Cost Constraint/Isocost Line

COST CONSTRAINT

$$C = wL + rK$$

$$(m = p_1x_1 + p_2x_2)$$

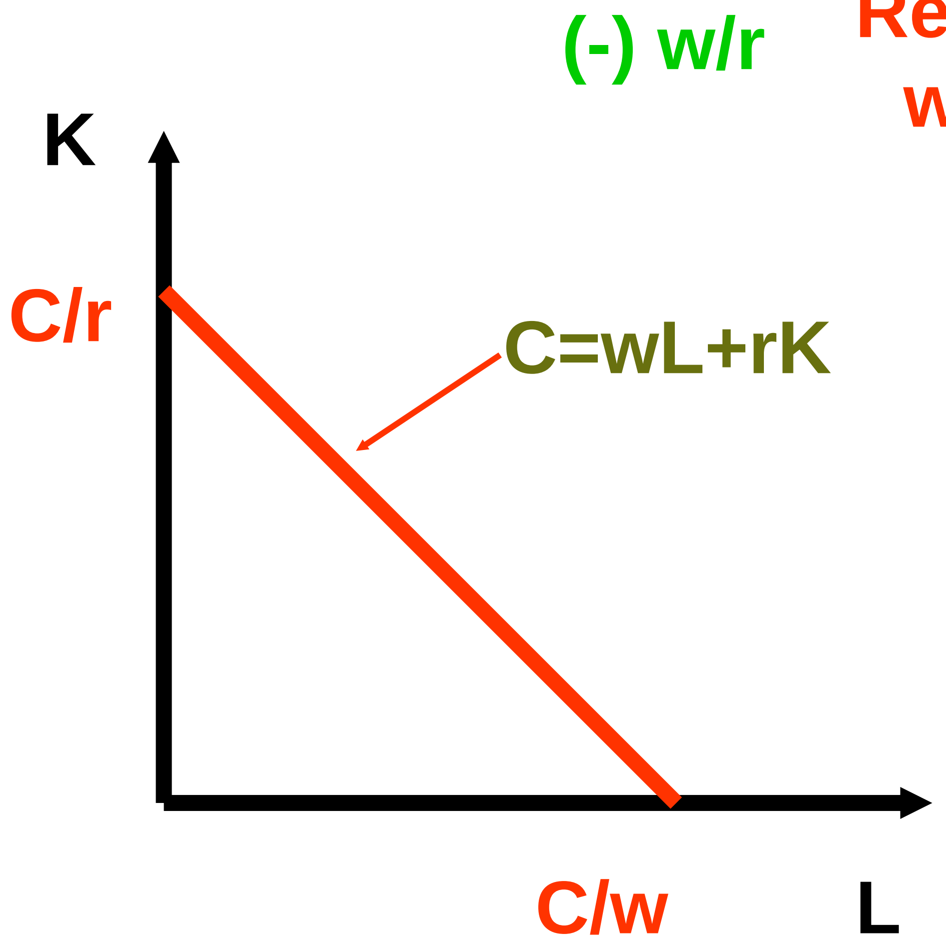
w: wage rate (including fringe benefits, holidays, PRSI, etc)

r: rental rate of capital

Rearranging:

