|E^*||V|^2)$ time, i.e., in $O(|V|^4)$ time in the worst case.*

Proof. Theorem 1 gives an $O(mn\min\{m, n\})$-time separation algorithm, where m is the number of tight constraints (6)–(7), and $n = |E^*| = O(|V|^2)$ is the number of fractional components in x^*. By virtue of Theorem 4, only $O(|V|)$ tight sets need be considered explicitly, hence $m = O(|V|)$ and the claim follows.

□

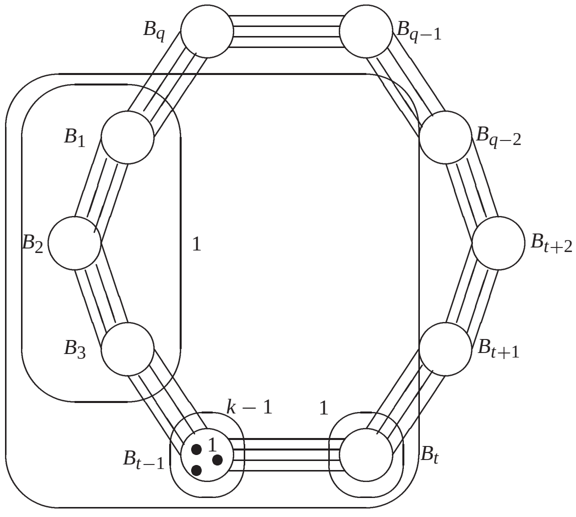

Fig. 2. Illustration for the proof of Theorem 4

The practical efficiency of the mod-k separation algorithm can be improved even further, as it turns out that one can always disregard all dominoes except one (arbitrarily chosen) in each necklace.

Theorem 6. *Let $\mathcal{B}$ contain, for each necklace in $\mathcal{F}(x^*)$, all beads and a single (arbitrary) domino. If any STSP mod-k cut is maximally violated by x^*, then there exists a maximally violated mod-k cut whose Chvátal-Gomory derivation uses SEC's associated with sets $S \in \mathcal{B}$ only.*

Proof. Let S be a tight set whose SEC is used in the derivation of some maximally violated mod-k cut. If S is not a bead of some necklace, then from Theorem 4 one can assume w.l.o.g. that S is the union of two consecutive beads, say B_i and B_{i+1}, of a certain necklace $B_1, \ldots, B_q$ with $q \geq 3$. Let $B_1 \cup B_2$ be the chosen domino in the necklace. For $i = 1, \ldots, q$, let $\mathcal{T}_i$ be the set consisting of the degree equations, the tight nonnegativity constraints, the SEC's associated with the beads in the given necklace, and the SEC associated with $B_i \cup B_{i+1}$. As in the proof of Theorem 4, we show that the SEC associated with $B_i \cup B_{i+1}$, $1 \leq i \leq q$, has the same coefficients, modulo k, as an inequality defined by a combination of elements in $\mathcal{T}_1$. If $i = 1$, this is clearly true. Assume by induction that it is true for $i = 1, \ldots, r-1$. We show that the SEC associated with $S = B_r \cup B_{r+1}$ has the same coefficients, modulo k, as an inequality defined by a combination of elements in $\mathcal{T}_{r-1}$. Adding together:

- the SEC's on B_{r-1} and B_{r+1},
- the SEC on $B_{r-1} \cup B_r$, multiplied by $k-1$,
- the degree equations on every node $v \in B_r$,
- the nonnegativity inequality $-x_e \leq 0$ for every $e \in E(B_r : B_{r+2} \cup \ldots \cup B_{q+r-2})$,

leads to

$$\alpha^T x := x(E(S)) + kx(E(B_r)) + kx(E(B_{r-1})) + kx(E(B_{r-1} : B_r))$$
$$\leq \alpha_0 := |S| + k|B_r| + k|B_{r-1}| - k - 1.$$

Therefore, by the inductive hypothesis, we can replace the SEC associated with $B_{r-1} \cup B_r$ with another inequality that has the same coefficients (modulo k), and is defined by a combination of elements in $\mathcal{T}_1$.

□

We next address the *Asymmetric TSP* (ATSP) polytope, defined as the convex hull of the characteristic vectors of all the Hamiltonian circuits of a given complete digraph $G = (V, A)$. For any $S \subseteq V$, let $\delta^+(S)$ and $\delta^-(S)$ denote the set of the arcs leaving and entering S, respectively, and let $A(S)$ denote the set of the arcs with both end nodes in S. When no confusion arises, for singleton sets we write v instead of $\{v\}$. For any $B, C \subseteq V$ we write $A(B : C)$ for $\delta^+(B) \cap \delta^-(C)$. For any real function $x : A \to R$ and for any $Q \subseteq A$, let $x(Q) := \sum_{(i,j)\in Q} x_{ij}$.

A widely-used ATSP formulation is based on the following constraints, called *out-* and *in-degree equations*, *SEC's*, and *nonnegativity constraints*, respectively:

$$x(\delta^+(v)) = 1, \quad \text{for all } v \in V \tag{9}$$
$$x(\delta^-(v)) = 1, \quad \text{for all } v \in V \tag{10}$$
$$x(A(S)) \leq |S| - 1, \quad \text{for all } S \subset V, |S| \geq 2 \tag{11}$$
$$-x_{ij} \leq 0, \quad \text{for all } (i, j) \in A. \tag{12}$$

An inequality $\alpha^T x \leq \alpha_0$ for the ATSP is called *symmetric* whenever $\alpha_{ij} = \alpha_{ji}$ for all $(i, j) \in A$.

