LECTURE 3

MULTI-VARIABLE OPTIMIZATION

QUESTIONS/ISSUES TO ADDRESSED:

1. Economic interpretation of first order and second order partial derivatives.

SKILLS TO BE MASTERED:

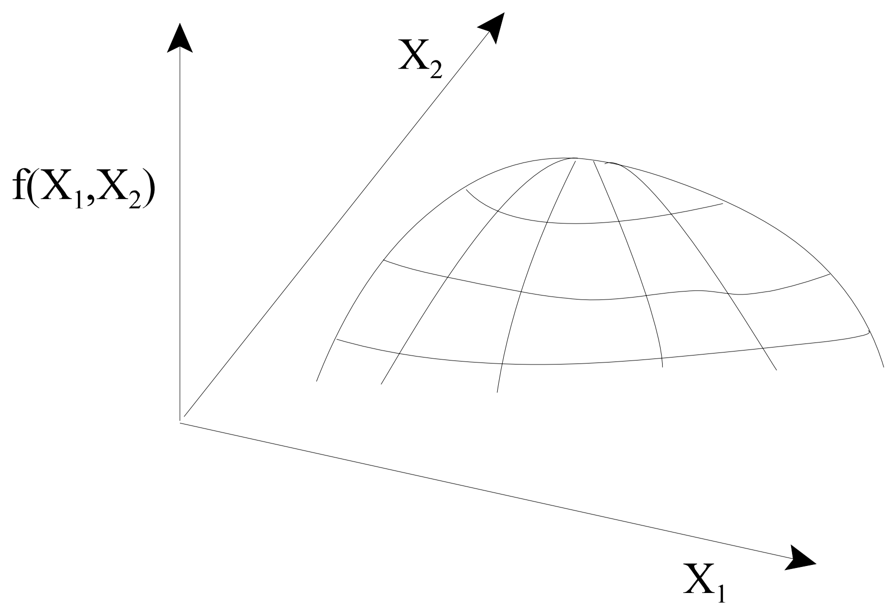
1. Partial differentiation.

2. Optimization of a function of several variables.

3. Second order conditions for optimization of multi-variable functions.

A Physical Illustration: Functions of 2 Variables

Maximization of a function of two variables is similar to climbing a hill:



Task: Derive some method that would enable an economic agent to find the maximum of a function of several variables.

Idea: As before, set “the slope” of the function to zero.

Problem: “Slope” not well defined in this context of a function of many variables.

Intuition: When climbing a mountain, the steepness of the ascent will depend on which direction we take.

A Solution

Special directions of interest are the ones we take along the path of a single variable holding all other variables constant.

The slope taken this way is called the *partial derivative* with respect to the variable we allow to change.

Devise an approach that utilizes these special directions to obtain the maximum of the function.

NOTATION

The partial derivatives of a function

$$elevation = f(longtitude, latitude)$$

with respect to its first argument (variable) are denoted by:

$$\frac{\partial\, elevation}{\partial\, longtitude} \quad \text{or} \quad \frac{\partial f}{\partial\, longtitude} \quad \text{or} \quad f_{longtitude} \quad \text{or} \quad f_1$$

Calculating Partial Derivatives

When taking the partial derivative with respect to a variable, we treat all the other variables as if they were constants.

Example 1:

$$f(x_1,x_2) = 5\,x_1^2 + e^{x_2}$$

$$f_{x_1} = 10\,x_1$$

$$f_{x_2} = e^{x_2}$$

General Observation

Additive separable functions are of the form

$$f(x_1, x_2) = g(x_1) + h(x_2)$$

and they have the property that partial derivatives depend only a single variable.

Example 2:

$$f(x_1,x_2) = \frac{\log(x_1)}{x_2}$$

$$f_{x_1} = \frac{1}{x_1}\,\frac{1}{x_2}$$

$$f_{x_2} = -\frac{\log(x_1)}{x_2^2}$$

General Observation

Even though the partial derivatives treat all the variables but one as constant, they are still, in general, functions of many variables i.e. their values will typically depend on the all variables of the f function.

Economic Example:

The marginal revenue from increasing quantity may depend on how high a product's quality is.

Similarly, the marginal revenue of increasing quality may depend on how many units the firm is selling.

Second Order Derivatives

Just like in the case of the functions of a single variable, we can take the derivative of the derivative.