$$K = C/r - (w/r)L$$

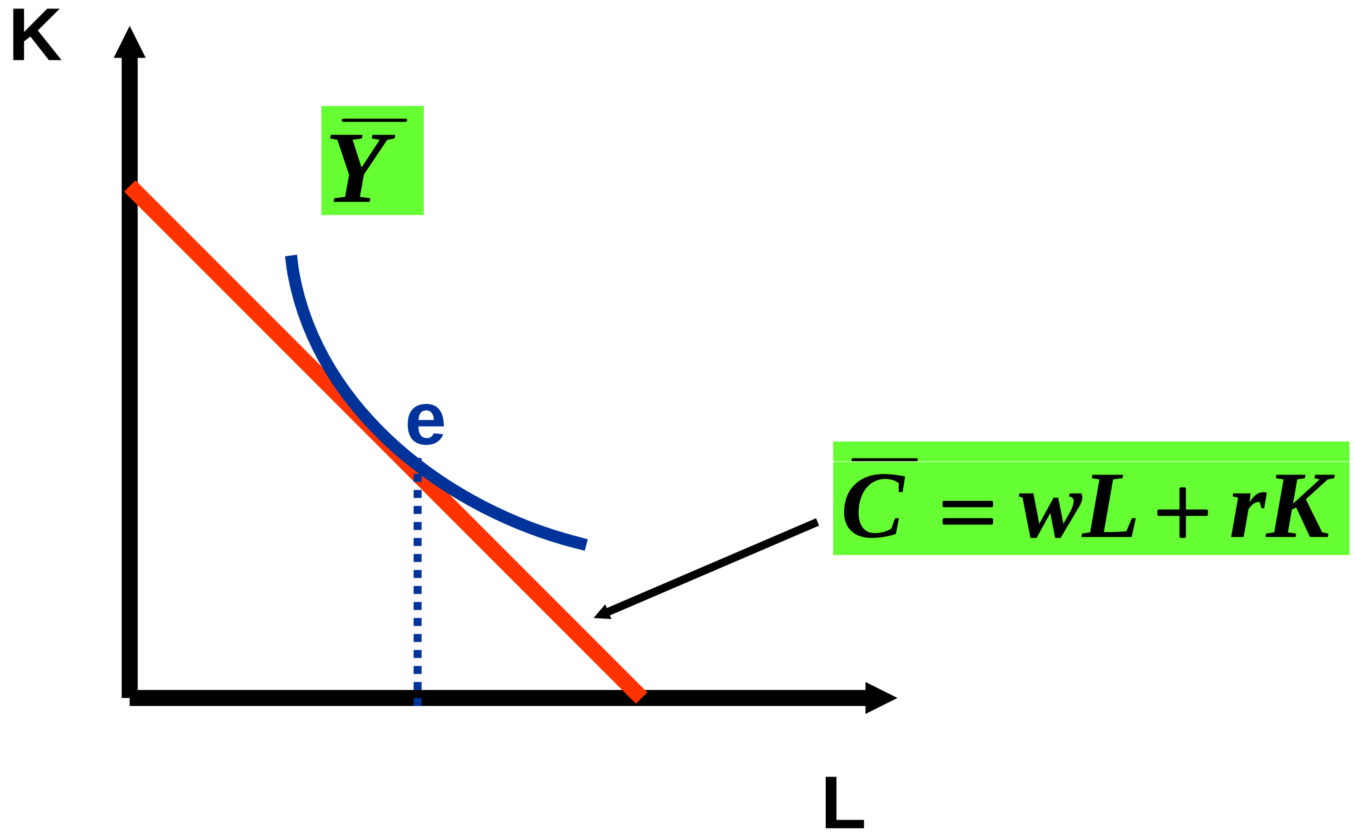
COST CONSTRAINT



Represents society's willingness to trade the factors of production

Slope of the isocost = $\Delta K/\Delta L = \delta K/\delta L$ = (-) w/r

EQUILIBRIUM



EQUILIBRIUM

We can either

Minimise cost subject to

$$Y = \bar{Y} \Rightarrow \bar{C} \text{ and } \quad e$$ **equilibrium**

or

Maximise output subject to

$$C = \bar{C} \Rightarrow \bar{Y} \text{ and } \quad e$$ **equilibrium**

EQUILIBRIUM

Lagrangian method

Minimise cost subject to output

$$L^* = wL + rK + \lambda\left(Y - f\left(K, L\right)\right)$$

or

Maximise output subject to costs

$$L^* = f\left(L, K\right) + \lambda\left(C - wL + rL\right)$$

Lagrange Method

- **Set up the problem (same as with utility maximisation subject to budget constraint).**
- **Find the first order conditions.**
- **It is easier to maximise output subject to a cost constraint than to minimise costs subject to an output constraint. The answer will be the same (<u>in essence</u>) either way.**

Example: Cobb-Douglas Equilibrium

Derive a demand function for Capital and Labour by maximising output subject to a cost constraint. Let L* be Lagrange, L be labour and K be capital.