Given a point x^* satisfying (9)–(12) with support $A^* := \{(i, j) \in A : x^*_{ij} > 0\}$, we call *tight* a node set S if the corresponding SEC is satisfied at equality. As SEC's are symmetric, one can always replace the given point x^* on G by a point y^* on the undirected counterpart of G, whose entries $y^*_{[i,j]}$ are obtained as $x^*_{ij} + x^*_{ji}$. By construction, a given set S is tight for x^* if and only if it is tight for y^*. Hence, as in the STSP, one can define in $O(|A^*||V| \log(|V|^2/|A^*|))$ time a family $\mathcal{F}(y^*)$ of $O(|V|)$ necklaces with the property that every tight set is the union of consecutive beads of some necklace in $\mathcal{F}(y^*)$.

Theorem 7. *If any ATSP mod-k cut is maximally violated by x^*, then there exists a maximally violated mod-k cut whose Chvátal-Gomory derivation uses SEC's associated with beads and dominoes (only) of necklaces of $\mathcal{F}(y^*)$.*

Proof. Same as the proof of Theorem 4, by replacing the degree equations on each node by the corresponding ATSP in- and out-degree equations.

□

As an immediate consequence, one has

Theorem 8. *For any given k, maximally violated mod-k cuts for the ATSP can be found in $O(|A^*||V|^2)$ time, i.e., in $O(|V|^4)$ time in the worst case.*

Proof. Analogous to that of Theorem 5.

□

As in the STSP, one obtains

Theorem 9. *Let $\mathcal{B}$ contain, for each necklace in $\mathcal{F}(y^*)$, all beads and a single (arbitrary) domino. If any ATSP mod-k cut is maximally violated by x^*, then there exists a maximally violated mod-k cut whose Chvátal-Gomory derivation uses SEC's associated with sets $S \in \mathcal{B}$ only.*

Proof. Analogous to that of Theorem 6, by replacing STSP degree equations by ATSP in- and out-degree equations.

□

4. mod-k cuts for the symmetric TSP

In this section we analyze facet-defining mod-k cuts for the symmetric TSP.

We first address mod-2 cuts that can be obtained from (6)–(8). A well known class of such cuts is that of *comb* inequalities, as introduced by Edmonds [14] in the context of matching theory, and extended by Chvátal [10] and by Grötschel and Padberg [20, 21] for the TSP. Comb inequalities are defined as follows; see Fig. 3 for an illustration. We are given a *handle* set $H \subset V$ and $t \geq 3$, t odd, *tooth* sets $T_1, \ldots, T_t \subset V$ such that $T_i \cap H \neq \emptyset$ and $T_i \setminus H \neq \emptyset$ hold for any $i = 1, \ldots, t$. The comb inequality associated with $H, T_1, \ldots, T_t$ reads:

$$x(E(H)) + \sum_{i=1}^{t} x(E(T_i)) \leq |H| + \sum_{i=1}^{t} (|T_i| - 1) - \frac{t+1}{2}. \tag{13}$$

The simplest case of comb inequalities arises for $|T_i| = 2$ for $i = 1, \ldots, t$, leading to the Edmonds' *2-matching constraints*. It is well known that comb inequalities define facets of the TSP polytope [22]. Also well known is that comb inequalities are mod-2

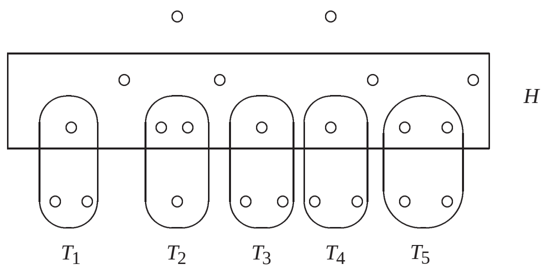

Fig. 3. A comb with $t = 5$ teeth

cuts obtained by weighing by $1/2$ and combining the following constraints:

$$\begin{aligned}
x(\delta(v)) &= 2, && \text{for all } v \in H \\
x(E(T_i)) &\le |T_i| - 1, && \text{for } i = 1, \dots, t \\
x(E(T_i \cap H)) &\le |T_i \cap H| - 1, && \text{for } i = 1, \dots, t \text{ such that } |T_i \cap H| \ge 2 \\
x(E(T_i \setminus H)) &\le |T_i \setminus H| - 1, && \text{for } i = 1, \dots, t \text{ such that } |T_i \setminus H| \ge 2 \\
-x_e &\le 0, && \text{for all } e \in \delta(H) \setminus \bigcup_{i=1}^{t} E(T_i).
\end{aligned}$$

As already mentioned, no polynomial-time exact separation algorithm for comb inequalities is known at present. A heuristic scheme for maximally violated comb inequalities has been recently proposed by Applegate, Bixby, Chvátal and Cook [1], and elaborated by Fleischer and Tardos [19] to give a polynomial-time exact method for the case of x^* with planar support. Here, comb separation is viewed as the problem of "building-up" a comb structure starting with a given set of dominoes. The interested reader is referred to [1] and [19] for a detailed description of the method.

Theorem 5 puts comb separation in a different light, in that it allows for efficient exact separation of maximally violated members of the family of mod-2 cuts which contains, among others, comb inequalities. One may wonder whether comb inequalities are the only TSP facet-defining mod-2 cuts with respect to formulation (6)–(8). This is not the case. In particular, we will show that the facet-defining *extended comb* inequalities of Naddef and Rinaldi [26] are facet-defining mod-2 cuts. We refer the reader to [23], Sect. 5.5, for a survey of known STSP facets, and in particular to pages 281–283, where extended comb inequalities (therein referred to as 2-*regular PWB* inequalities) are described.

