Production Theory 2

Returns-to-Scale

- Marginal product describe the change in output level as a single input level changes. (Short-run)
- Returns-to-scale describes how the output level changes as all input levels change, e.g. all input levels doubled. (Long-run)

Returns-to-Scale

If, for any input bundle $(x_1,\dots,x_n)$,

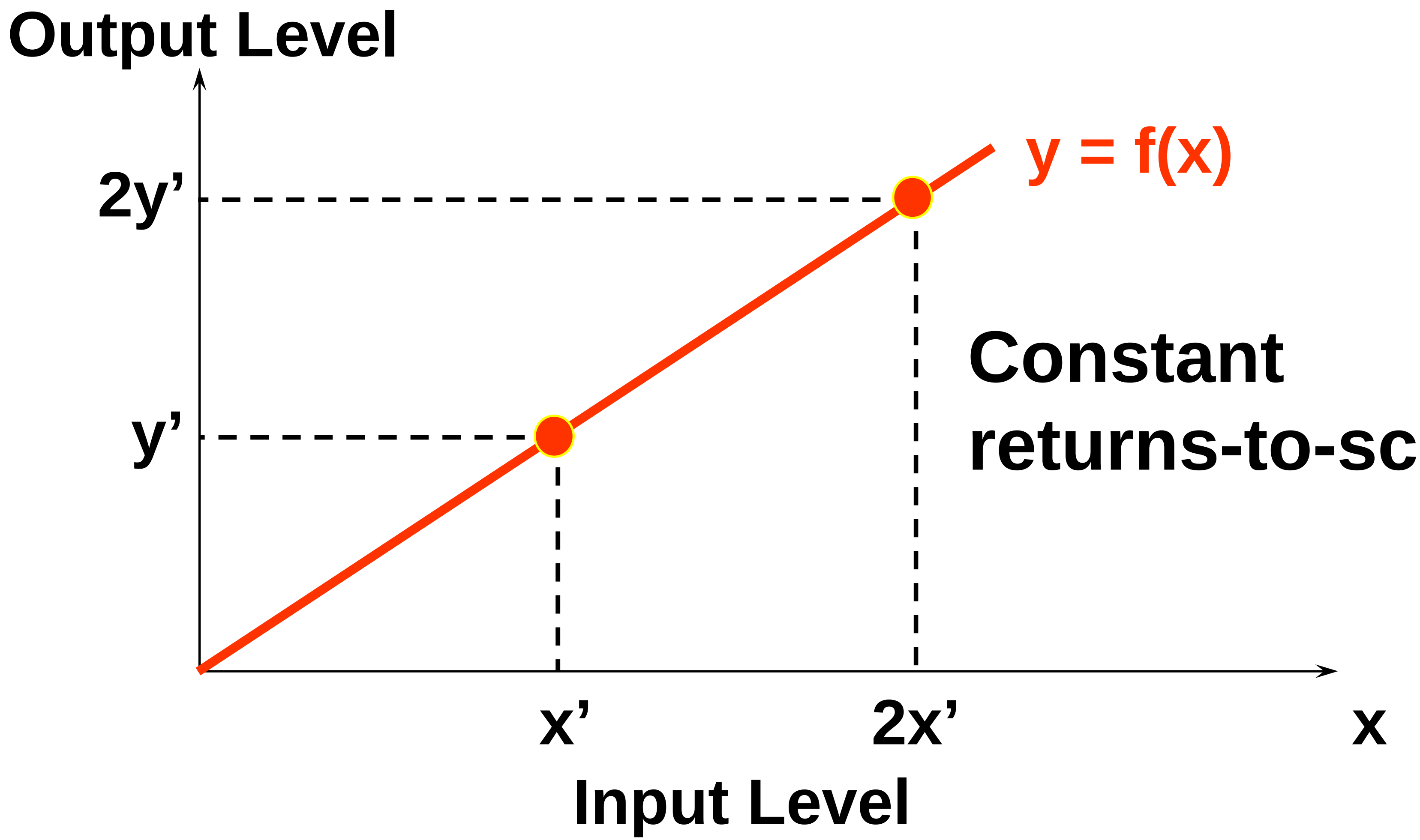
$$f(tx_1, tx_2, \cdots, tx_n) = t.f(x_1, x_2, \cdots, x_n)$$

then the technology described by the production function f exhibits constant returns-to-scale, e.g. doubling all input levels doubles the output level (t=2).

Note: Books often (confusingly) replace t with k.