Set up the problem

$$L^{*} = AK^{a}L^{(1-a)} + \lambda\left(\bar{C} - wL - rK\right)$$

Multiply out the part in brackets

$$L^{*} = AK^{a}L^{(1-a)} + \lambda\bar{C} - \lambda wL - \lambda rK$$

Cobb-Douglas: Equilibrium

Find the First Order Conditions by differentiation with respect to K, L and λ

$$\frac{\partial L^*}{\partial K} = AaK^{a-1}L^{(1-a)} - \lambda r = 0$$

1st FOC

$$\frac{\partial L^*}{\partial L} = A(1-a)K^{a}L^{-a} - \lambda w = 0$$

2nd FOC

$$\frac{\partial L^*}{\partial \lambda} = \overline{C} - wL - rK = 0$$

3rd FOC

Cobb-Douglas: Equilibrium

Rearrange the 1st FOC and the 2nd FOC so that λ is on the left hand side of both equations

$$-\lambda = \frac{AaK^{a-1}L^{(1-a)}}{r}$$

1st FOC

$$-\lambda = \frac{A(1-a)K^{a}L^{-a}}{w}$$

2nd FOC

Cobb-Douglas: Equilibrium

We now have two equations both equal to -λ so we can get rid of λ

$$\frac{AaK^{a-1}L^{(1-a)}}{r} = \frac{A(1-a)K^{a}L^{-a}}{w}$$

(Aside: Notice that we can find Y nested within these equations.)

Cobb-Douglas: Equilibrium

$$\frac{AaK^{a-1}L^{(1-a)}}{r} = \frac{A(1-a)K^{a}L^{-a}}{w}$$

$$\frac{AaK^{a}K^{-1}L^{(1-a)}}{r} = \frac{A(1-a)K^{a}L^{-a}L^{1}}{wL}$$

We have multiplied by L/L=1

$$\frac{AaK^{-1}K^{a}L^{(1-a)}}{r} = \frac{A(1-a)K^{a}L^{1-a}}{wL}$$

$$\frac{aK^{-1}}{r} = \frac{(1-a)}{wL}$$

Cobb-Douglas: Equilibrium

$$\frac{a}{rK} = \frac{(1-a)}{wL}$$

Note: $K^{-1} = 1/K$

$$rK = \frac{awL}{(1-a)}$$

Return to the 3rd FOC and replace rk

Cobb-Douglas: Equilibrium

$$rK = \frac{awL}{(1-a)}$$

$$\bar{C} - wL - rK = 0$$

$$wL + \frac{awL}{(1-a)} = C$$

$$wL\left(1 + \frac{a}{(1-a)}\right) = C$$

Cobb-Douglas: Equilibrium

$$wL\left(1 + \frac{a}{(1-a)}\right) = \overline{C}$$

Simplify again.

$$\overline{C} = \frac{wL}{(1-a)}$$

$$L = \frac{(1-a)\overline{C}}{w}$$

Demand function for Labour

Cobb-Douglas: Equilibrium

$$\frac{a}{rK} = \frac{(1-a)}{wL}$$

Rearrange so that wL is on the left hand side

$$wL = \frac{(1-a)}{a} rK$$

Now go back to the 3rd FOC and replace wl and follow the same procedure as before to solve for the demand for Capital

