LECTURE 10: SINGLE INPUT COST FUNCTIONS

ANSWERS AND SOLUTIONS

True/False Questions

False_ When a firm is using only one input, the cost function is simply the price of that input times how many units of that input the firm is using.

True_ When a firm is using only one input, the cost function is simply the price of that input times how many units of that input the firm needs to produce a given level of output.

False_ If a firm has no fixed costs, and its average cost is decreasing for all levels of output, then its marginal cost is also decreasing for all levels of output.

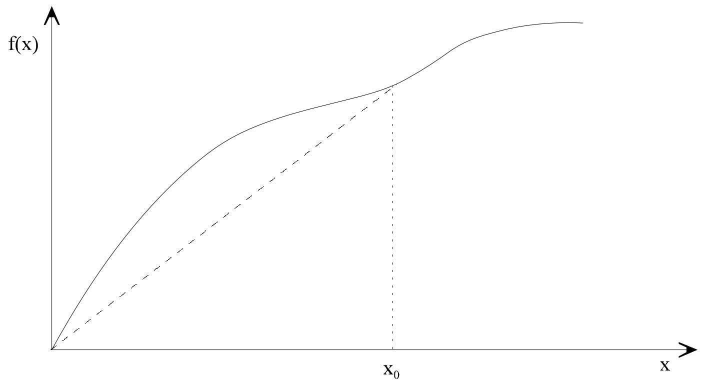
True_ If a firm's average cost is decreasing for all levels of output, then its marginal cost is less than its average cost for all levels of output.

False_ If there are no fixed costs, then average cost cannot be higher than marginal cost for all output levels.

False_ Average cost cannot be lower than marginal cost for all output levels.

Short Questions

1. Consider carefully the figure of this single-input production function, where the production function is given by the solid black line, and the average product at input level x_0 is given by the slope of the dashed line.



Notice that as x_0 moves to the right, the slope of the dashed line goes down, and as x_0 moves to the left, the slope of the dashed line goes up.

On the basis of the above information, and anything else that you can discern from the graph, answer the following questions. Provide a brief explanation of your answer.

a. Is the marginal cost associated with this production function increasing for all output levels?

No. For a single input production function, marginal cost is increasing when marginal product is decreasing. In this production function, marginal product is increasing for input levels around x_0 (the slope of the production function is increasing in that region). Thus, for input levels around x_0 marginal cost is decreasing.

b. Does this production function has decreasing returns to scale for all output levels?

Yes. Average product is decreasing for all levels of output, which directly implies that returns to scale are decreasing.

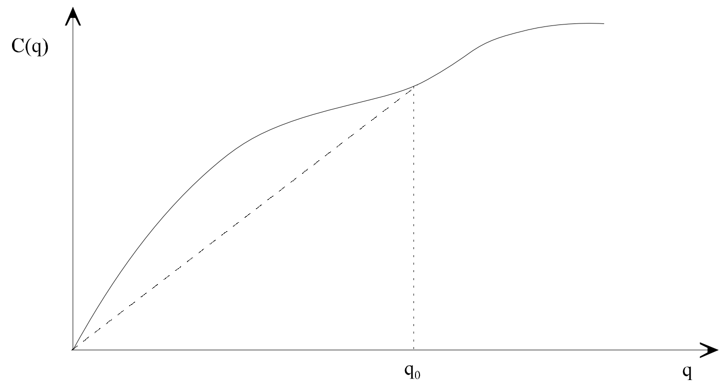
c. Is the average cost associated with this production function increasing for all output levels?

Yes. The average cost is increasing if (and only if) the average product is decreasing. Since average product is decreasing for all output levels, average cost is increasing for all output levels.

d. If the price of input *x* becomes sufficiently low, the this production function will have increasing returns to scale. Is this true, and why (or why not?)

No, it is not true. The price of an input has nothing to do with returns to scale.

2. Consider carefully the figure of this cost function of a single-input firm. The cost function is given by the solid black line, while the average cost at output level q_0 is given by the slope of the dashed line.



Notice that as q_0 moves to the right, the slope of the dashed line goes down, and as q_0 moves to the left, the slope of the dashed line goes up.