Returns-to-Scale

One input



Constant returns-to-scale

Returns-to-Scale

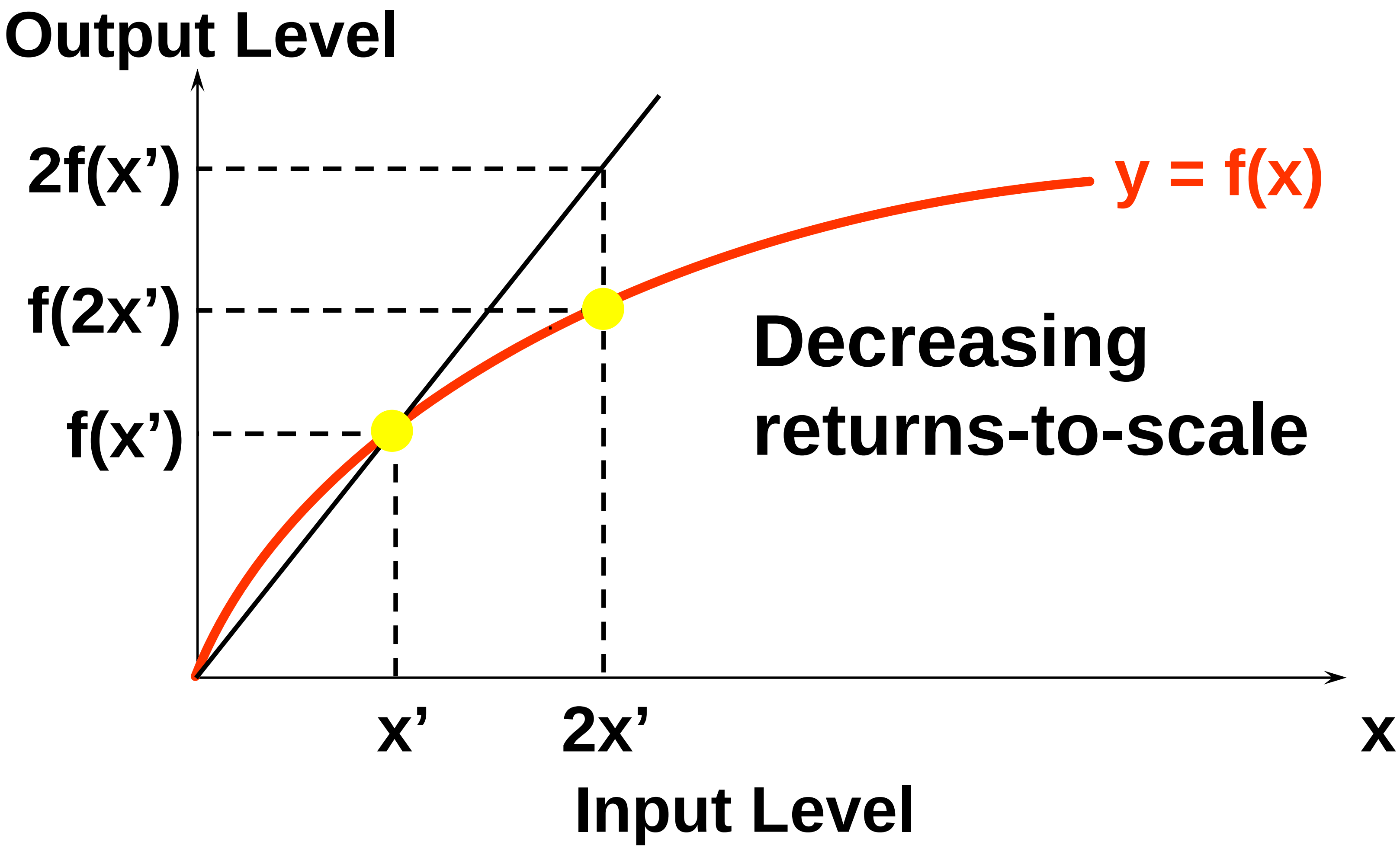
If, for any input bundle $(x_1,\ldots,x_n)$,

$$f(tx_1, tx_2, \cdots, tx_n) < tf(x_1, x_2, \cdots, x_n)$$

then the technology exhibits decreasing returns-to-scale, e.g. doubling all input levels less than doubles the output level (t=2).

Returns-to-Scale

<u>One</u> input



Returns-to-Scale

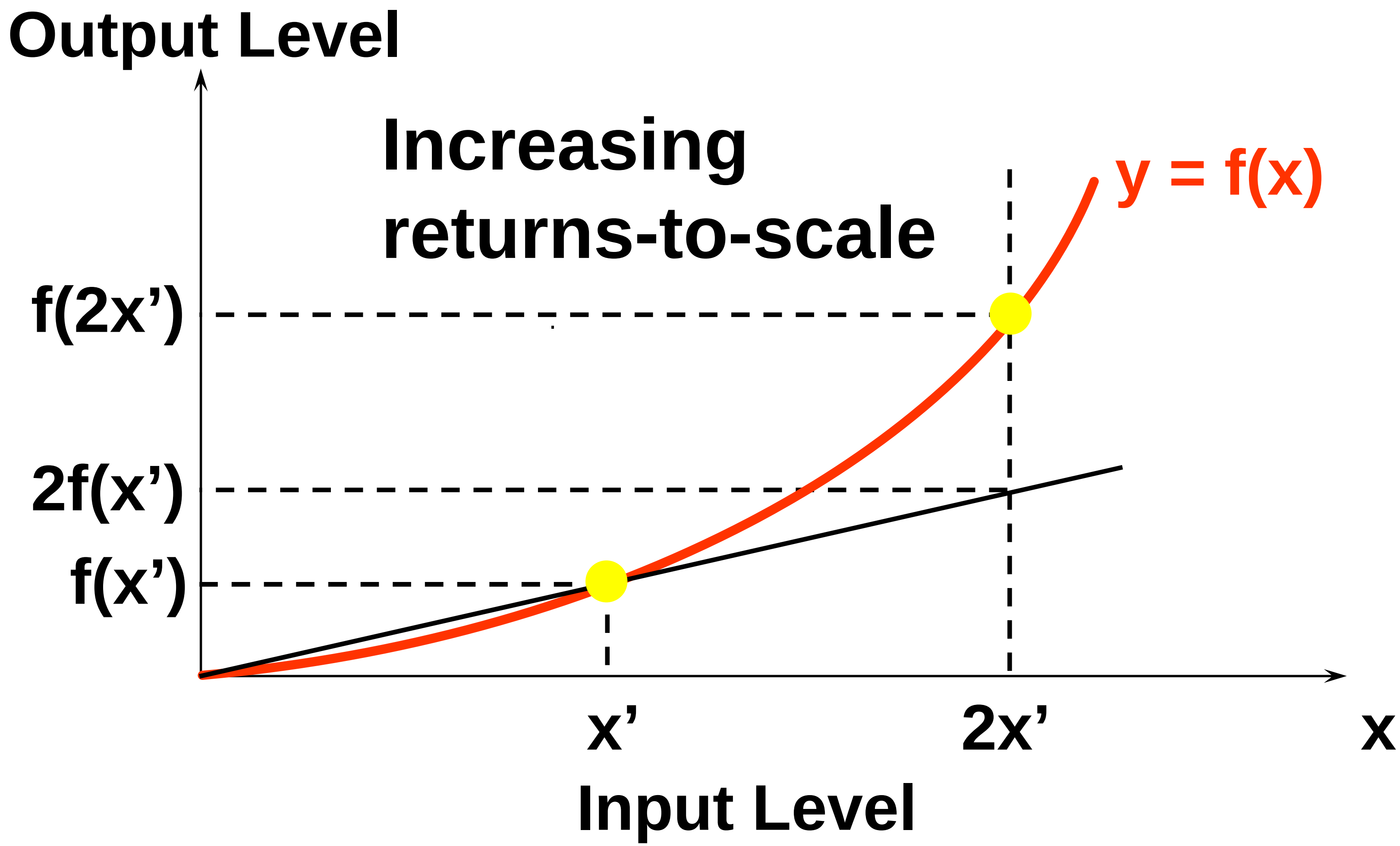
If, for any input bundle $(x_1,\ldots,x_n)$,