The basic (called simple) extended comb inequalities are as follows; see Fig. 4 for an illustration. Let $H \subset V$ be the *handle* of the comb, and let $T_1, T_2, \dots, T_t$, where $t \ge 3$ odd, be disjoint *tooth* sets such that $|A_i| = |B_i| \ge 1$ for $i = 1, \dots, t$, where $A_i := T_i \cap H$ and $B_i := T_i \setminus H$. The simple extended comb inequality, written in $\le$ form,

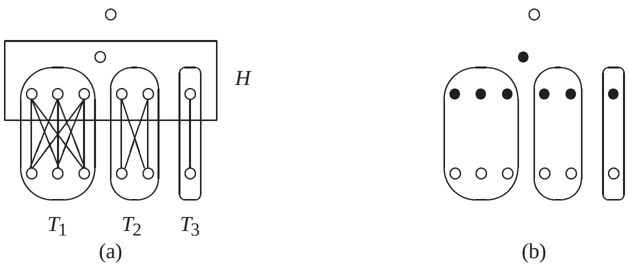

Fig. 4. (**a**) The support graph of a simple extended comb inequality; all the drawn edges, as well as the edges in $E(H)$, have coefficient 1. (**b**) A mod-2 derivation, obtained by combining the degree equations on the black nodes and the SEC's on the sets drawn in continuous line (the nonnegativity inequalities used in the derivation are not indicated)

reads

$$x(E(H)) + \sum_{i=1}^{t} x(E(A_i : B_i)) \leq |H| + \sum_{i=1}^{t} \frac{|T_i|}{2} - \frac{t+1}{2}. \tag{14}$$

This inequality is a mod-2 cut obtained by combining the following constraints weighed by $1/2$ and rounding down the right-hand side:

$$\begin{aligned} x(\delta(v)) &= 2, && \text{for all } v \in H \\ x(E(T_i)) &\leq |T_i| - 1, && \text{for } i = 1, \ldots, t \\ -x_e &\leq 0, && \text{for all } e \in \left(\delta(H) \setminus \bigcup_{i=1}^{t} E(T_i)\right) \cup \left(\bigcup_{i=1}^{t} E(A_i) \cup E(B_i)\right). \end{aligned}$$

Requirement $|A_i| = |B_i|$ expresses a necessary condition to obtain a facet-defining mod-2 cut. Indeed, it is well known that the replacement of H by $V \setminus H$ yields an equivalent inequality. Hence, suppose w.l.o.g. $|A_i| > |B_i|$ for a certain i, i.e., $|B_i| \leq (|T_i|-1)/2$. Then, in the above mod-2 derivation one could replace the SEC $x(E(T_i)) \leq |T_i| - 1$ and the nonnegativity constraints for $e \in E(A_i) \cup E(B_i) \cup E(B_i : H \setminus A_i)$ by the degree equations on the nodes in B_i and the nonnegativity constraints for $e \in E(B_i : V \setminus H)$ so as to obtain a mod-2 cut with left-hand side coefficients increased for the edges in $E(B_i) \cup E(B_i : H \setminus A_i)$ and unchanged for the remaining edges, and with a right-hand side value before rounding decreased by $1/2(|T_i| - 1 - 2|B_i|) \geq 0$.

Nonsimple extended comb inequalities are obtained by replacing every node v by a nonempty node set Q_v and by setting the coefficient of all $e \in E(Q_v)$ to β_v, where $\beta_v = 2$ if $v \in H \cap T_i$ for some i; $\beta_v = 0$ if $v \notin H \cup T_1 \cup \ldots \cup T_t$; $\beta_v = 1$ otherwise. The right-hand side is increased by $\sum_v \beta_v(|Q_v| - 1)$. Let $\tilde{H} := \bigcup_{v \in H} Q_v$ and $\tilde{T}_i := \bigcup_{v \in T_i} Q_v$ for all $i = 1, \ldots, t$. The resulting inequality turns out to be a mod-2 cut, obtained by weighing by $1/2$ and combining

$$\begin{aligned} x(\delta(v)) &= 2, && \text{for all } v \in \tilde{H} \\ x(E(\tilde{T}_i)) &\leq |\tilde{T}_i| - 1, && \text{for } i = 1, \ldots, t \\ x(E(Q_v)) &\leq |Q_v| - 1, && \text{for all } v \in \bigcup_{i=1}^{t} T_i \text{ such that } |Q_v| \geq 2 \end{aligned}$$

and the nonnegativity constraints $-x_e \leq 0$ for a convenient set of edges. The above discussion is summarized in the following theorem.

Theorem 10. *Extended comb inequalities are STSP mod-2 cuts.*

Extended comb inequalities can be derived from 2-matching constraints by means of two general lifting operations, called *edge-cloning* and *0-node lifting*. These operations have been studied by Naddef and Rinaldi [27] who proved that, under mild assumptions, they preserve the facet-defining property of the original inequality. Interestingly, at least for the case of extended comb inequalities both operations do not increase the Chvátal rank [29] of the starting inequality, and also preserve the property of being a mod-2 cut. One may wonder whether this property is true in general. A partial answer to this question will be given in Sect. 5, where we study the two operations in the more general context of the asymmetric TSP.