Demand for capital

$$K = \frac{a\bar{C}}{r}$$

EQUILIBRIUM

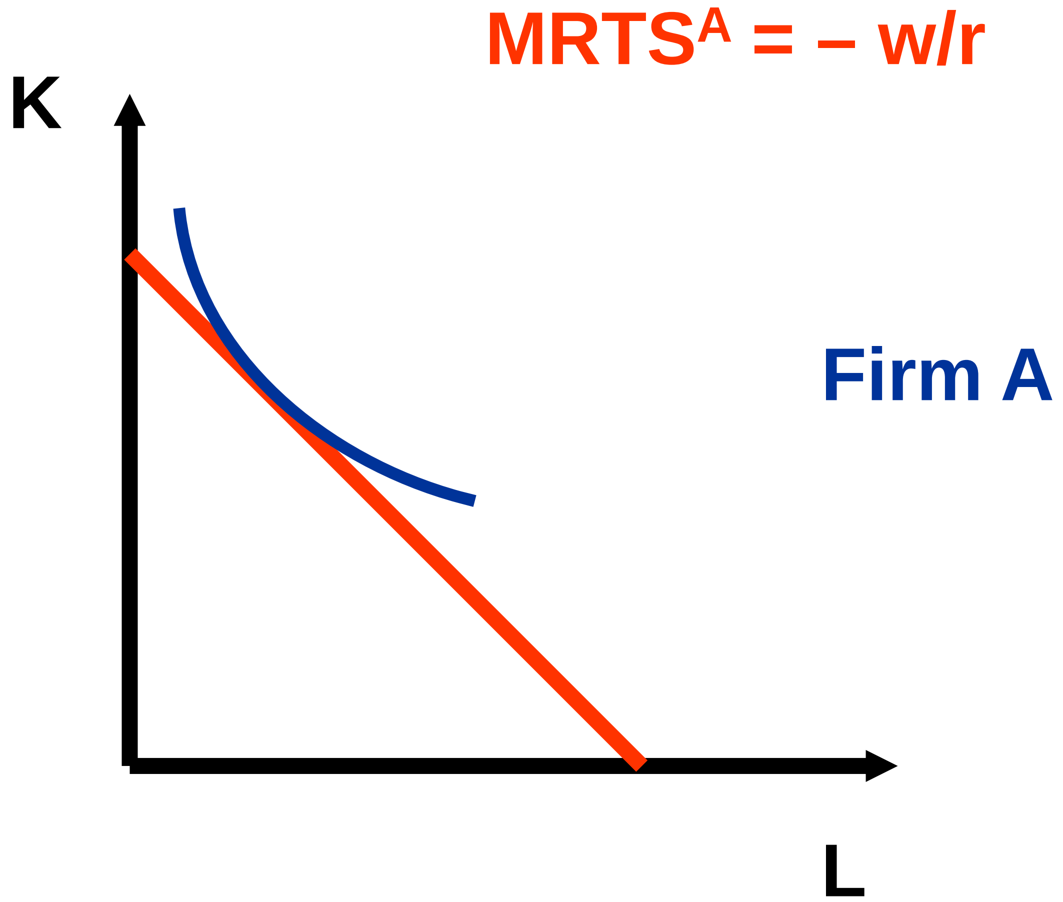
Slope of isoquant = Slope of isocost

MRTS = (-) w/r

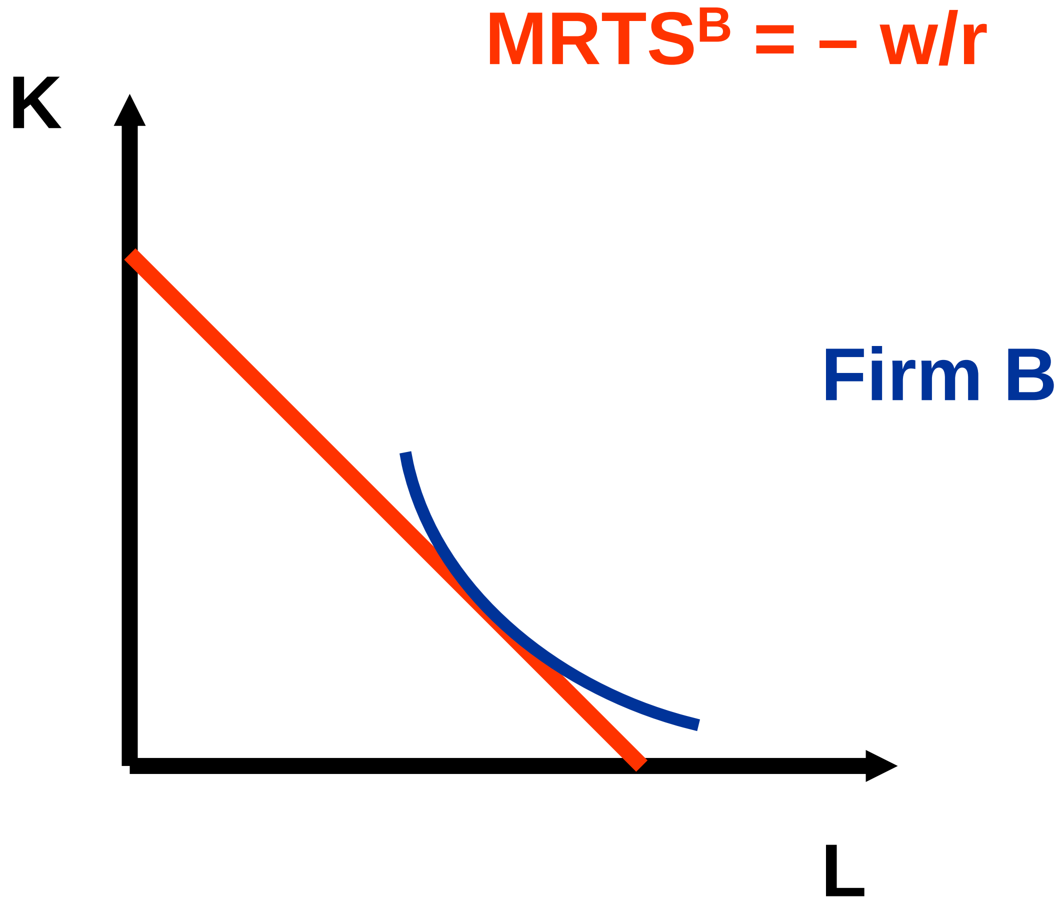
or

MP_L/MP_K = (-) w/r

Aside: General Equilibrium



Aside: General Equilibrium



Aside: General Equilibrium

$MRTS^A = -w/r$

$MRTS^B = -w/r$

$MRTS^A = MRTS^B$

We will return to this later when doing general equilibrium.

The Cost-Minimization Problem

- **A firm is a cost-minimizer if it produces any given output level y $\geq$ 0 at smallest possible total cost.**
- **c(y) denotes the firm's smallest possible total cost for producing y units of output.**
- **c(y) is the firm's total cost function.**

The Cost-Minimization Problem

- **Consider a firm using two inputs to make one output.**
- **The production function is**

  $$y = f(K,L)$$
- **Take the output level $y \geq 0$ as given.**
- **Given the input prices r and w, the cost of an input bundle (K,L) is $rK + wL$.**

The Cost-Minimization Problem

- For given r, w and y, the firm's cost-minimization problem is to solve

$$\min_{K,L\geq 0} (rK + wL)$$

subject to $$f(K,L) = y$$

Here we are minimising costs subject to a output constraint. Usually you will not be asked to work this out in detail, as the working out is tedious.

The Cost-Minimization Problem

- **The levels K*(r,w,y) and L*(r,w,y) in the least-costly input bundle are referred to as the firm's conditional demands for inputs 1 and 2.**
- **The (smallest possible) total cost for producing y output units is therefore**