$$f(tx_1, tx_2, \cdots, tx_n) > tf(x_1, x_2, \cdots, x_n)$$

then the technology exhibits increasing returns-to-scale, e.g. doubling all input levels more than doubles the output level (t=2).

Returns-to-Scale

One input



Returns-to-Scale: Example

The Cobb-Douglas production function is

$$y = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}.$$

$$(kx_1)^{a_1}(kx_2)^{a_2} \cdots (kx_n)^{a_n} = k^{a_1+\cdots+a_n} y.$$

The Cobb-Douglas technology's returns-to-scale is

constant if $a_1 + \ldots + a_n = 1$

increasing if $a_1 + \ldots + a_n > 1$

decreasing if $a_1 + \ldots + a_n < 1$.

Short-Run: Marginal Product

- **A marginal product is the rate-of-change of output as one input level increases, holding all other input levels fixed.**
- **Marginal product diminishes because the other input levels are fixed, so the increasing input's units each have less and less of other inputs with which to work.**

Long-Run: Returns-to-Scale

- **When all input levels are increased proportionately, there need be no such "crowding out" as each input will always have the same amount of other inputs with which to work. Input productivities need not fall and so returns-to-scale can be constant or even increasing.**

Homogenous Production Function

A production function is homogeneous of degree α if

$$F(tK, tL) = t^{\alpha} F(K,L) \text{ for all } t.$$

If $\alpha = 1$ CRS

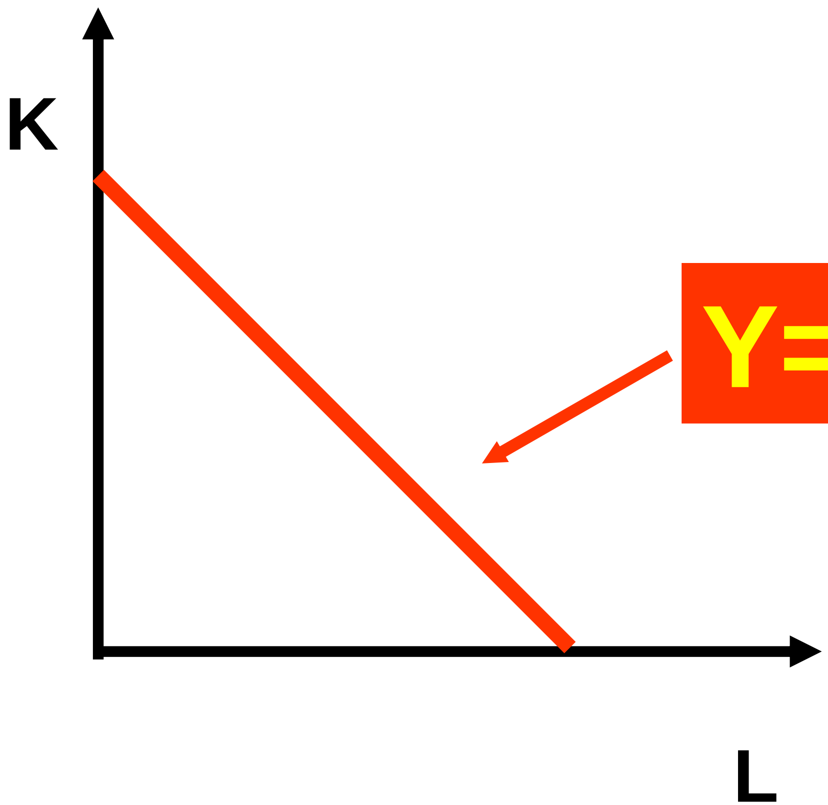
If $\alpha > 1$ IRS

If $\alpha < 1$ DRS

Note: Not all production functions are homogeneous. ($Y = 1 + L + K$)