A family $\mathcal{N}$ of sets $S_1, \dots, S_k \subseteq V$ is called *nested* (or *laminar*) if, for all i, j, $S_i \cap S_j \neq \emptyset$ implies $S_i \subseteq S_j$ or $S_j \subseteq S_i$. The node sets associated with SEC's with nonzero multipliers in the Chvátal-Gomory derivation of an extended comb inequality define a nested family $\mathcal{N}$ with nesting degree not greater than 2, in the sense that $\mathcal{N}$ does not contain 3 subsets $S_{i_1} \subset S_{i_2} \subset S_{i_3}$. Actually, it is easy to show that any mod-k cut can be derived by only using SEC's associated with subsets defining a nested family. Interestingly, there are mod-2 facet-defining STSP cuts whose Chvátal-Gomory derivation involves SEC's with nesting level greater than 2. Here is an example. Consider the fractional point x^* of Fig. 5(a). It is not hard to check by complete enumeration that x^* maximally violates no extended comb inequality. However, x^* maximally violates the mod-2 cut $\alpha^T x \leq \alpha_0$ whose derivation is illustrated in Fig. 5(b). We have verified through a computer program that $\alpha^T x \leq \alpha_0$ is facet defining for the STSP and ATSP polytopes for $|V| \geq 12$.

We next address a simple case of facet-defining mod-3 cut, arising for $|V| = 8$ and obtained by combining the following inequalities:

$$
\begin{array}{lll}
1/3 \text{ times} & x(\delta(v)) = 2, & \text{for } v = 6, 7 \\
2/3 \text{ times} & x(\delta(v)) = 2, & \text{for } v = 3, 4 \\
1/3 \text{ times} & x(E(S)) \leq |S| - 1, & \text{for } S = \{2, 6\}, \{3, 7\}, \{3, 4, 7, 8\} \\
2/3 \text{ times} & x(E(S)) \leq |S| - 1, & \text{for } S = \{4, 8\}, \{2, 3, 4, 6, 7, 8\}, \{2, 3, 4, 5, 6, 7, 8\}
\end{array}
$$

plus $1/3$ or $2/3$ times the nonnegativity constraints $-x_e \leq 0$ for a convenient set of edges. The resulting inequality reads $\alpha^T x \leq \alpha_0$, with $\alpha_0 = 13$ and α as in Fig. 6(a), where we report a matrix whose entry α_{ij}, $i < j$, gives the coefficient of edge $(i, j) \in E$. By multiplying $\alpha^T x \leq \alpha_0$ by -2 and adding 2, 4, 4, 1, 3, 3, 3 times the degree equations on nodes $2, 3, \dots, 8$, one obtains the equivalent form $\beta^T x \geq \beta_0$, where $\beta_0 = 40 - 2\alpha_0 = 14$ and β is as in Fig. 6(b). This latter inequality coincides with the one called NEW1 in Christof, Jünger and Reinelt [9]. A generalization of this inequality will be presented in the next section.

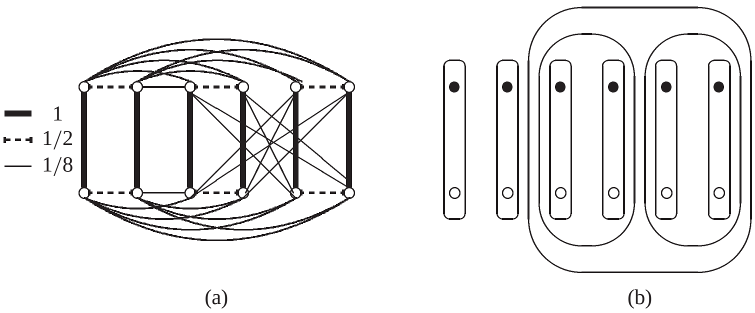

Fig. 5. (**a**) A fractional point x^* which violates maximally no extended comb inequality. (**b**) The derivation of a mod-2 cut which is maximally violated by x^*

$$\begin{bmatrix} - & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ - & - & 2 & 2 & 0 & 2 & 1 & 1 \\ - & - & - & 3 & 1 & 2 & 3 & 2 \\ - & - & - & - & 1 & 2 & 2 & 3 \\ - & - & - & - & - & 1 & 1 & 0 \\ - & - & - & - & - & - & 2 & 1 \\ - & - & - & - & - & - & - & 2 \\ - & - & - & - & - & - & - & - \end{bmatrix} \qquad \begin{bmatrix} - & 2 & 4 & 4 & 1 & 3 & 3 & 3 \\ - & - & 2 & 2 & 3 & 1 & 3 & 3 \\ - & - & - & 2 & 3 & 3 & 1 & 3 \\ - & - & - & - & 3 & 3 & 3 & 1 \\ - & - & - & - & - & 2 & 2 & 4 \\ - & - & - & - & - & - & 2 & 4 \\ - & - & - & - & - & - & - & 2 \\ - & - & - & - & - & - & - & - \end{bmatrix}$$

(a) $\alpha x \leq 13$ (b) $\beta x \geq 14$

Fig. 6. Equivalent forms of inequality NEW1

5. mod-k cuts for the asymmetric TSP

In this section we address classes of facet-defining mod-k cuts for the asymmetric TSP. It is well known that every symmetric ATSP inequality $\alpha^T x \leq \alpha_0$ has an obvious counterpart $\overline{\alpha}^T y \leq \alpha_0$ for the STSP, obtained by defining $\overline{\alpha}_{[i,j]} := \alpha_{ij} (= \alpha_{ji})$ for all undirected edges $[i, j]$ of the STSP undirected graph. Moreover, it is easy to show [16] that $\overline{\alpha}^T y \leq \alpha_0$ is facet defining for the STSP polytope if $\alpha^T x \leq \alpha_0$ is facet defining for the ATSP polytope, while the converse does not hold. In view of this correspondence, most of the properties of ATSP mod-k cuts we will analyze in this section extend to STSP. This is true whenever, for each node, the Chvátal-Gomory multipliers for the associated in- and out-degree equations coincide.