$$c(r,w,y) = rK^*(r,w,y) + wL^*(r,w,y)$$

Conditional Input Demands

- **Given r, w and y, how is the least costly input bundle located?**
- **And how is the total cost function computed?**

A Cobb-Douglas Example of Cost Minimization

- **A firm's Cobb-Douglas production function is** $y = f(K,L) = K^{1/3}L^{2/3}$
- **Input prices are r and w.**
- **What are the firm's conditional input demand functions?**

K*(r,w,y), L*(r,w,y)

A Cobb-Douglas Example of Cost Minimization

At the input bundle (K*,L*) which minimizes the cost of producing y output units:

(a) $$y = (K^*)^{1/3}(L^*)^{2/3}$$ **and**

(b) $$-\frac{w}{r} = -\frac{\partial y/\partial L}{\partial y/\partial K} = -\frac{(2/3)(K^*)^{1/3}(L^*)^{-1/3}}{(1/3)(K^*)^{-2/3}(L^*)^{2/3}}$$

$$= -\frac{2K^*}{L^*}$$

A Cobb-Douglas Example of Cost Minimization

(a) $y = (K^*)^{1/3}(L^*)^{2/3}$

(b) $\dfrac{w}{r} = \dfrac{2K^*}{L^*}$

A Cobb-Douglas Example of Cost Minimization

(a) $y = (K^*)^{1/3}(L^*)^{2/3}$ **(b)** $\frac{w}{r} = \frac{2K^*}{L^*}$

From (b), $L^* = \frac{2r}{w} K^*$

Now substitute into (a) to get

$$y = (K^*)^{1/3}\left(\frac{2r}{w}K^*\right)^{2/3} = \left(\frac{2r}{w}\right)^{2/3} K^*$$

So $K^* = \left(\frac{w}{2r}\right)^{2/3} y$ **is the firm's conditional demand for Capital.**

A Cobb-Douglas Example of Cost Minimization

Since $L^* = \frac{2r}{w} K^*$ **and** $K^* = \left(\frac{w}{2r}\right)^{2/3} y$

$$L^* = \frac{2r}{w}\left(\frac{w}{2r}\right)^{2/3} y$$

is the firm's conditional demand for input 2.