Perfect Substitutes

Constant Returns to Scale: Show

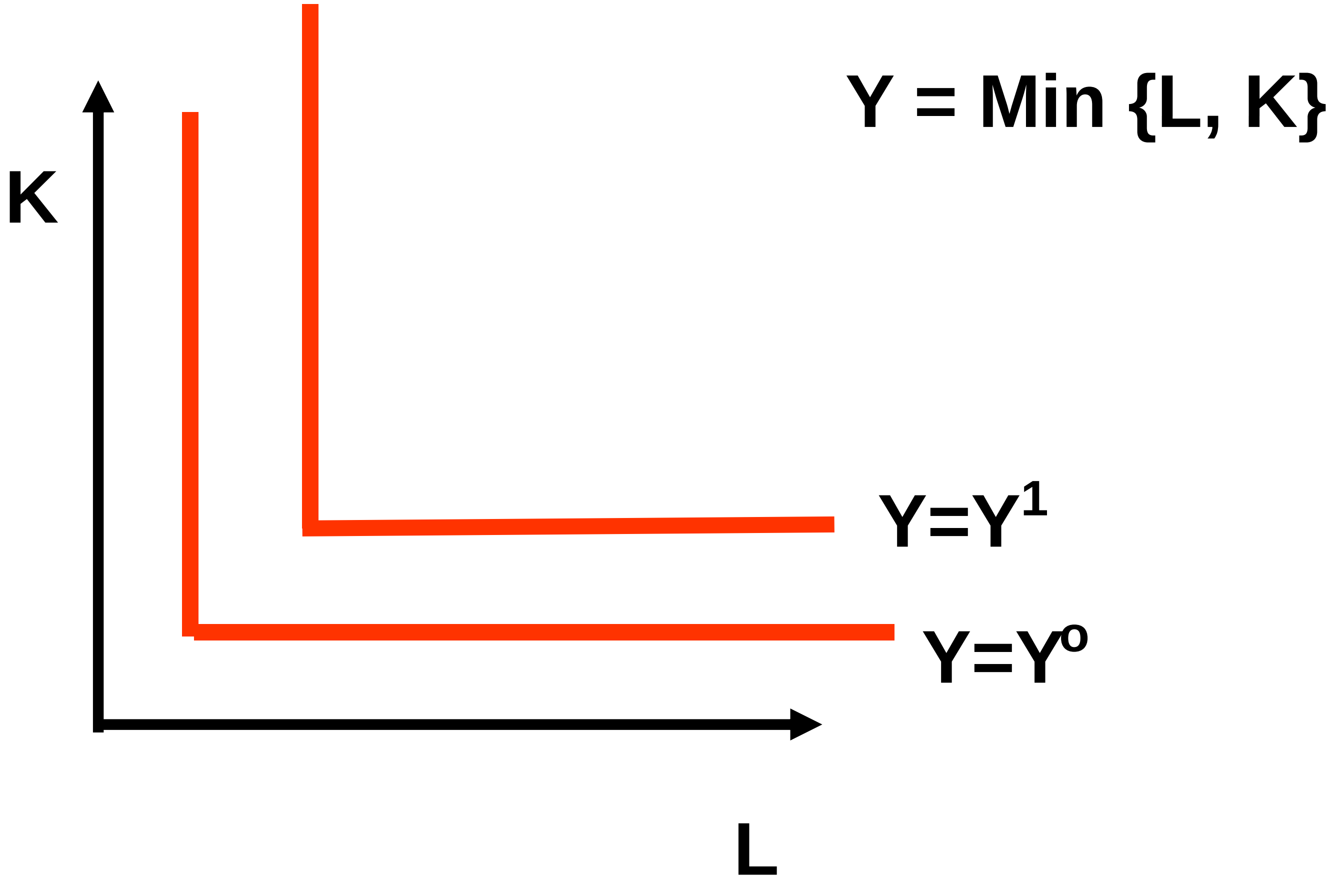


Y=aK + bL

MRTS= (-) b/a

Perfect Complements

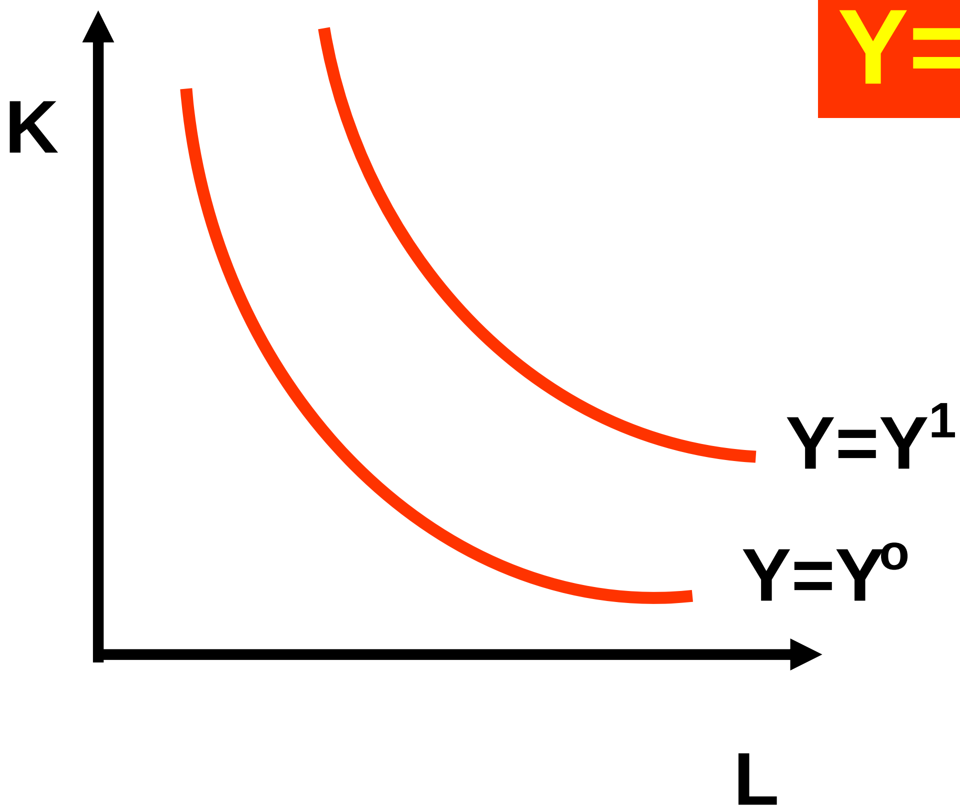
Constant Returns to Scale: Show



Cobb-Douglas

Homogeneous of degree (♋ + ♌)

$Y=AK^{\alpha}L^{\beta}$



Properties of Cobb-Douglas Production Function

$$Y = AK^{\alpha}L^{\beta}$$

The Cobb-Douglas is homogeneous of degree $\varepsilon = (\alpha + \beta)$.