Answer the following questions given the information provided above in the text or the figure. <u>Provide a brief explanation of your answer.</u>

a. The marginal cost of this cost function is decreasing for some range of output, while increasing in others. True or False?

It is true. The marginal cost is given by the slope of the cost function. The slope of this cost function is decreasing for low output levels, increasing for intermediate output levels (around q_0), and decreasing again for higher output levels.

b. Is the average product of this firm everywhere increasing, everywhere decreasing, or decreasing for some range of output and increasing for some range of output?

The average product is increasing for all output levels. One can see from the graph and the discussion about the slope of the dashed line, that the average cost is decreasing for all output levels. When the average cost is decreasing, the average product is increasing.

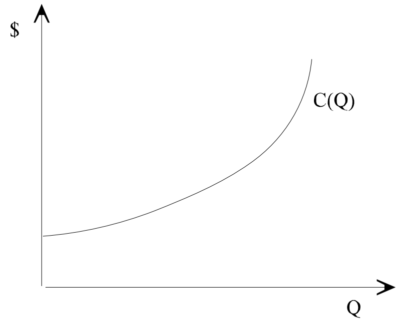
c. The production function of this firm is characterized by increasing returns to scale. True or False?

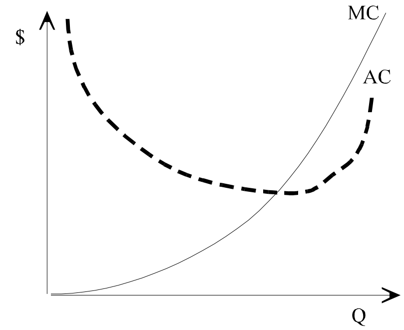
True. Since average product is increasing, this means that the production function has increasing returns to scale.

d. Will an increase in the price of this firm's input (i) leave the cost function unchanged, (ii) shift it upwards by the same amount for all output levels, (iii) shift it up proportionately to the current cost incurred to produce a given output level, or (iv) shift it up proportionately to the output level produced?

The third of these statements is the correct one. An increase in the input price increases costs proportionately from their current values.

3. Consider the figure for this total cost function, given in the figure below on the left.





This production function has only a single input.

a. Is this production function characterized by increasing, constant, or decreasing returns to scale? Explain.

Returns to scale can be inferred by looking at what happens to average cost as output increases. For low levels of output, average cost is declining (recall that average cost is the slope of a ray from the origin to the cost function). For higher levels of output, average cost is increasing. Therefore, for low levels of output, this production function is characterized by *increasing* returns to scale, while for high levels of output it is characterized by *decreasing* returns to scale.

b. Is the marginal product of this production function increasing for every output level, decreasing for every output level, or constant. Explain.

The marginal product of a single input production function is increasing if the marginal cost associated with that production function is decreasing, and conversely, the marginal product of a single input production function is decreasing if the marginal cost is increasing. The marginal cost is the slope of the cost function. It can be seen that the marginal cost is basically zero at Q=0, and then it is steadily increases.

c. In the blank figure on the left above, draw the average cost and marginal cost functions that correspond to the cost function on the right.

The marginal cost is in the thin solid line, while the average cost is in the thick dashed line. Observe that AC starts from infinity (as there are some fixed costs), then declines, and finally increases again. MC starts from zero and continuously increases. MC and AC intersect at the minimum of the AC curve.

Problems

In the short run, the firm's amount of capital equipment is fixed at $K = 16$. Therefore, the firm's short run production function is given by

$$q = 16\sqrt{L}$$

The cost of capital is $r = \$1$, and the wage rate for L is $w = \$4$. Given that the firm must finance 16 of capital (it is a fixed input) at a price of 1, its fixed costs are equal to 16. In addition to the fixed costs, the firm will incur costs for the labor it decides to hire.

a. What is the firm's (short-run) demand for labor as a function of output produced, i.e., what is the input requirement of labor to product output q?

Since labor is the only input that the firm can vary in the short-run, there is a unique amount of labor that would allow the firm to produce any required level of output. This can be found by simply solving the short run production function for L.

$$q = 16\sqrt{L} \quad \Rightarrow$$

$$q^2 = 256\,L \quad \Rightarrow$$

$$L = \frac{q^2}{256}$$

b. Calculate the firm's short-run total cost curve.