We first study two known ATSP lifting operations, and their interpretation in the context of mod-k cuts. When applied to symmetric inequalities, both operations produce a symmetric inequality, hence our analysis in the ATSP context extends immediately to the STSP case.

Suppose we are given a valid ATSP inequality $\alpha^T x \leq \alpha_0$ with integer coefficients. For each $v \in V$ we define the *loop coefficient*

$$\alpha_{vv} := \max\{\alpha_{iv} + \alpha_{vj} - \alpha_{ij} : i, j \in V, |\{i, j, v\}| = 3\} \tag{15}$$

(note however that $(v, v) \notin A$). A basic ATSP lifting operation is *clique-lifting* (also called *node cloning*), as defined in Balas and Fischetti [6]. This operation is the ATSP counterpart of the *0-node lifting* operation for the STSP, as introduced by Naddef and Rinaldi [27]. Let $G' = (V', A')$ be the complete digraph whose node set V' is obtained from V by replacing each $v \in V$ by a nonempty node set Q_v. For all $(i, j) \in A'$, let $\beta_{ij} := \alpha_{h(i)h(j)}$, where $Q_{h(i)}$ and $Q_{h(j)}$ denote the node sets containing i and j, respectively. The inequality $\beta^T x' \leq \beta_0$, where $\beta_0 := \alpha_0 + \sum_{v \in V} \alpha_{vv}(|Q_v| - 1)$, is said to be obtained from $\alpha^T x \leq \alpha_0$ through clique-lifting. An inequality which cannot be obtained through clique lifting is called *primitive*. It is shown in [6] that the validity of $\alpha^T x \leq \alpha_0$ for the ATSP polytope on G, say $P(G)$, implies the validity of $\beta^T x' \leq \beta_0$ for the ATSP polytope on G', say $P(G')$. Moreover, if $\alpha^T x \leq \alpha_0$ defines a facet of $P(G)$, then $\beta^T x' \leq \beta_0$ also defines a facet of $P(G')$.

Suppose now a rank-1 Chvátal-Gomory derivation of $\alpha^T x \leq \alpha_0$ is known, i.e., we are given multipliers u_v^+, u_v^-, and $\lambda_S \geq 0$ associated with constraints (9), (10) and (11), respectively, such that

$$\alpha_{ij} = \left\lfloor u_i^+ + u_j^- + \sum_{S \subset V: i,j \in S} \lambda_S \right\rfloor, \qquad \text{for all } (i, j) \in A$$

and

$$\alpha_0 = \left\lfloor \sum_{i \in V} (u_i^+ + u_i^-) + \sum_{S \subset V: |S| \geq 2} \lambda_S \right\rfloor.$$

Notice that Chvátal-Gomory multipliers for the nonnegativity constraints $-x_{ij} \leq 0$ are not given explicitly, as they can easily be defined as the fractional part of $u_i^+ + u_i^- + \sum_{S \subset V: i,j \in S} \lambda_S$. We aim at extending this derivation to the clique-lifted inequality $\beta^T x' \leq \beta_0$. To this end, for each $v \in V$ we define

$$\lambda_v := \alpha_{vv} - \left(u_v^+ + u_v^- + \sum_{S \subset V: v \in S, |S| \geq 2} \lambda_S \right) \tag{16}$$

where α_{vv} is computed as in (15).

Theorem 11. *Let $\alpha^T x \leq \alpha_0$ be an ATSP mod-k cut with multipliers u_v^+, u_v^-, and $\lambda_S \geq 0$ associated with constraints (9), (10) and (11), respectively, and for all $v \in V$ define λ_v as in (16). If $\lambda_v \geq 0$ for all $v \in V$, then every inequality $\beta^T x' \leq \beta_0$ obtained from $\alpha^T x \leq \alpha_0$ through clique lifting is a mod-k cut.*

Proof. We claim that $\beta^T x' \leq \beta_0$ can be obtained by combining:

u_v^+ times	$x'(\delta^+(w)) = 1,$	for all $w \in Q_v, v \in V$
u_v^- times	$x'(\delta^-(w)) = 1,$	for all $w \in Q_v, v \in V$
λ_S times	$x'(A'(\bigcup_{v \in S} Q_v)) \leq \lvert \bigcup_{v \in S} Q_v \rvert - 1,$	for all $S \subset V, \lvert S \rvert \geq 2$
λ_v times	$x'(A'(Q_v)) \leq \lvert Q_v \rvert - 1,$	for all $v \in V$ such that $\lvert Q_v \rvert \geq 2$

and then rounding down the resulting coefficients. This is clear for the left-hand side coefficients. As to the right-hand side, its increase with respect to α_0 is given by the integer quantity

$$\sum_{v \in V} (|Q_v| - 1) \cdot \left(u_v^+ + u_v^- + \sum_{S \subset V: v \in S, |S| \geq 2} \lambda_S + \lambda_v \right) = \sum_{v \in V} (|Q_v| - 1) \alpha_{vv},$$

as required. Noting that, by definition, $k\lambda_v$ is integer for all $v \in V$ concludes the proof. □

Corollary 1. *Let $\alpha^T y \leq \alpha_0$ be an STSP mod-k cut with multipliers u_v and $\lambda_S \geq 0$ associated with constraints (6) and (7), respectively, and for all $v \in V$ define λ_v as in (16), where $u_v^+ := u_v^- := u_v$ for all $v \in V$. If $\lambda_v \geq 0$ for all $v \in V$, then every inequality $\beta^T y' \leq \beta_0$ obtained from $\alpha^T y \leq \alpha_0$ through clique lifting is a mod-k cut.*

The application of Theorem 11 requires $\lambda_v \geq 0$ for all $v \in V$, a condition that can easily be checked once a mod-k derivation is given. This condition holds for all the facet-defining TSP mod-k cuts we are aware of.