Properties of Cobb-Douglas Production Function

Proof: Given $Y=K^{\alpha}L^{\beta}$ now introduce t

$$Y=(tK)^{\alpha}(tL)^{\beta}$$

$$Y= t^{\alpha} K^{\alpha} t^{\beta} L^{\beta}$$

$$Y=t^{\alpha+\beta} \boxed{K^{\alpha} L^{\beta}}$$

$$Y= t^{\alpha+\beta} Y$$

$$Y=t^{\varepsilon}Y \quad \text{as} \quad \varepsilon=\alpha+\beta$$

If $\varepsilon =1$	$(\alpha+\beta=1)$	then	CRS
If $\varepsilon >1$	$(\alpha+\beta>1)$	then	IRS
If $\varepsilon <1$	$(\alpha+\beta<1)$	then	DRS

Properties of Cobb-Douglas Production Function

Output Elasticity Y=AK^α^L^β^

$$\frac{\partial Y}{\partial K} \cdot \frac{K}{Y} = \alpha$$

For Capital (show)

$$\frac{\partial Y}{\partial L} \cdot \frac{L}{Y} = \beta$$

For Labour (show)

Properties of Cobb-Douglas Production Function

$Y=AK^{\alpha}L^{\beta}$

Marginal Product of Capital (show)

$$\alpha . AP_k$$

Marginal Product of Labour (show)

$$\beta . AP_L$$

Properties of Cobb-Douglas Production Function

$Y=AK^{\alpha}L^{\beta}$

Marginal Rate of Technical Substitution (MRTS = TRS)

$$\frac{\beta}{\alpha}\frac{K}{L}$$

Show

Properties of Cobb-Douglas Production Function

$Y=AK^{\alpha}L^{\beta}$

Euler's theorem:

$$MP_K K + MP_L L = (\varepsilon)Y$$

Where ε is the degree of homogeneity (show)

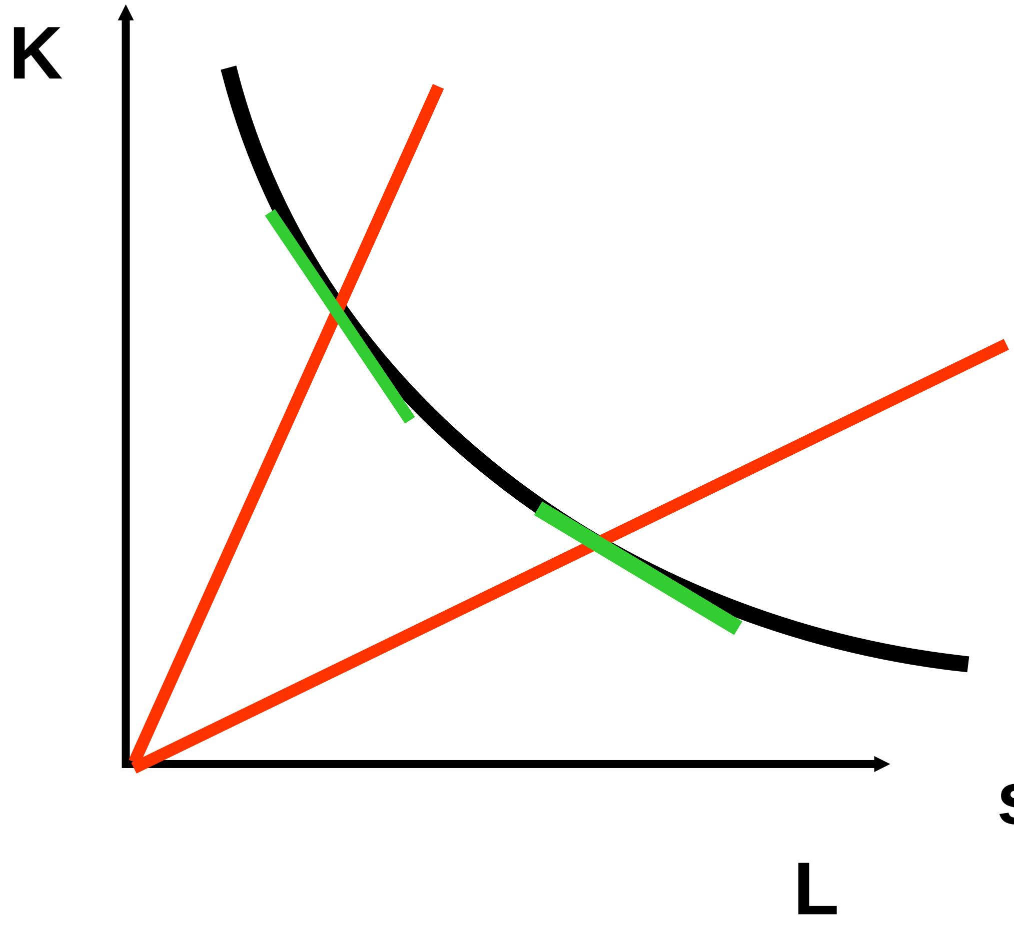
Elasticity of Substitution

- **The Elasticity of Substitution is the ratio of the proportionate change in factor proportions to the proportionate change in the slope of the isoquant.**
- **Intuition: If a small change in the slope of the isoquant leads to a large change in the K/L ratio then capital and labour are highly substitutable.**