The short-run total cost is given by:

$$SRTC = w\,L + r\,\bar{K}$$

Substituting in for w, r, and $\bar{K}$ we get:

$$SRTC = 4\,\frac{q^2}{256} + 1 \quad 16$$

$$= \frac{q^2}{64} + 16$$

c. Calculate the short-run average cost curve.

$$SRAC = \frac{SRTC}{q}$$

$$= \frac{\frac{q^2}{64} + 16}{q}$$

$$= \frac{q}{64} + \frac{16}{q}$$

d. What is the firm's short-run marginal cost function ?

$$SRMC = \frac{d\,SRTC}{d\,q}$$

$$= 2\,\frac{q}{64}$$

$$= \frac{q}{32}$$

e. Which output level minimizes the short run average cost ? [The answer should be a number.]

Recall that when marginal cost is less than average cost, average cost is falling, and conversely when marginal cost exceeds average cost, average cost is increasing. Thus, when average cost is at a minimum (i.e., the slope of the average cost function is zero), marginal cost must be equal to average cost. The fastest way to find the answer is to use this fact. Equating *SRMC* with *SRAC* we get:

$$SRMC = SRAC \quad \Rightarrow$$

$$\frac{q}{32} = \frac{q}{64} + \frac{16}{q} \quad \Rightarrow$$

$$\frac{2\,q}{64} = \frac{q}{64} + \frac{16}{q} \quad \Rightarrow$$

$$\frac{q}{64} = \frac{16}{q} \quad \Rightarrow$$

$$q^2 = 1024 \quad \Rightarrow$$

$$q_{AC_{\min}} = 32$$

Alternatively, one could also directly differentiate the short-run average cost function and find the output level that minimizes it. The same answer would have been obtained using this alternate method.

2. A firm drills for oil in the Texas Panhandle. The amount of oil the firm can extract from a reservoir depends on the number of wells that it drills and is given by the equation

$$X = \alpha \log(1 + 4D)$$

where X is the amount of oil extracted, D is the number of wells drilled, and α is a parameter that reflects the efficiency of drilling in terms of producing oil. Each well has a cost of 5.

a. Suppose $\alpha = 5$. How many wells does the firm have to drill to extract 50 units of oil? [Note: $\log(x) = y \Rightarrow y = e^x$]

If $\alpha = 5$, then the production function that relates the number of wells to the oil extracted from the ground is

$$X = 5 \log(1 + 4D)$$

For X=50, the above equation becomes

$$50 = 5 \log(1 + 4D)$$

Solving for D, we get

$$10 = \log(1 + 4D) \quad \Rightarrow$$

$$e^{10} = 1 + 4D \quad \Rightarrow$$

$$D = \frac{e^{10} - 1}{4} \approx 5506$$

The firm will need to drill 5,506 wells to obtain the required level of oil!!

b. Suppose now that a technological advance doubles the oil that can be extracted from any given number of wells, i.e., suppose that the value of α is now equal to 10. How many well would the firm need to extract 50 units of oil from such a reservoir?

If $\alpha = 10$, then the production function that relates the number of wells to the oil extracted from the ground is