The second lifting operation we address is *2-cycle cloning*, as studied in [16]. This operation is the ATSP counterpart of the *edge-cloning* procedure introduced by Naddef and Rinaldi [27] for the STSP. Given a valid ATSP inequality $\alpha^T x \leq \alpha_0$ with integer coefficients, let h and k be two distinct nodes such that

$$\Delta(h,k) := (\alpha_{hh} + \alpha_{kk}) - (\alpha_{hk} + \alpha_{kh}) > 0.$$

Define a new complete digraph $G' = (V', A')$ where V' is obtained from V by introducing two new nodes, say h' and k', and let $\beta_{ij} := \alpha_{ij}$ for all $(i, j) \in A$; $\beta_{h'j} := \alpha_{hj}$, $\beta_{jh'} := \alpha_{jh}$, $\beta_{k'j} := \alpha_{kj}$, and $\beta_{jk'} := \alpha_{jk}$ for all $j \in V \setminus \{h,k\}$; $\beta_{hk'} := \beta_{h'k} := \alpha_{hk}$; $\beta_{kh'} := \beta_{k'h} := \alpha_{kh}$; $\beta_{k'h'} := \alpha_{kh}$ and $\beta_{h'k'} := \alpha_{hk}$; $\beta_{hh'} := \beta_{h'h} := \alpha_{hh} - \Delta(h,k)$; and $\beta_{kk'} := \beta_{k'k} := \alpha_{kk} - \Delta(h,k)$. The inequality $\beta^T x' \leq \beta_0$, where $\beta_0 := \alpha_0 + \alpha_{hh} + \alpha_{kk} - \Delta(h,k)$, is said to be obtained from $\alpha^T x \leq \alpha_0$ by *cloning the 2-cycle* induced by h, k.

The 2-cycle lifted inequality $\beta^T x' \leq \beta_0$ can be thought of as obtained from $\alpha^T x \leq \alpha_0$ by first applying clique-lifting to replace nodes h and k by $\{h, h'\}$ and $\{k, k'\}$, respectively, and then by adding the nonvalid inequality $-(x'_{hh'} + x'_{h'h} + x'_{kk'} + x'_{k'k}) \leq -1$ weighed by $\Delta(h,k)$. As a result, $\beta^T x' \leq \beta_0$ is not guaranteed to be valid for $P(G')$. If $\beta^T x' \leq \beta_0$ is valid for $P(G')$, however, then it is facet defining whenever $\alpha^T x \leq \alpha_0$ defines a facet of $P(G)$, as proved in [16].

Application of 2-cycle cloning requires checking the validity of the resulting inequality, which is an NP-complete problem in general. The following theorem gives easily-checkable sufficient conditions for validity in case a mod-k derivation of $\alpha^T x \leq \alpha_0$ is available.

Theorem 12. *Let $\alpha^T x \leq \alpha_0$ be an ATSP mod-k cut with multipliers u_v^+, u_v^-, and $\lambda_S \geq 0$ associated with constraints (9), (10) and (11), respectively, and for all $v \in V$ define λ_v as in (16). If $\lambda_h > 0$, $\lambda_k > 0$, $\lambda_h + \lambda_k = 1$, and $\Delta(h,k) = 1$, then the 2-cycle cloned inequality $\beta^T x' \leq \beta_0$ is a valid mod-k cut.*

Proof. Let $Q_v := \{v\}$ for all $v \in V \setminus \{h,k\}$, $Q_h := \{h, h'\}$, and $Q_k := \{k, k'\}$. One can readily check that $\beta^T x' \leq \beta_0$ can be obtained by combining and rounding the following inequalities:

$$\begin{array}{lll} u_v^+ \text{ times} & x'(\delta^+(w)) = 1, & \text{for all } w \in Q_v, v \in V \\ u_v^- \text{ times} & x'(\delta^-(w)) = 1, & \text{for all } w \in Q_v, v \in V \\ \lambda_S \text{ times} & x'(A'(\bigcup_{v \in S} Q_v)) \leq |\bigcup_{v \in S} Q_v| - 1, & \text{for all } S \subset V, |S| \geq 2. \end{array}$$

$\square$

Corollary 2. *Let $\alpha^T y \leq \alpha_0$ be an STSP mod-k cut with multipliers u_v and $\lambda_S \geq 0$ associated with constraints (6) and (7), respectively, and for all $v \in V$ define λ_v as in (16), where $u_v^+ := u_v^- := u_v$ for all $v \in V$. If $\lambda_h > 0$, $\lambda_k > 0$, $\lambda_h + \lambda_k = 1$, and $\Delta(h,k) = 1$, then the 2-cycle (or edge) cloned inequality $\beta^T y' \leq \beta_0$ is a valid mod-k cut.*

The conditions in Theorem 12 are satisfied by several choices of h and k for most TSP facet-defining mod-k cuts, in their primitive (simple) form.

