LECTURE 5: PRINCIPLES OF PRODUCTION FUNCTIONS - SINGLE INPUT FUNCTIONS

ANSWERS AND SOLUTIONS

True/False Questions

True_ If a production function has an elasticity of scale of 3, this means that increasing all inputs by 1% will increase output by 3%.

False_ The marginal product of an input *x* is higher than the average product of *x*.

False_ Consider a single-input production function. If the marginal product exceeds the average product for all levels of output and there are no fixed inputs, then this production function is characterized by decreasing returns to scale.

False_ Consider a single-input production function. If the marginal product is increasing at some level of output *q*, then the production function is characterized by increasing returns to scale at that level of output.

False_ If a production function has an elasticity of scale of 2, this means that increasing all inputs by 2% will increase output by 1%.

False_ The average product of *x* is higher than the marginal product of *x*.

True_ If the average product of a single input production function is increasing for all levels of output, then this production function is characterized by increasing returns to scale.

True_ If the marginal product of a single input production function exceeds the average product for all levels of output and there are no fixed inputs, then this production function is characterized by increasing returns to scale.

True_ Consider a single-input production function. If the average product is increasing at some level of output *q*, then the production function is characterized by increasing returns to scale at that level of output.

True_ The average product of a single input production function is given by $q^{0.5}$ where *q* is the output level. This production function is characterized by increasing returns to scale.

Short Questions

1. Explain what is meant by “decreasing returns to scale”.

Decreasing returns to scale means that if all inputs are increased by 1%, output will be increased by less than 1%.

Problems

1. The production function of an oil-field is given by

$$Q = \log(1 + \beta D)$$

where Q is the quantity of oil produced, D reflects the amount of drilling in the field (measured in thousands of feet of extraction holes drilled into the oil-field), and β is a positive number that indexes the effectiveness of drilling.

a. What is the marginal product of additional drilling? [Assume for simplicity that D is a real number, i.e, that the firm can drill increments of thousands of feet.] Are there decreasing returns to increasing drilling in that field?

The marginal product of additional drilling is

$$\frac{dQ}{dD} = \frac{\beta}{1 + \beta D}$$

This is clearly decreasing in the amount of drilling done. Therefore, there are diminishing returns to drilling in this field.

b. Suppose that an other oil-field yields oil on the basis of the production function

$$Q = 2 \log(1 + \beta D)$$

That is, this oil-field is more productive than the first one. Suppose that the firm that owns the oil-fields has drilled 50 thousand feet in the first field and 120 thousand feet in the second field. For what values of β would the firm choose to expand drilling in the first (relatively unproductive) field rather than the second?

Denote by Q_1 and D_1 the output and drilling in field 1, and by Q_2 and D_2 the output and drilling in field 2. The marginal product of drilling in field 1 is

$$\frac{\beta}{1 + \beta D_1}$$

and the marginal product of drilling in field 2 is

$$\frac{2\ \beta}{1\ +\ \beta\ D_2}$$

The firm would choose to drill in the first field if

$$\frac{\beta}{1\ +\ \beta\ D_1} > \frac{2\ \beta}{1\ +\ \beta\ D_2}$$

Plugging in the values for D_1 and D_2 we have

$$\frac{\beta}{1\ +\ \beta\ 50} > \frac{2\ \beta}{1\ +\ \beta\ 120} \quad \Rightarrow$$

$$1\ +\ 120\ \beta > 2\ +\ 100\ \beta \quad \Rightarrow$$

$$20\ \beta > 1 \quad \Rightarrow$$

$$\beta > \frac{1}{20}$$

Therefore, if β is greater than 0.05 the firm will choose to expand its drilling on field 1.

Discussion: How is it possible for the firm to prefer to expand production in the relatively unproductive field? This might sound counter-intuitive. The reason is that drilling in both fields yields diminishing returns. Field 2 is more productive, in that its production function is above the production function of field 1 (i.e., if you drill the same amount in both fields, field 2 will yield more oil). However, if you drill enough in field 2 the marginal value of the additional drilling will eventually become very small . . . possibly smaller than that of field 1, provided you only drill a little in that field.

c. How does the effectiveness of drilling technology affect that decision? That is, will the firm choose to expand drilling in the first field of the second when drilling technology is effective or when it is ineffective?

