LECTURE 2

SINGLE VARIABLE OPTIMIZATION

QUESTIONS/ISSUES TO ADDRESSED:

1. How did calculus made its way into Economics?

2. Why is the optimization hypothesis widely used?

3. How should one view optimization-based models?

4. Why is single variable optimization not enough?

Skills to be Mastered:

1. Differentiation of functions of a single variable.

2. Calculating the maximum and the minimum of functions of a single variable.

How Did Calculus Make it Into Economics

The Assumption of Optimal Behavior:

Economic agents are seeking to achieve something that is “best” or “optimal” from their point of view.

Examples:

– Managers maximize profits or utility.

– Consumers maximize utility.

– Governments maximize output or chances of re-election.

Issues:

- People do often do not do what is best for them.

- People do not solve maximization problems before they take decisions.

Do these concerns invalidate the optimization hypothesis?

- If the economic agents' deviations from optimality are not systematic, the optimal solution may describe the average behavior.

- If economic players arrive at a near optimum behavior via heuristics, optimization-based economics would still have descriptive power.

- Optimization based economics can be used in a normative manner, even it sometimes does not describe actual behavior well.

WHY IS THE OPTIMIZATION HYPOTHESIS SO WIDELY USED?

a. This assumption, though not literally true, is a good approximation for "average" behavior in many cases.

b. The concept is precise.

c. There are a lot of mathematical techniques that are of widespread use that can be readily used to explore problems that are written as maximization problems.

ILLUSTRATIVE MOTIVATING EXAMPLE

Consider a manager of firm the profits of which are a function of the output it produces.

Nothing else, except output, affects the firms profits.

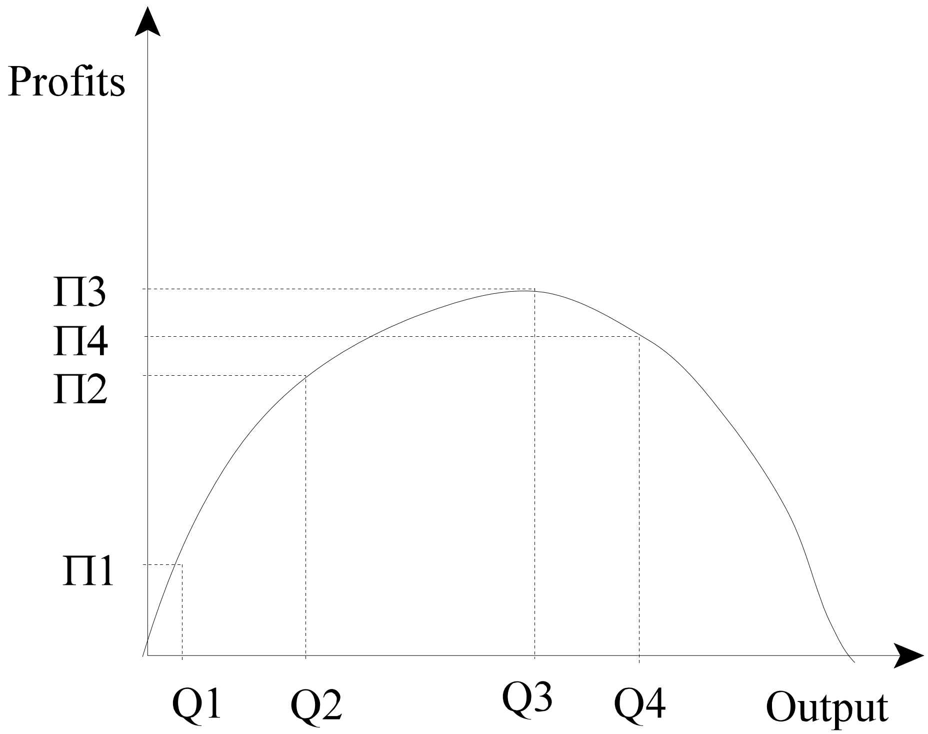
What are some reasonable properties of this profit function ?

- At zero output the firm makes zero profit.
- As output increases, its profits rise.
- At large levels of output the firm can only sell it all the output by dropping its price.

  In turn, this results in lower profits.

THEREFORE: There must a particular level of output which results in maximum profits.

This is depicted in the figure below.



Is this sufficient to describe a simple economic model of this firm?

Not yet.

What is needed is a behavioral assumption.

Assumption: The manager wants to maximize the firm’s profits.

Other assumptions consistent with an optimization framework are possible.

IMPLEMENTATION: FINDING THE OPTIMAL OUTPUT

– Trial and Error.

– Direct Optimization.

Our Approach:

In this course, we will abstract from the precise manner in which the manager finds the optimum level of output.

We will presume that he knows the shape of his profit function and picks the profit maximizing output, q_3, right away.

Observation:

The profit maximizing level of output corresponds to the level of output where the slope of the profit function is zero.