We next analyze specific classes of mod-k cuts for the ATSP. We start with mod-2 cuts, which include the following large class of facet-defining cuts introduced in Balas and Fischetti [6], called *SD-inequalities*. The primitive members of this class are as follows. Let $H \subset V$ be the *handle* of the SD-inequality, and let $T_1, \dots, T_t$ be $t \geq 1$ disjoint *tooth* sets such that $|T_i \cap H| = |T_i \setminus H| = 1$. Moreover, let S and D be (possibly empty) disjoint subsets of $V \setminus (H \cup T_1 \cup \ldots \cup T_t)$, called the set of *source* and *destination* nodes, such that $|S| + |D| + t$ is odd. The primitive SD-inequalities have the form:

$$x(A(S \cup H : D \cup H)) + \sum_{i=1}^{t} x(A(T_i)) \leq |H| + \frac{|S| + |D| + t - 1}{2}, \qquad (17)$$

and are facet defining for the ATSP polytope whenever $||S| - |D|| \leq \max\{t - 3, 0\}$, except in 3 pathological cases arising for $|V| \leq 6$; see [6]. A larger class of facet-defining cuts can be obtained by applying 0 or more times 2-cycle cloning on the pairs (h, k) with $h \in T_i \setminus H$ and $k \in T_i \cap H$ for a certain i, and then clique lifting. These inequalities are called *extended SD-inequalities*. Notice that 2-cycle cloning can also be applied to pairs (h, k) with $h \in S$ and $k \in D$, but the resulting inequality is still a primitive SD-inequality. 2-cycle cloning cannot be applied to other (h, k) pairs, instead; e.g., for $h \in S$ and $k \in T_i \setminus H$ condition $\Delta(h, k) = 1$ is not satisfied.

Theorem 13. *Extended SD-inequalities are ATSP mod-*2 *cuts.*

Proof. It is known that primitive SD-inequalities are mod-2 cuts associated with the following nonzero multipliers:

- $u_v^+ = 1/2$, for all $v \in H \cup S$,
- $u_v^- = 1/2$, for all $v \in H \cup D$,
- $\lambda_W = 1/2$, for all $W \in \{T_1, \dots, T_t\}$.

Loop coefficients (15) are easily computed as: $\alpha_{vv} = 2$ for $v \in H \cap T_i$ $(i = 1, \dots, t)$; $\alpha_{vv} = 0$ for $v \in V \setminus (S \cup D \cup H \cup \bigcup_{i=1}^{t} T_i)$, and $\alpha_{vv} = 1$ for all other nodes v. According to (16), one then has $\lambda_v = 1/2$ for all $v \in S \cup D \cup \bigcup_{i=1}^{t} T_i$ and $\lambda_v = 0$ for all other nodes v. The claim then follows from Theorems 11 and 12.

□

We next address ATSP mod-3 cuts, which include the following *C3-inequalities*:

$$x(A(W_1)) + x(A(W_2)) + \sum_{j \in W_2} x_{i_1 j} + x_{i_2 i_1} + x_{i_3 i_1} + x_{i_3 i_2} \leq |W_1| + |W_2| - 1, \qquad (18)$$

where i_1, i_2 and i_3 are distinct nodes, and W_1 and W_2 are disjoint node sets with $i_1 \in W_1$, $i_2 \in W_2$, $i_3 \notin W_1 \cup W_2$, $|W_1| \geq 2$, and $|W_2| \geq 2$. Primitive C3-inequalities arise when $W_1 = \{i_1, j_1\}$ and $W_2 = \{i_2, j_2\}$ for some $j_1, j_2 \in V$. C3-inequalities have been introduced by Grötschel and Padberg (see [22]), and proved to be facet defining by Fischetti [15]. A larger class of facet-defining cuts can be obtained from primitive C3-inequalities by applying 0 or more times 2-cycle cloning on the pairs (i_1, j_1) and (i_2, j_2), and then by using clique lifting. We call the resulting inequalities *extended C3-inequalities*.

Theorem 14. *Extended C3-inequalities are ATSP mod-*3 *cuts.*

Proof. We first prove that primitive C3-inequalities are mod-3 cuts. To this end, it suffices to consider the following nonzero Chvátal-Gomory multipliers:

- $u_v^+ = 2/3$, for $v = i_1$,
- $u_v^- = 2/3$, for $v \in \{i_1, i_2\}$,
- $u_v^+ = 1/3$, for $v \in \{i_2, i_3\}$,
- $u_v^- = 1/3$, for $v = j_2$,
- $\lambda_S = 1/3$, for $S \in \{W_1, W_2\}$.

Loop coefficients (15) for a primitive C3-inequality are computed as: $\alpha_{vv} = 2$ for $v \in \{i_1, i_2\}$; $\alpha_{vv} = 1$ for $v \in \{j_1, j_2, i_3\}$; and $\alpha_{vv} = 0$ for all other nodes v. From (16) one then obtains: $\lambda_v = 2/3$ for $v \in \{j_1, i_2, i_3\}$; $\lambda_v = 1/3$ for $v \in \{i_1, j_2\}$; and $\lambda_v = 0$ for the remaining v. The claim then follows from Theorems 11 and 12.

□

Notice that switching the multipliers of the in- and out-degree equations in the Chvátal-Gomory derivation of an extended C3-inequality provides a different (still facet-defining) mod-3 cut.

Another example of an ATSP mod-3 cut is given by the asymmetric counterpart of inequality NEW1 of Fig. 6. A larger class of cuts, which we call *extended NEW1 inequalities*, can be obtained from inequality NEW1, in the $\leq$ form given in Fig. 6(a), by applying 0 or more times 2-cycle cloning on the pairs $(2, 6)$, $(3, 7)$ and $(4, 8)$, and then by using clique lifting.