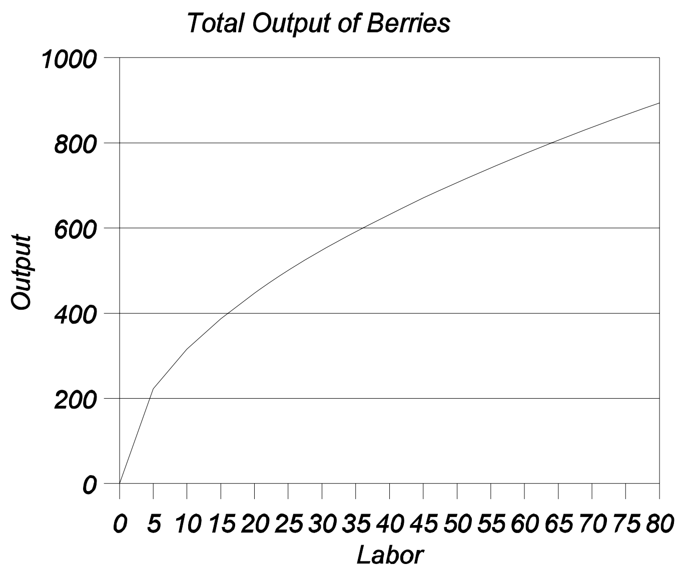
It follows from the solution in part (b) that if the drilling technology is good enough, the firm will expand drilling in field 1; if it is not good enough, it will choose to expand drilling in field 2.

2. The number of berries harvested in a field depends on the number of workers employed in that field. Let the quantity harvested be given by

$$q = 100\sqrt{L}$$

where L the number of labor-hours used.

a. Graph the relationship between q and L, with q on the vertical axis and L on the horizontal axis. Does this production function exhibit decreasing, constant, or increasing returns to scale ?



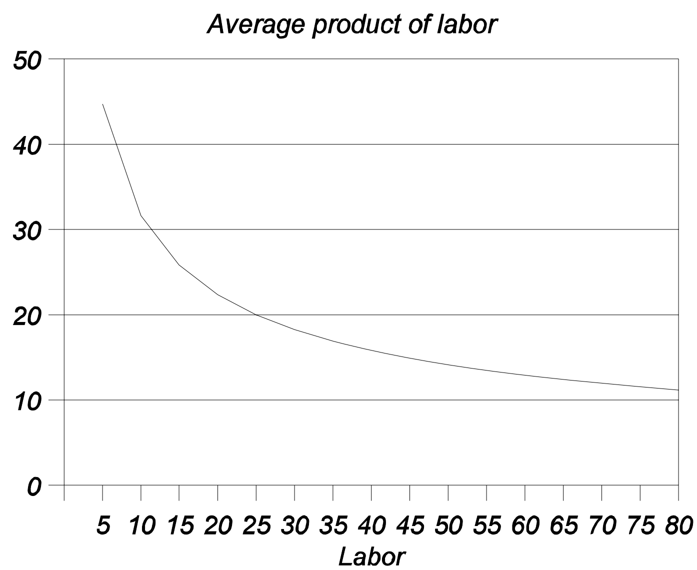
This production function exhibits decreasing returns to scale, as increasing labor inputs (the only production input) results in a less than proportional increase in the output.

b. What is the average productivity of labor in this farm? Graph this relationship and show that AP_L diminishes as labor input increases.

The average productivity of labor is the total output divided by the amount of labor employed:

$$AP_L = \frac{q}{L} = \frac{100\sqrt{L}}{L} = \frac{100}{\sqrt{L}}$$

The relationship of the average product to output is graphed below.



It is apparent, both from the formula and the graph, that for higher values of L, the average product is lower.

c. Derive the marginal productivity of labor for this farm. Show that $MP_L < AP_L$ for all values of *L*.

The marginal productivity of labor is the slope of the total output with respect to labor.

$$MP_L = \frac{dq}{dL} = \frac{50}{\sqrt{L}}$$

It can be readily seen that $MP_L = 0.5\ AP_L$.

3. Consider the production function

$$f(x) = \begin{cases} \alpha\sqrt{x} - \beta & \text{if } \alpha\sqrt{x} - \beta \geq 0 \\ 0 & \text{if } \alpha\sqrt{x} - \beta < 0 \end{cases}$$

The parameter β captures the notion of the set-up cost: you need to use a critical level of input before you get any positive output.