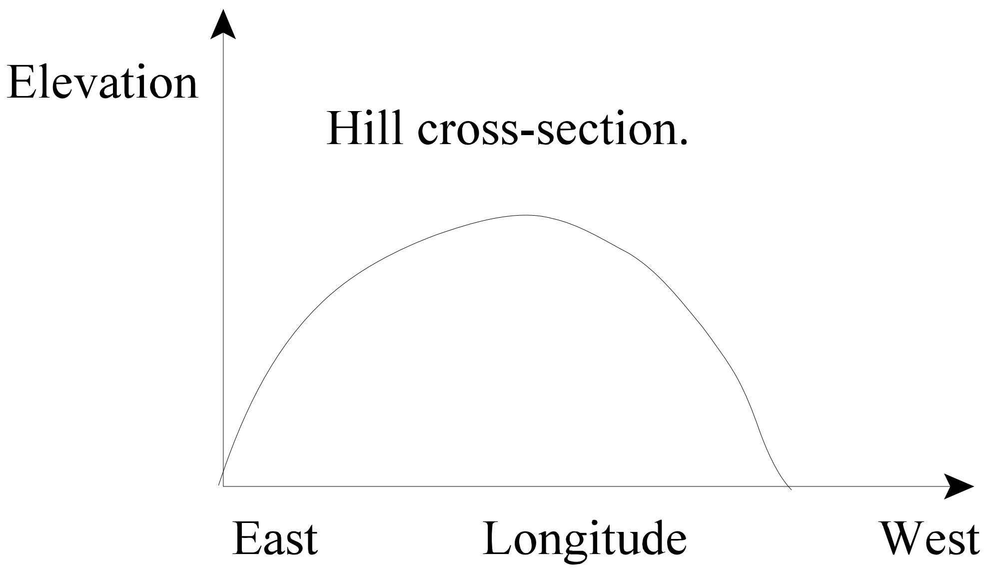
The slope of the slope with respect to a particular dimension is called the *second order partial derivative* with respect to that dimension.

It is denoted by:

$$\frac{\partial\left(\frac{\partial f}{\partial x_1}\right)}{\partial x_1} \equiv \frac{\partial^2 f}{\partial x_1^2} \equiv f_{11}$$

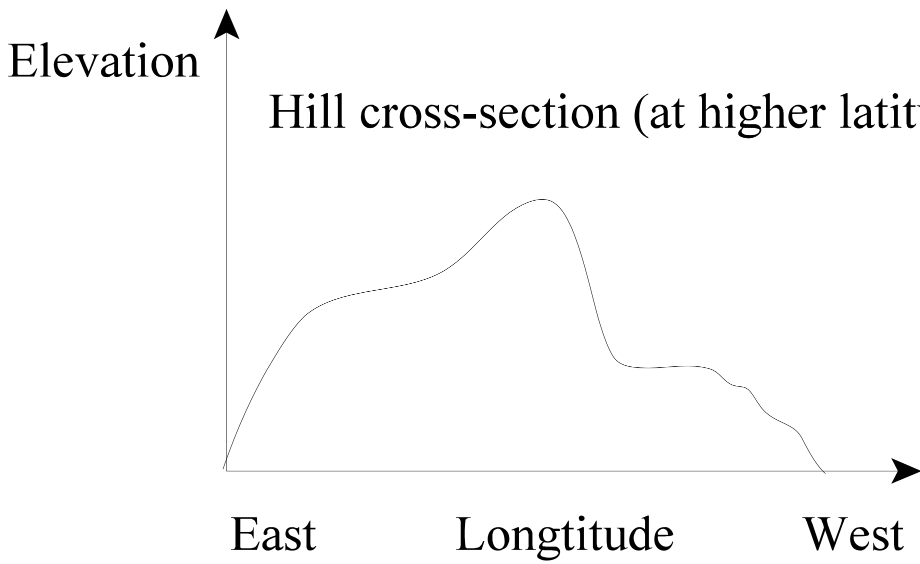
A PHYSICAL ILLUSTRATION

Consider a slice of a hill along the East-West dimension.



How can one obtain the first and second partial derivatives from this graph?

Further north, the slice of the hill in the East-West dimension would be different.



The partial derivative with respect to Longitude will also be different.

It is in this way that the partial derivative with respect to one variable can be a function of the other variable.

Unlike the case of functions of a single variable, we can also take the second order *cross-partial derivative*.