$$X = 10 \log(1 + 4D)$$

For X=50, the above equation becomes

$$50 = 10 \log(1 + 4D)$$

Solving for D, we get

$$5 = \log(1 + 4D) \Rightarrow$$

$$e^5 = 1 + 4D \Rightarrow$$

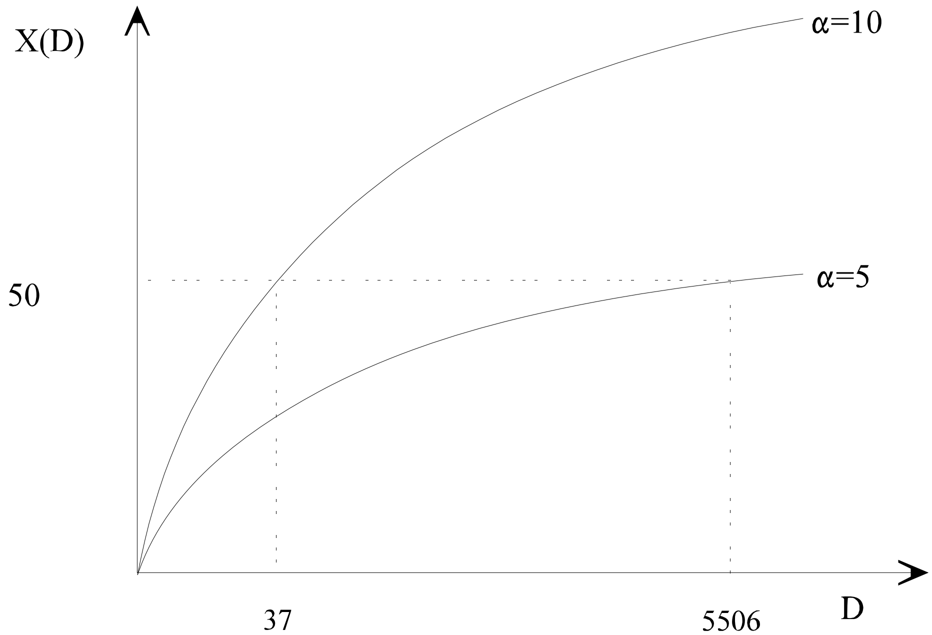
$$D = \frac{e^5 - 1}{4} \approx 37$$

If the technology of extraction improves in efficiency, the firm would need to drill only 37 wells!

c. Why do you think is there such a big difference between the answers in (a) and (b). Why isn't the answer in (b) simply half the answer in (a)?

The reason for the large difference in the answers in (a) and (b) has to do with the fact that this production function has strong diminishing returns to scale. The amount of oil that the marginal well can extract becomes very small for high values of D . . . the first few wells are much more productive that the subsequent wells. Thus an increase in the productivity of these early wells by a factor of 2 can substitute for a huge amount of subsequent drilling.

This can also be seen graphically in the following figure.



It can be easily seen that a doubling of the output for any given value of D reduces the number of wells needed to produce a given amount of output more than proportionately.

d. By how much (in percentage terms) does this doubling in drill efficiency reduce the costs of extracting 50 units of oil?

Because the only costs are the costs of drilling, the percentage reduction in costs of extracting 50 units of oil is equal to the percentage reduction in the number of wells needed to be drilled. This reduction can be obtained from

$$\frac{5506-37}{5506} \approx 0.993$$

i.e., costs are been reduced by approximately 99.3%.

3. The oil extracted from an oil field depends on the number of wells drilled on that field, and on the field's remaining resources or capacity. Suppose that for field A this relationship be given by

$$q_A = \log(1+D_A)$$

where D_A is the number wells drilled in field *A*. Another field, denoted by *B*, is much bigger and for the same number of wells, the oil extracted is three times that of field *A*. In particular, for field *B* oil extracted is given by

$$q_B = 3 \log(1 + D_B)$$

where D_B is the number of wells drilled in field *B*.

a. What is the marginal product of wells drilled in each of the two fields?

The marginal product of wells drilled is given by the derivative of output with respect to wells. For oil field *A* it is equal to

$$\frac{dq_A}{dD_A} = \frac{1}{1 + D_A}$$

and for oilfield B it is equal to

$$\frac{d\,q_B}{d\,D_B} = \frac{3}{1 + D_B}$$

b. Suppose there are 3 wells already drilled in field *A* and 14 wells drilled in field *B*. The oil company has resources to drill an additional well. In which oil-field should it drill it to get the largest increase in extracted oil?

It should drill that extra well in the oil field that the marginal product of wells is highest. Since $D_A = 3$ and $D_B = 14$, the marginal product of wells in oil field *A* is equal to 1/4, while the marginal product of wells in oilfield *B* is 3/15=1/5. [To obtain this just plug 3 and 14 in the solutions to part (a) above.]