Elasticity of Substitution

- $\sigma = \dfrac{\text{\% Change in K/L}}{\text{\% Change in Slope of Isoquant}}$
- $\sigma = \dfrac{\text{\% Change in K/L}}{\text{\% Change in MRTS}}$

Elasticity of Substitution



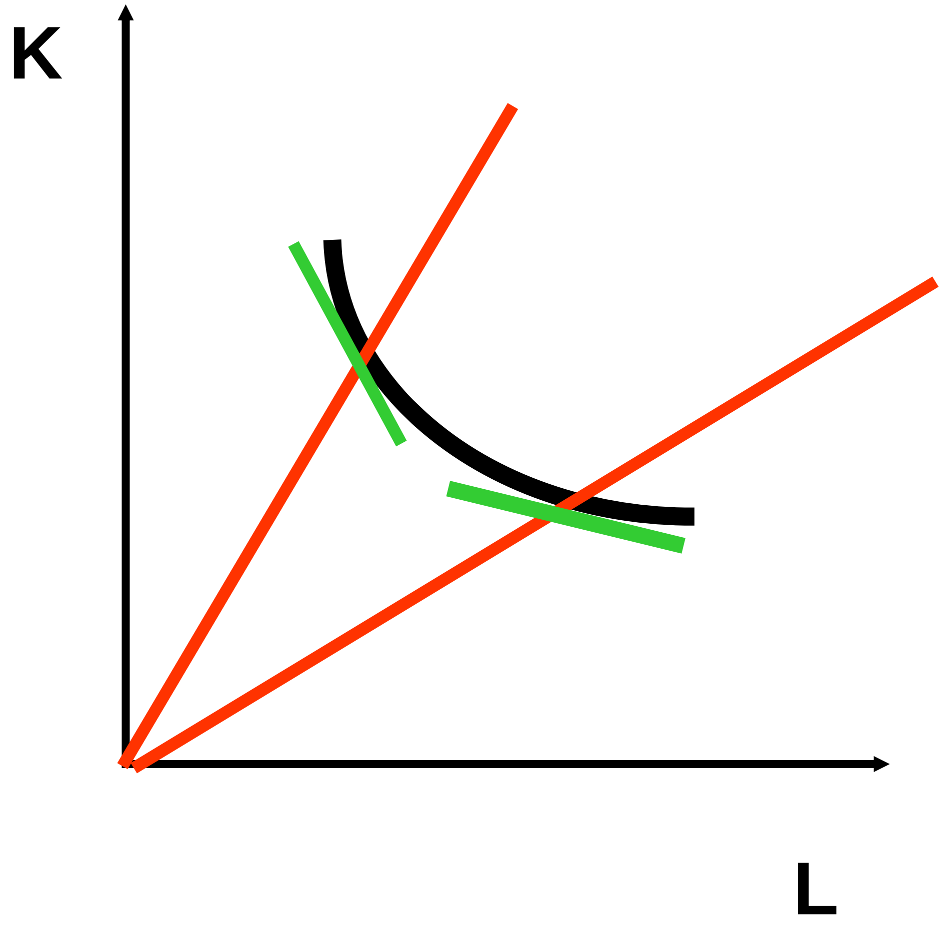
A small change in the MRTS

Large change in K/L

High σ

K and L are highly substitutable for each other

Elasticity of Substitution



A large change in the MRTS

Small change in K/L

Low σ

K and L are not highly substitutable for each other

Elasticity of Substitution

$$\sigma = \frac{\dfrac{\partial (K/L)}{(K/L)}}{\dfrac{\partial(\partial K/\partial L)}{(\partial K/\partial L)}}$$

Properties of Cobb-Douglas Production Function

$Y = AK^{\alpha}L^{\beta}$

The elasticity of substitution = 1

$$\sigma = \frac{\dfrac{\partial (K/L)}{(K/L)}}{\dfrac{\partial (\partial K/\partial L)}{(\partial K/\partial L)}}$$

Show

Properties of Cobb-Douglas Production Function

- **In equilibrium, MRTS = w/r and so the formula for σ reduces to,**

$$\sigma = \frac{\%\Delta \ in \ K/L}{\%\Delta \ in \ w/r}$$

Useful for Revision Purposes:
Not Obvious Now

Properties of Cobb-Douglas Production Function

- **For the Cobb-Douglas, σ=1 means that a 10% change in the factor price ratio leads to a 10% change in the opposite direction in the factor input ratio.**

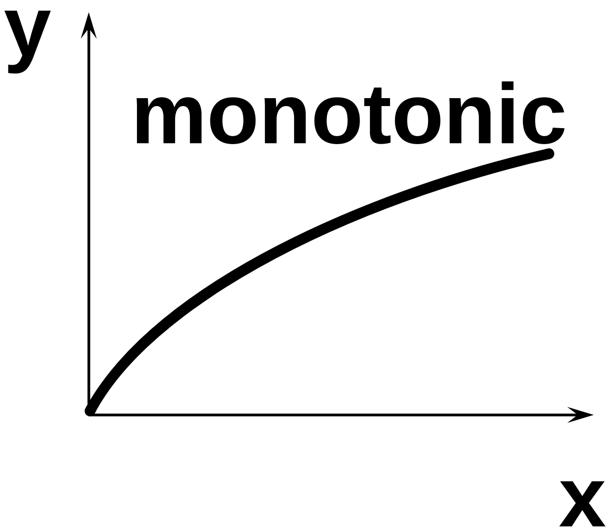
Useful For Revision Purposes:
Not Obvious Now