Idea:

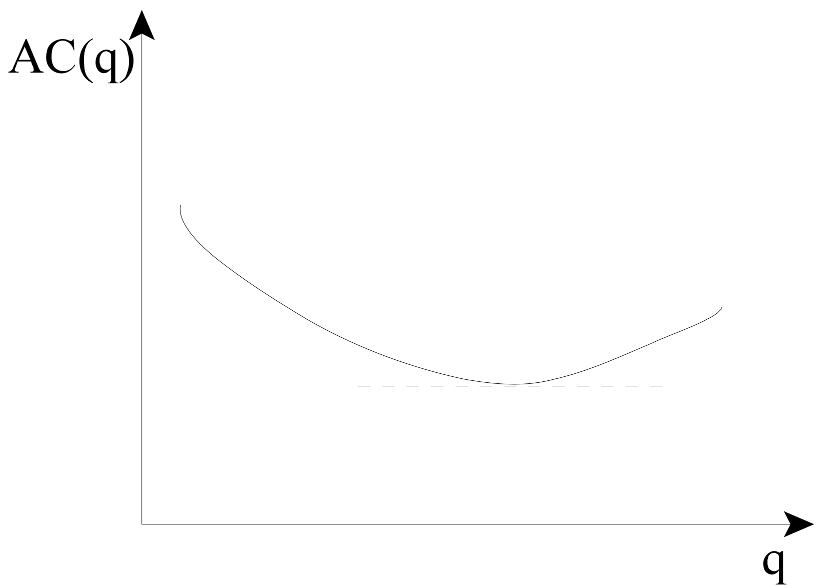
If we could find a function for the slope, then we could solve the equation

$$\text{slope}\ (q) = 0$$

and find the optimum output.

This function for the slope exists and is called the *derivative*.

Second Order Derivatives



The slope of this function is zero at a point that is clearly not the maximum.

Indeed, setting the first derivative to zero corresponds to a local minimum.

Key Result:

Slope = 0 is a necessary condition for a maximum [if we are at a maximum, then the slope must be equal to zero], but not a sufficient condition [sometimes the slope will be equal to zero, but we will not be at a maximum]

We can gain some insight by looking at the slope of the slope.

This is known as the *second derivative* of a function.

A. For a maximum, say maximizing profits, we need

i. First Order Condition:

$$\frac{d\,f(x)}{d\,x} = 0$$

ii. Second Order Condition:

$$\frac{d^2 f(x)}{d\,x^2} < 0$$

B. For a minimum, say minimizing costs, we need:

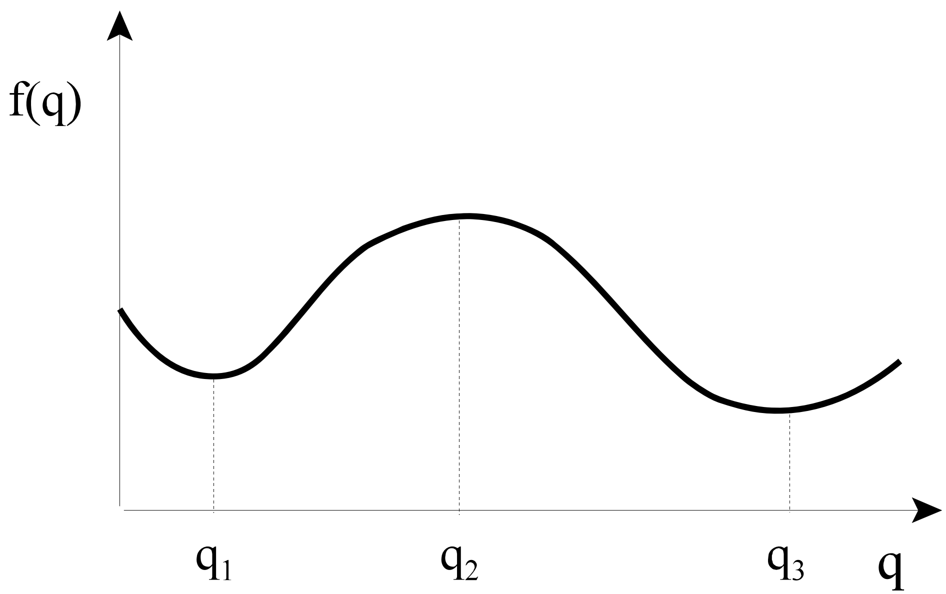
i. First Order Condition:

$$\frac{d f(x)}{d x} = 0$$

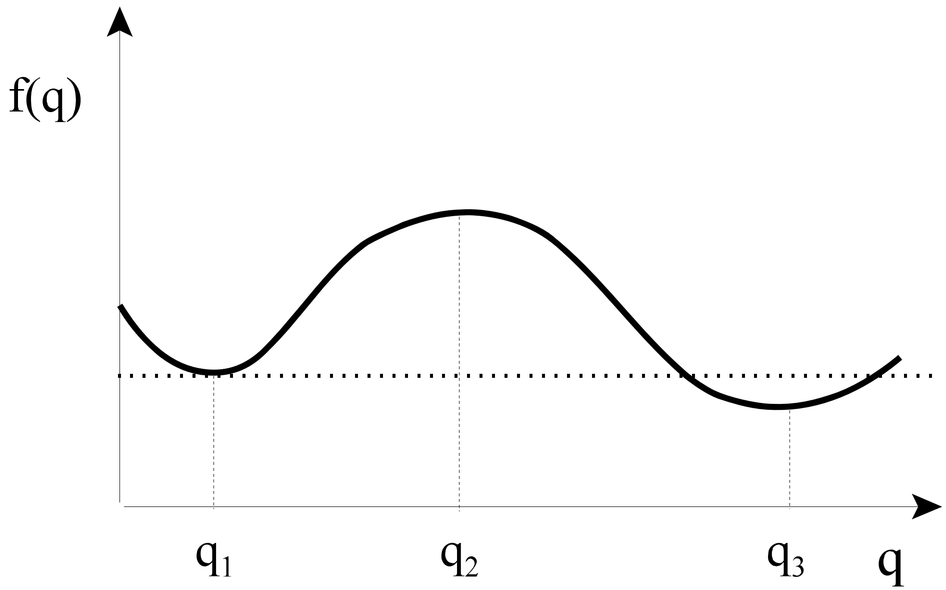
ii. Second Order Condition:

$$\frac{d^2 f(x)}{d x^2} > 0$$

MULTIPLE LOCAL MINIMA AND LOCAL MAXIMA



Identifying the global minimum



Some Rules of Differentiation

1. If b is a constant, that is, it does not depend on x, then:

$$\frac{db}{dx} = 0$$

Example:

$$\frac{d\,5}{dx} = 0$$

2. If a and b are constants and b is not equal to zero:

$$\frac{d(ax^{b})}{dx} = b\ a\ x^{b-1}$$

Application: Maximizing a profit function.

The variable x can be output, quality, or some other variable at the control of the firm.

3. Derivative of the logarithmic function.

$$\frac{d\log(x)}{dx} = \frac{1}{x}$$

[Note: All logs in this course are natural logs, and will be denoted by log(x).]

4. Derivative of the exponential function.

$$\frac{d\,e^{x}}{d\,x} = e^{x}$$

The exponential function is the only function that is equal to its own derivative.

5. Derivative of the sum of two functions.

$$\frac{d\,[\,f(x) + g(x)\,]}{d\,x} = f^{\prime}(x) + g^{\prime}(x)$$

Application: Consider a firm that sells to two markets.

$f(x)$ = revenue from market 1, $g(x)$ = revenue from market 2

x = a variable that affects the revenue of both markets, such as national advertising or product R&D.

We wish to know how a change in R&D affects total revenue.

6. Product Rule. Derivative of product of two functions.

$$\frac{d\,[f(x)\ g(x)]}{d\,x} = f^{/}(x)\ g(x) + f(x)\ g^{/}(x)$$

Application: Consider the profit function of a firm.

Denote by,

$f(x)$ = profit per unit output, x = output [$g(x) = x$]

We want to know how total profit is affected by a change in output.

7. Derivative of a ratio of two functions.

$$\frac{d\,\frac{f(x)}{g(x)}}{d\,x} = \frac{f'(x)\;g(x)\;-\;f(x)\;g'(x)}{g(x)^2}$$

Application: Denote by

$f(x)$ = profit of a firm, x = capital stock of a firm [$g(x) = x$]

Then, the ratio $f(x)/g(x)$ is the rate of return of the firm.

We want to know how the rate of return is affected by an increase in the capital stock.

8. Chain Rule.

$$\frac{d\,G(f(x))}{d\,x} = \frac{d\,G}{d\,f(x)}\,\frac{d\,f}{d\,x}$$ with some abuse of notation.

Application: Denote by x = capital stock of a firm,

$f(x)$ = output level of the firm

Capital stock affects how much output the firm is producing which in turn determines the firm's profits.

Firm chooses capital to maximize profits.

WHY ISN'T SINGLE VARIABLE CALCULUS ENOUGH?

Observation:

Economic functions seldom involve a single variable.

Examples:

i. A firm hires labor and capital. Its profit function can then be thought as a function of these two inputs, as opposed to being a function of output.

ii. A firm chooses its output level and the quality of its products. Total profits are a function of both of these variables.

iii. A firm may jointly produce two products. [say, computer monitors and t.v. sets.] The production level of each product affects the marginal cost of both.

iv. A firm chooses the price of its product and the location of where to sell it. For instance, a gas station choosing a particular location on a road.

This necessitates the extension of our single variable calculus techniques to multiple dimensions.

We develop this extension in the next lecture.