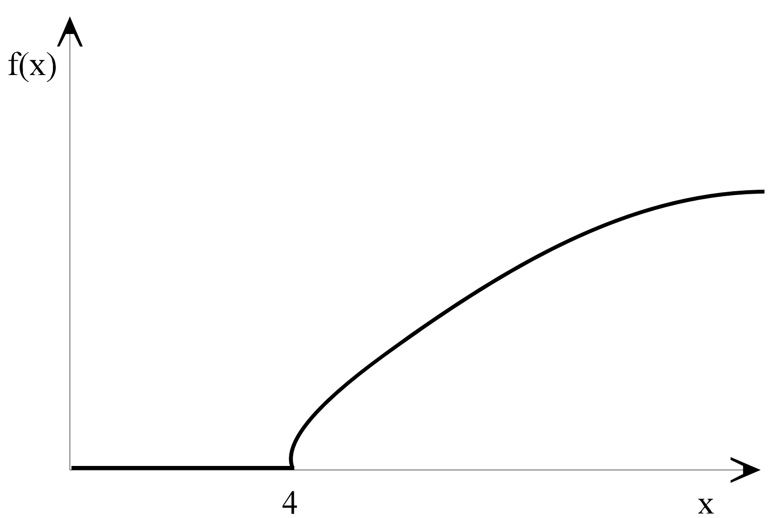
a. Graph this production function for $\alpha = 1$ and $\beta = 2$. Label any intercepts. Show work.

For these parameter values, the production function is given by

$$f(x) = \begin{cases} \sqrt{x} - 2 & \text{if } \sqrt{x} - 2 \geq 0 \\ 0 & \text{if } \sqrt{x} - 2 < 0 \end{cases}$$

This is basically the square root function, shifted down by an amount equal to 2.

Output is zero until $\sqrt{x} > 2 \Rightarrow x > 4$. From then onwards, output is increasing, but at a decreasing rate (because of the square root).



b. Calculate the marginal product and the average product of x. Treat α and β as unknown parameters.

The marginal product of x is equal to the slope of the production function, and thus given by

$$MP(x) = \begin{cases} \dfrac{\alpha}{2} \dfrac{1}{\sqrt{x}} & \text{if } \alpha \sqrt{x} - \beta \geq 0 \\ 0 & \text{if } \alpha \sqrt{x} - \beta < 0 \end{cases}$$

The average product of x is equal to output divided by x, and thus given by

$$f(x) = \begin{cases} \alpha \dfrac{1}{\sqrt{x}} - \dfrac{\beta}{x} & \text{if } \alpha \sqrt{x} - \beta \geq 0 \\ 0 & \text{if } \alpha \sqrt{x} - \beta < 0 \end{cases}$$

c. Does this production function have increasing or decreasing returns to scale? Base your answer and explanation on your graph in part (a). [Using math is fine, but it is the hard way to show this, and you don't need to do it.]

Returns to scale are increasing if the average product is increasing, and decreasing if the average product is decreasing. The average product is the slope of a ray from the origin to the production function. It is clear from the figure above that the slope of such a ray from the origin is initially zero, then increases, but eventually decreases. Therefore, for low input levels the production function has constant returns to scale (output is constant at zero!), for intermediate levels of input it has increasing returns to scale, and for higher levels of inputs it has decreasing returns to scale.

4. Consider the production function

$$f(x) = \begin{cases} \alpha\, x^2 - \beta & \text{if } \alpha\, x^2 - \beta \geq 0 \\ 0 & \text{if } \alpha\, x^2 - \beta < 0 \end{cases}$$

The parameter β captures the notion of the set-up cost: you need to use a critical level of input before you get any positive output.

a. Graph this production function for $\alpha = 1$ and $\beta = 2$.