Theorem 15. *Extended NEW1 inequalities are ATSP mod-*3 *cuts.*

Proof. We have shown in Sect. 4 that the NEW1 inequality is a mod-3 cut associated with the following nonzero multipliers:

- $u_v^+ = u_v^- = 1/3$, for $v = 6, 7$,
- $u_v^+ = u_v^- = 2/3$, for $v = 3, 4$,
- $\lambda_S = 1/3$, for $S = \{2, 6\}, \{3, 7\}, \{3, 4, 7, 8\}$,
- $\lambda_S = 2/3$, for $S = \{4, 8\}, \{2, 3, 4, 6, 7, 8\}, \{2, 3, 4, 5, 6, 7, 8\}$.

Loop coefficients are as follows: $\alpha_{11} = 0$; $\alpha_{22} = 2$; $\alpha_{33} = \alpha_{44} = 4$; $\alpha_{55} = 1$; $\alpha_{66} = \alpha_{77} = \alpha_{88} = 3$. Hence one obtains: $\lambda_1 = 0$; $\lambda_2 = \lambda_4 = \lambda_5 = \lambda_7 = 1/3$; $\lambda_3 = \lambda_6 = \lambda_8 = 2/3$. The claim then follows from Theorems 11 and 12.

□

We verified by enumeration that inequality NEW1 is facet defining for the ATSP, and hence for the STSP. Therefore, extended NEW1 inequalities define facets of both the ATSP and STSP polytopes.

We finally address ATSP mod-k cuts for arbitrary $k \geq 2$. In particular, we consider the following D_{k+1}^+*-inequalities*:

$$\sum_{j=1}^{k} x_{i_j i_{j+1}} + x_{i_{k+1} i_1} + 2 \sum_{j=3}^{k+1} x_{i_1 i_j} + \sum_{j=4}^{k+1} \sum_{t=3}^{j-1} x_{i_j i_t} \leq k \tag{19}$$

where $i_1, i_2, \ldots, i_{k+1}$ are distinct nodes, and $2 \leq k \leq n-2$. D^+_{k+1}-inequalities have been introduced by Grötschel and Padberg (see [22]), and proved to be facet defining for the ATSP polytope by Fischetti [15]. These authors also addressed D^-_{k+1}-*inequalities*, obtained from D^+_{k+1}-inequalities by switching the coefficients α_{ij} and α_{ji} for all i, j. We call *extended* D^+_{k+1}- *and* D^-_{k+1}-*inequalities* the facet-defining cuts obtained from D^+_{k+1}- and D^-_{k+1}-inequalities, respectively, by applying 0 or more times 2-cycle cloning on the pair (i_1, i_{k+1}), and by using clique lifting on the resulting cuts.

Theorem 16. *Extended D^+_{k+1}- and D^-_{k+1}-inequalities are ATSP mod-k cuts.*

Proof. Consider the D^+_{k+1}-inequalities (D^-_{k+1}-inequalities can be dealt with similarly). It is known that D^+_{k+1}-inequalities are mod-k cuts associated with the following non-negative multipliers:

- $u^+_{i_1} = (k-1)/k$,
- $u^-_{i_j} = (k-j+2)/k$, for $j = 3, \ldots, k+1$,
- $\lambda_S = 1/k$, for $S \in \{Z_1, \ldots, Z_k\}$, where $Z_j := \{i_{j+1}, i_{j+2}, \ldots, i_{k+1}\} \cup \{i_1\}$.

We have $\alpha_{vv} = 2$ for $v \in \{i_3, i_4, \ldots, i_{k+1}\} \cup \{i_1\}$; $\alpha_{vv} = 1$ for $v = i_2$; and $\alpha_{vv} = 0$ for the remaining v. Hence, $\lambda_{i_1} = 1/k$, $\lambda_{i_2} = \lambda_{i_3} = \ldots = \lambda_{i_{k+1}} = (k-1)/k$, and $\lambda_v = 0$ for the remaining v. The claim then follows from Theorems 11 and 12.

□

6. Conclusions

We have investigated the separation of mod-k cuts, i.e., of Chvátal-Gomory cuts obtained with multipliers in $\{0, 1/k, \ldots, (k-1)/k\}$. Even for $k = 2$, this problem is NP-complete. However, for any given k, we have shown how to find efficiently maximally violated mod-k cuts, i.e., mod-k cuts violated by $(k-1)/k$ by the given point x^* to be separated. Separation of maximally violated mod-k cuts for the symmetric and asymmetric TSP has been studied. We have proposed an $O(|V|^2|E^*|)$-time exact separation procedure, where $|V|$ and $|E^*|$ are the number of nodes and the number of edges in the support graph of x^*. We have also investigated several families of facet-defining mod-k cuts for both the symmetric and asymmetric TSP.

Recent developments in cutting-plane algorithms, such as the work of Balas, Ceria and Cornuéjols [2,3] and Balas, Ceria, Cornuéjols and Natraj [4] on lift-and-project (disjunctive) cuts and Gomory cuts, put the emphasis on separation of large classes of inequalities which are not given explicitly. The approach developed in this paper provides still another tool for tackling hard problems.

Future theoretical research should be devoted to the study of the structure of undominated (facet-defining) mod-k TSP cuts. One should also address mod-k cuts for other combinatorial problems. Furthermore, the practical use of the separation methods herein proposed should be investigated.

Acknowledgements. The work of the first two authors was partially supported by MURST and by CNR, Italy. We thank an anonymous referee for his/her helpful comments.

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