This is defined as

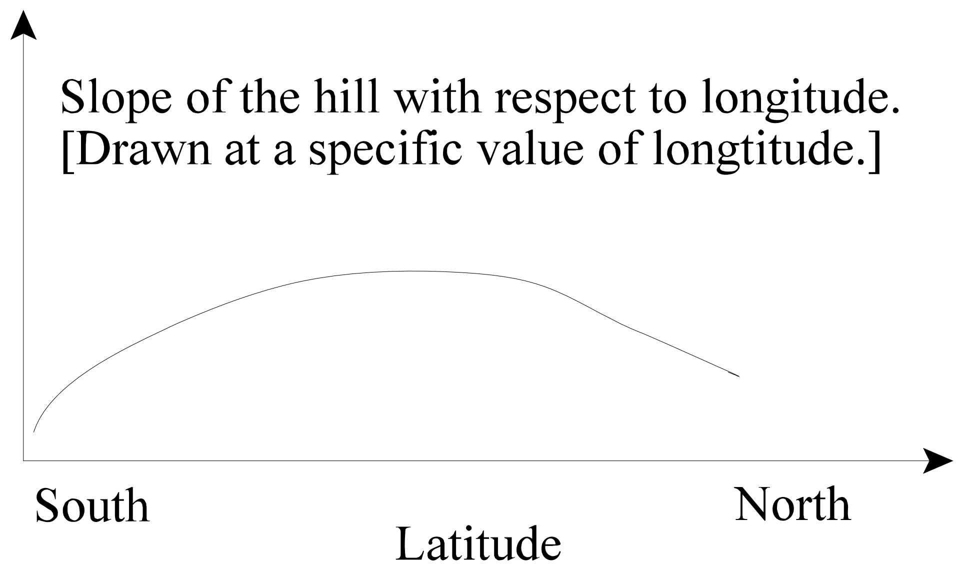
$$\frac{\partial\left(\dfrac{\partial f}{\partial x_1}\right)}{\partial\ x_2} \equiv \frac{\partial^2 f}{\partial x_1\,\partial x_2} \equiv f_{12}$$

This tells us how the slope of the function with respect to x_1 changes as we move along the x_2 direction.

Physical Example:

Suppose we plot the slope of a hill with respect to Longitude as a function of Latitude for a particular value of Longitude.

The slope of this curve, shown below, is the cross-partial derivative.



ECONOMIC ILLUSTRATIONS OF CROSS-PARTIAL DERIVATIVES

i. How much the marginal revenue of output change as we improve quality.

ii. How would the returns to quality change as we increase output.

iii. How does the marginal cost of product A change as we increase production of product B.

iv. How would the marginal revenue of firm A change as firm B increases its output.

Back to the numerical examples:

1. $f(x_1,x_2) = 5\,x_1^2 + e^{x_2}$

$f_{x_1} = 10\,x_1$ $\qquad$ $f_{x_2} = e^{x_2}$

$f_{x_1,x_1} = 10$ $\qquad$ $f_{x_2,x_2} = e^{x_2}$

$f_{x_1,x_2} = 0$ $\qquad$ $f_{x_2,x_1} = 0$

General Observation

The cross partial derivative is zero for all additive separable functions, i.e., for all functions of the form $f(x_1, x_2) = g(x_1) + h(x_2)$.

2. $$f(x_1,x_2) = \frac{\log(x_1)}{x_2}$$

$$f_{x_1} = \frac{1}{x_1}\frac{1}{x_2} \qquad f_{x_2} = -\frac{\log(x_1)}{x_2^2}$$

$$f_{x_1,x_1} = -\frac{1}{x_1^2}\frac{1}{x_2} \qquad f_{x_2,x_2} = 2\frac{\log(x_1)}{x_2^3}$$

$$f_{x_1,x_2} = -\frac{1}{x_1}\frac{1}{x_2^2} \qquad f_{x_2,x_1} = -\frac{1}{x_1}\frac{1}{x_2^2}$$

General Observation

The cross-partial derivatives are independent of the order they are taken.

$$f_{x_1,x_2} = f_{x_2,x_1}$$

This is demonstrated by the examples above.

TOTAL DIFFERENTIAL

What about the total change in elevation as we move in some combination of longitude and latitude?

We can use a concept analogous to the one for a single variable function,

$$\Delta\, elevation \;\approx\; slope_{longtitude}\;\Delta\, longtitude \;+\; slope_{latitude}\;\Delta\; latitude$$

or in using calculus notation (for small changes) we have:

$$d\, elevation \;=\; f_{longtitude}\;\; d\, longtitude \;+\; f_{latitude}\;\; d\, latitude$$

This expression on the right hand side is called *the total differential*.

Obtaining the Maximum of the Function

Insight:

The top of the hill is a location where, no matter which direction we take a (small) step, we have no change in elevation.

For this to be the case, all partial derivatives must be equal to zero.

Otherwise, as we can see from the concept of the total differential, we would take a step along (backwards or forwards) the direction that the derivative is non-zero and we would experience a change in altitude.

Implication:

All the partial derivatives being equal to zero is a necessary condition to be at the top of hill, i.e., to have maximized profits or whatever else objective function we have.

Caveat:

Observe that all partial derivatives being equal to zero is also a necessary condition to be at the minimum of a function.

“Definition”

The First Order Conditions of maximization <u>or minimization</u> are given by setting all partial derivatives equal to zero.

These conditions can be solved for the unknown variables to yield the optimum.

Implementation:

In general this is feasible, since a function of 2 variables, it will have 2 partial derivatives and, thus, 2 equations, which could be solved for the 2 unknowns.