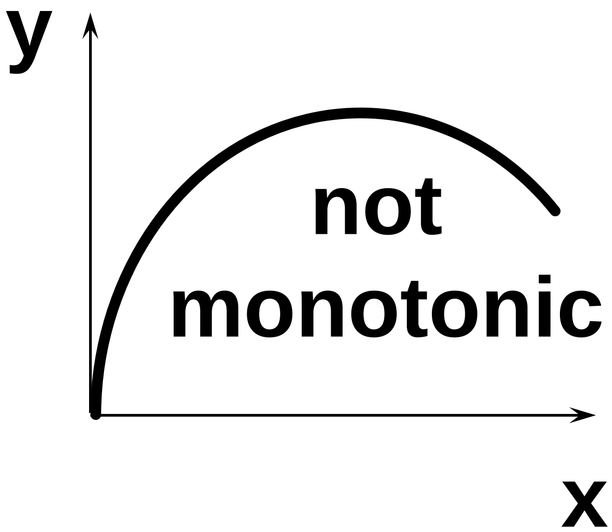
Well-Behaved Technologies

- A **well-behaved technology is**
  - **monotonic, and**
  - **convex.**

Well-Behaved Technologies - Monotonicity

- **Monotonicity: More of any input generates more output.**

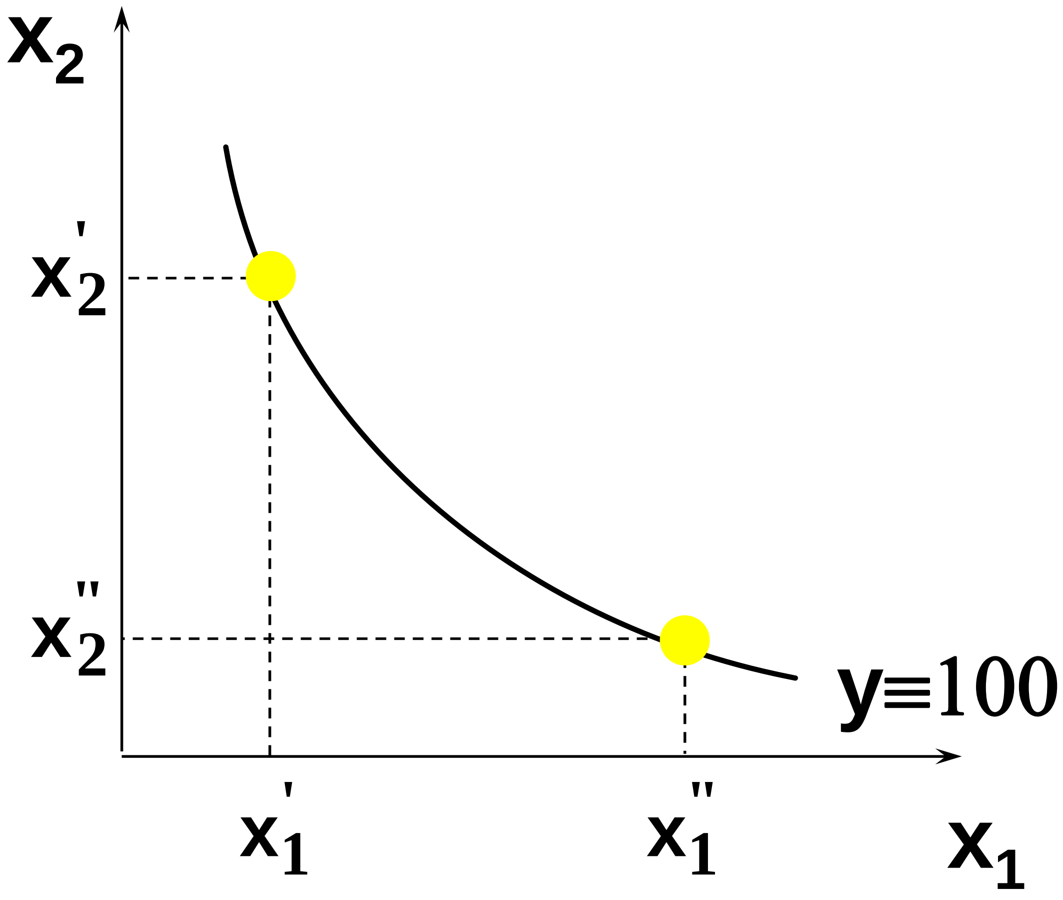




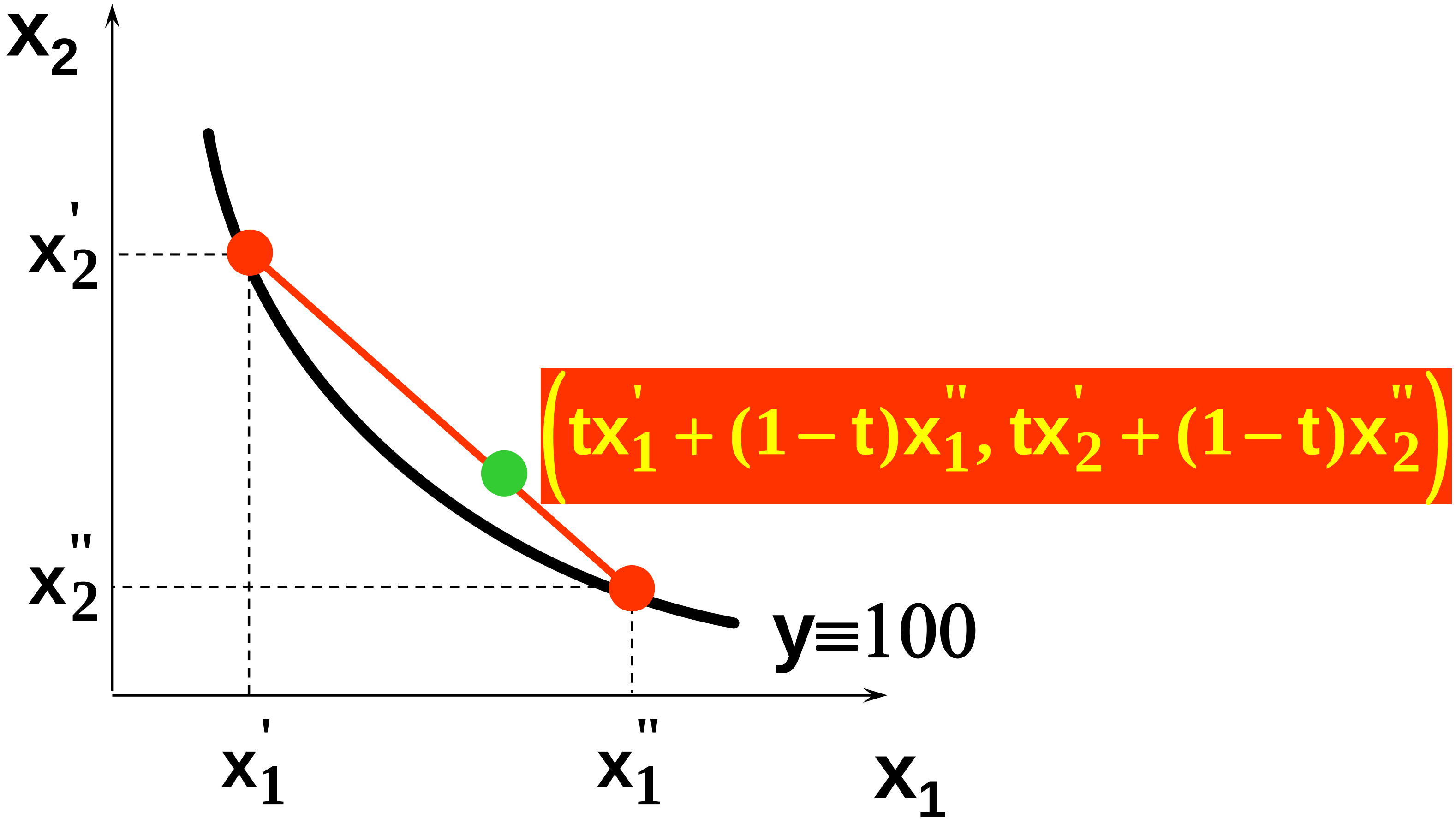
Well-Behaved Technologies - Convexity