Therefore, the firm should drill that additional well to oil field *A*, since 1/4 > 1/5. Despite that for the same number of wells oil field *B* is 3 times as productive as oilfield *A*, drilling in oilfield makes sense. This is because there are diminishing returns to scale to each of the two fields and field *B* has many more wells drilled already compared to field *A*.

c. If each well costs 5 thousand dollars to drill, what is the cost (in thousands of dollars) of drilling enough wells to obtain 4 units of oil from well *A* and 9 units of oil from well *B*? [Allow for fractional wells.]

The number of wells required to produce output q_A and q_B in the two fields is obtained by solving the production functions for the number of wells drilled. For oilfield *A* this yields

$$q_A = \log(1+D_A) \Rightarrow$$

$$e^{q_A} = 1 + D_A \Rightarrow$$

$$D_A = e^{q_A} - 1$$

Repeating these steps for oilfield B yields

$$q_B = 3 \log(1+D_B) \Rightarrow$$

$$\frac{q_B}{3} = \log(1+D_B) \Rightarrow$$

$$e^{\frac{q_B}{3}} = 1 + D_B \Rightarrow$$

$$D_B = e^{\frac{q_B}{3}} - 1$$

Plugging 4 and 9 for q_A and q_B respectively, and using the fact that each well has a cost of 5, we can obtain the total cost of the wells required to produce these output levels from the expression

$$\begin{aligned} Cost &= 5\ (e^4 - 1 + e^{9/3} - 1) \\ &= 5\ (e^4 + e^3 - 2) \\ &\approx 363.4 \end{aligned}$$

4. Consider the production function

$$f(x) = \begin{cases} 3x^2 - \beta & \text{if } 3x^2 - \beta \geq 0 \quad \text{(or, equivalently, if } x \geq \sqrt{\beta/3}\text{)} \\ 0 & \text{if } 3x^2 - \beta < 0 \quad \text{(or, equivalently, if } x < \sqrt{\beta/3}\text{)} \end{cases}$$

The positive parameter β captures the notion of the set-up cost: you need to use a critical level of input before you get any positive output.

a. Calculate the marginal product and the average product of x. Treat β as an unknown (positive) parameter.

The marginal product of x is given by the derivative of $f(x)$ with respect to x. It is equal to

$$MP_x = \frac{df(x)}{dx} = 6x \qquad \text{if} \qquad x \geq \sqrt{\beta/3}$$

and zero if $x < \sqrt{\beta/3}$. The average product of x is given by the ratio of $f(x)$ over x.

$$AP_x = \frac{f(x)}{x} = 3x - \frac{\beta}{x} \qquad \text{if} \qquad x \geq \sqrt{\beta/3}$$

and zero if $x < \sqrt{\beta/3}$.

b. Show mathematically whether this production function has increasing, decreasing, or constant returns to scale?

This production is increasing returns to scale if the average product is increasing in x, decreasing returns to scale if the average product is decreasing in x, and constant returns to scale if the average product is a constant. To determine which is the case, we can differentiate the average product with respect to x, to get

$$\frac{dAP_x}{dx} = 3 + \frac{\beta}{x^2} \qquad \text{if} \qquad x \geq \sqrt{\beta/3}$$

and zero if $x < \sqrt{\beta/3}$.

Since the parameter β is a positive number, and x is positive (you cannot consumer negative inputs!), the production function is increasing returns to scale for $x \geq \sqrt{\beta/3}$, and constant returns to scale for $x < \sqrt{\beta/3}$.

c. Suppose that each unit of input x has a cost of w. What is the cost of producing q units of output?

To identify the cost of producing q units of output, one first needs to compute the input required to produce q units of output (where q is a positive number; clearly, the lowest cost of producing zero units of output is zero). This can be obtained by solving the production function for the input x, to get

$$q = 3x^2 - \beta \quad \Rightarrow$$

$$q + \beta = 3x^2 \quad \Rightarrow$$

$$x^2 = \frac{q + \beta}{3} \quad \Rightarrow$$

$$x = \sqrt{\frac{q + \beta}{3}}$$

Since each unit of x costs w, the total cost of producing q units of output is

$$C(q) = w\,x$$

$$= w\sqrt{\frac{q + \beta}{3}}$$