Similarly, for 3 and 4 (or more) variable functions.

$$\frac{\partial f(x_1, x_2, x_3)}{\partial x_1} = 0$$

$$\frac{\partial f(x_1, x_2, x_3)}{\partial x_2} = 0 \quad \text{solve for} \quad x_1,\ x_2,\ x_3$$

$$\frac{\partial f(x_1, x_2, x_3)}{\partial x_3} = 0$$

A NUMERICAL EXAMPLE: SEPARABLE FUNCTIONS

The equation below can be thought of representing the elevation of a hill.

$$y = -(x_1 - 1)^2 - (x_2 - 2)^2 + 10$$

What is the maximum? Can you tell?

The quadratic terms expand to

$$y = -x_1^2 + 2x_1 - 1 - x_2^2 + 4x_2 - 4 + 10$$

$$= -x_1^2 + 2x_1 - x_2^2 + 4x_2 + 5$$

The First Order Conditions are:

$$\frac{\partial y}{\partial x_1} = 0 \quad \Rightarrow \quad -2x_1 + 2 = 0$$

and $$\frac{\partial y}{\partial x_2} = 0 \quad \Rightarrow \quad -2x_2 + 4 = 0$$

Solving for x_1 and x_2 yields:

$$x_1 = 1$$

and

$$x_2 = 2$$

Discussion

This system is easy to solve because each equation can be solved independently of the other.

This is true for all additive separable objective functions.

However, this most objectives functions we will consider will not be additive separable, and the equations of the first order conditions would have to be solved jointly.

Let us see one such example.

NUMERICAL EXAMPLE: NON-SEPARABLE FUNCTIONS

The equation below can be thought of representing a firm's profit function

$$y = -50 + 4\,q_1 - 2\,q_1^2 + 3\,q_2 - q_2^2 - q_1\,q_2$$

where q_1 and q_2 are output levels of two products.

The First Order Conditions are

$$\frac{\partial y}{\partial q_1} = 0 \quad \Rightarrow \quad 4 - 4q_1 - q_2 = 0$$

$$\frac{\partial y}{\partial q_2} = 0 \quad \Rightarrow \quad 3 - 2q_2 - q_1 = 0$$

The substitution method is a reliable, brute force way to solve such systems.

(a) Solve the first equation for one of the variables (say, q_1).

$$q_1 = \frac{4 - q_2}{4}$$

(b) Substitute the answer into the second equation.

$$3 - 2q_2 - \frac{4 - q_2}{4} = 0 \quad \Rightarrow$$

$$12 - 8q_2 - 4 + q_2 = 0$$

(c) Solve for the remaining variable (q_2).

$$q_2 = \frac{8}{7}$$

(d) Substitute back into (a) to get the answer for the first variable (q_1).

$$q_1 = \frac{4 - \frac{8}{7}}{4}$$

$$= \frac{\frac{20}{7}}{4}$$

$$= \frac{5}{7}$$

We will sometimes use shortcuts, but knowing a method that reliably works is useful.

SECOND ORDER CONDITIONS.

As in the case of maximization of a function of a single variable, the First Order Conditions can yield either a maximum or a minimum.

To determine which one of the two it is, we must consider the Second Order Conditions.

These involve both the second partial derivatives and the cross-partial derivatives.

For a two variable maximization problem, the SOCs require that

$$\frac{\partial^2 f(x_1, x_2)}{\partial x_1^2} < 0 \qquad \text{(as is the case for single variable SOC)}$$

$$\frac{\partial^2 f(x_1, x_2)}{\partial x_2^2} < 0 \qquad \text{(as is the case for single variable SOC)}$$

$$\frac{\partial^2 f(x_1, x_2)}{\partial x_1^2} \frac{\partial^2 f(x_1, x_2)}{\partial x_2^2} - \left(\frac{\partial^2 f(x_1, x_2)}{\partial x_1 \partial x_2} \right)^2 > 0$$

It turns out that if the first and third inequalities are satisfied, so is the second. Therefore, the second inequality does not need to be checked separately.

For a two variable minimization problem, the SOCs require that

$$\frac{\partial^2 f(x_1, x_2)}{\partial x_1^2} > 0 \qquad \text{(as is the case for single variable SOC)}$$

$$\frac{\partial^2 f(x_1, x_2)}{\partial x_2^2} > 0 \qquad \text{(as is the case for single variable SOC)}$$

$$\frac{\partial^2 f(x_1, x_2)}{\partial x_1^2} \frac{\partial^2 f(x_1, x_2)}{\partial x_2^2} - \left(\frac{\partial^2 f(x_1, x_2)}{\partial x_1 \partial x_2} \right)^2 > 0$$

As before, the first and third condition imply the second.

For the most part (but not always), we will not consider the Second Order Conditions for functions of several variables in this course.

Unless otherwise noted, for this course the First Order Conditions will yield

- the maximum when the objective is to maximize the function
- the minimum when the objective is to minimize the function