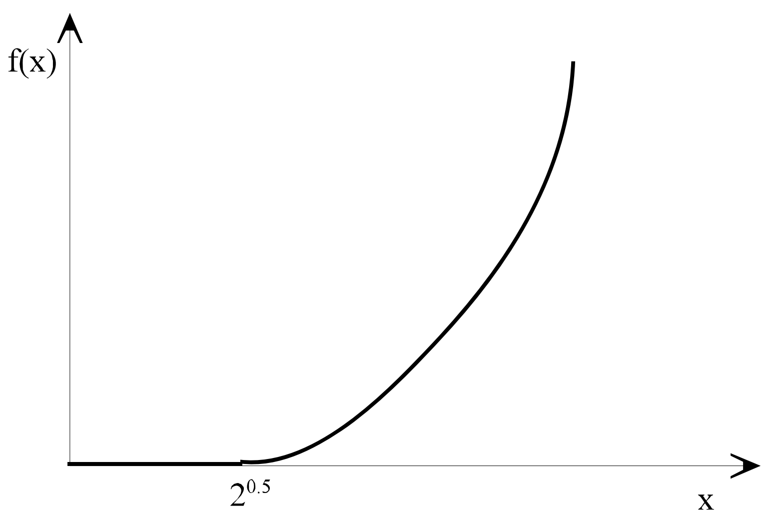
Setting $\alpha = 1$ and $\beta = 2$ in the above production function we obtain

$$f(x) = \begin{cases} x^2 - 2 & \text{if } x^2 - 2 \geq 0 \\ 0 & \text{if } x^2 - 2 < 0 \end{cases} \Rightarrow$$

$$f(x) = \begin{cases} x^2 - 2 & \text{if } x \geq \sqrt{2} \\ 0 & \text{if } x < \sqrt{2} \end{cases}$$

There is no output until the input level reaches $\sqrt{2}$, afterwards output is increasing at an increasing rate. In fact, the production function looks like the quadratic function shifted downwards by an amount equal to $\sqrt{2}$.

The production function is plotted below:



b. Calculate the marginal product and the average product of x. Treat α and β as unknown parameters.

Notice that for input level x, the output produced is positive if

$$\alpha x^2 - \beta > 0 \quad \Rightarrow$$

$$x > \sqrt{\frac{\beta}{\alpha}}$$

For lower levels of input use, output is zero. The marginal product is given by the slope of the above function. Thus,

$$MP(x) = \frac{df(x)}{dx}$$

$$= \begin{cases} 2 \alpha x & \text{if } x \geq \sqrt{\frac{\beta}{\alpha}} \\ 0 & \text{if } x < \sqrt{\frac{\beta}{\alpha}} \end{cases}$$

Average product is the slope of the line from the origin to the production function. Thus,

$$AP(x) = \frac{f(x)}{x}$$

$$= \begin{cases} \alpha x - \frac{\beta}{x} & \text{if } x \geq \sqrt{\frac{\beta}{\alpha}} \\ 0 & \text{if } x < \sqrt{\frac{\alpha}{\beta}} \end{cases}$$

c. Does this production function have increasing or decreasing returns to scale? Base your answer and explanation on your graph in part (a). [Using math is fine, but it is the hard way to show this, and you don't need to do it.]

The production function has increasing returns to scale if average product is increasing, decreasing returns to scale if average product is decreasing, and constant returns to scale is average product is constant.

For input levels that are less than $\sqrt{2}$, average product is constant. Hence, for this range of input, returns to scale are constant.

For input levels that are higher than $\sqrt{2}$, average product is increasing. This can be seen either directly from the figure (the slope of a ray from the origin is increasing as we increase *x*), or by taking the derivative of the average product function. Hence, for this range of inputs, the production function has increasing returns to scale.

d. Suppose you did not know the values of parameters α and β but you only knew that they were positive constants. Would you still have been able to reach the same the conclusion as that obtained in part (c) above?

Yes. For values of *x* that are less than $\sqrt{\alpha/\beta}$, the production function exhibits constant returns to scale, while for higher input levels, returns to scale are positive, as the general shape of the production function would not depend on the values of α and β. This can formally be seen by differentiating the average product of the production function with respect to *x*. This yields

$$\frac{d\,AP(x)}{dx} = \begin{cases} \alpha + \dfrac{\beta}{x^2} & \text{if } x \geq \sqrt{\dfrac{\beta}{\alpha}} \\[2ex] 0 & \text{if } x < \sqrt{\dfrac{\alpha}{\beta}} \end{cases}$$

Notice that for $x > \sqrt{\alpha/\beta}$, $AP(x)$ increases with *x* regardless of the values of α and β.