- **Convexity: If the input bundles x' and x" both provide y units of output then the mixture tx' + (1-t)x" provides at least y units of output, for any 0 < t < 1.**

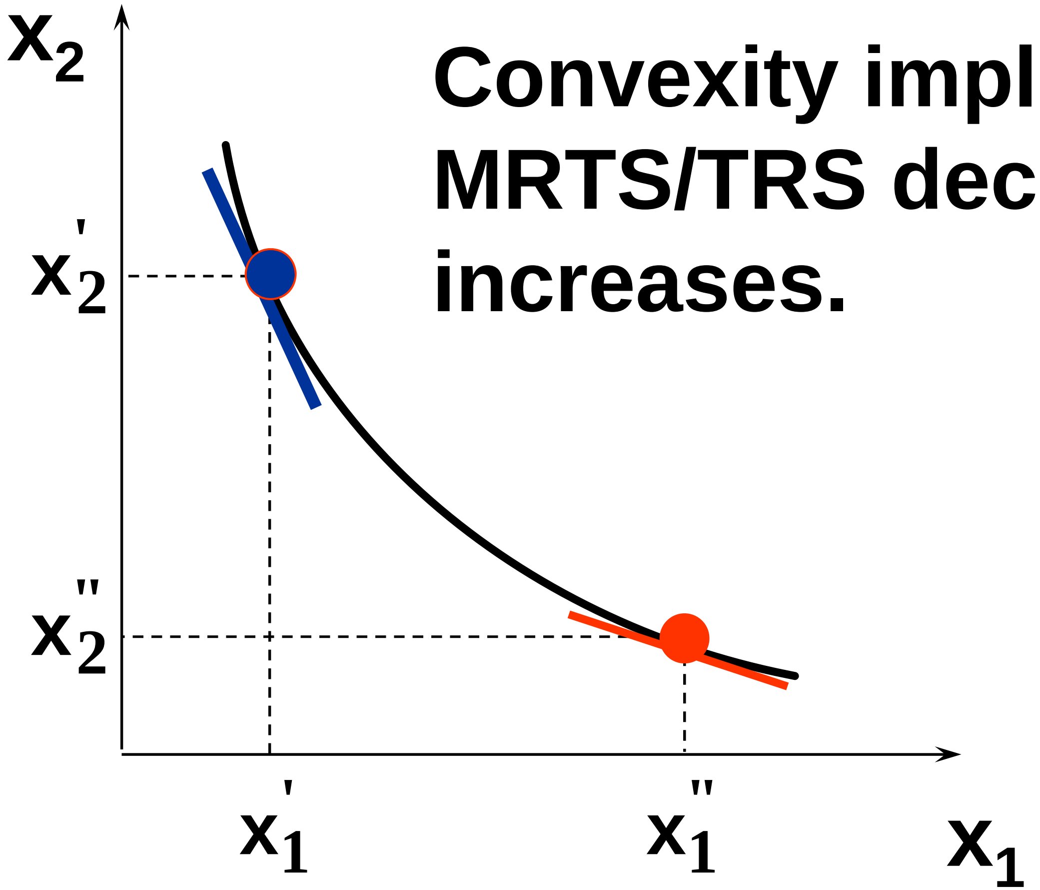
Well-Behaved Technologies - Convexity



Well-Behaved Technologies - Convexity



Well-Behaved Technologies - Convexity



Convexity implies that the MRTS/TRS decreases as x_1 increases.