$$L^* = \left(\frac{2r}{w}\right)^{1/3} y$$

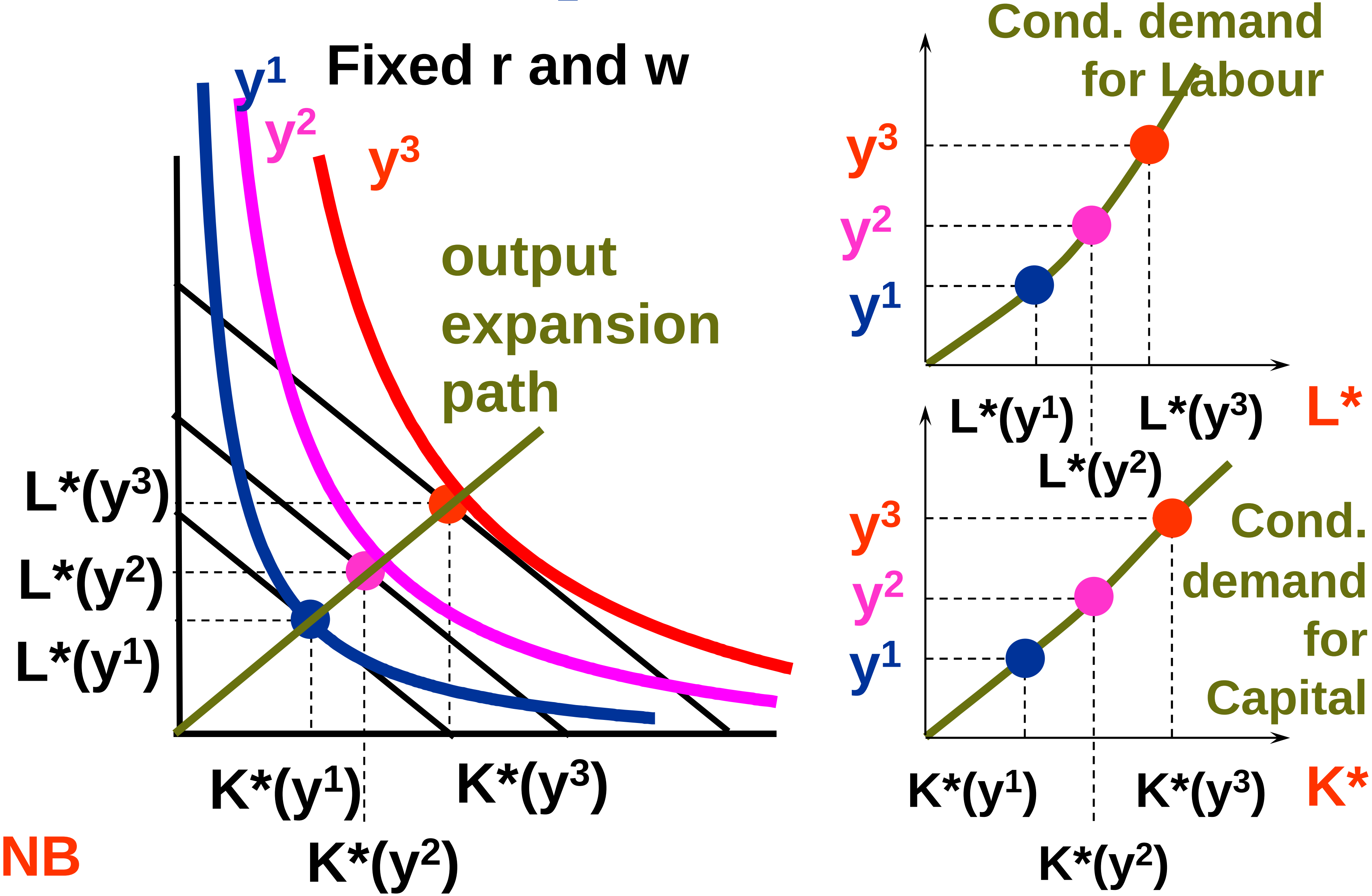
A Cobb-Douglas Example of Cost Minimization

So the cheapest input bundle yielding y output units is

$$\left(K^*(r,w,y), L^*(r,w,y)\right)$$

$$= \left(\left(\frac{w}{2r} \right)^{2/3} y, \left(\frac{2r}{w} \right)^{1/3} y \right)$$

Conditional Input Demand Curves

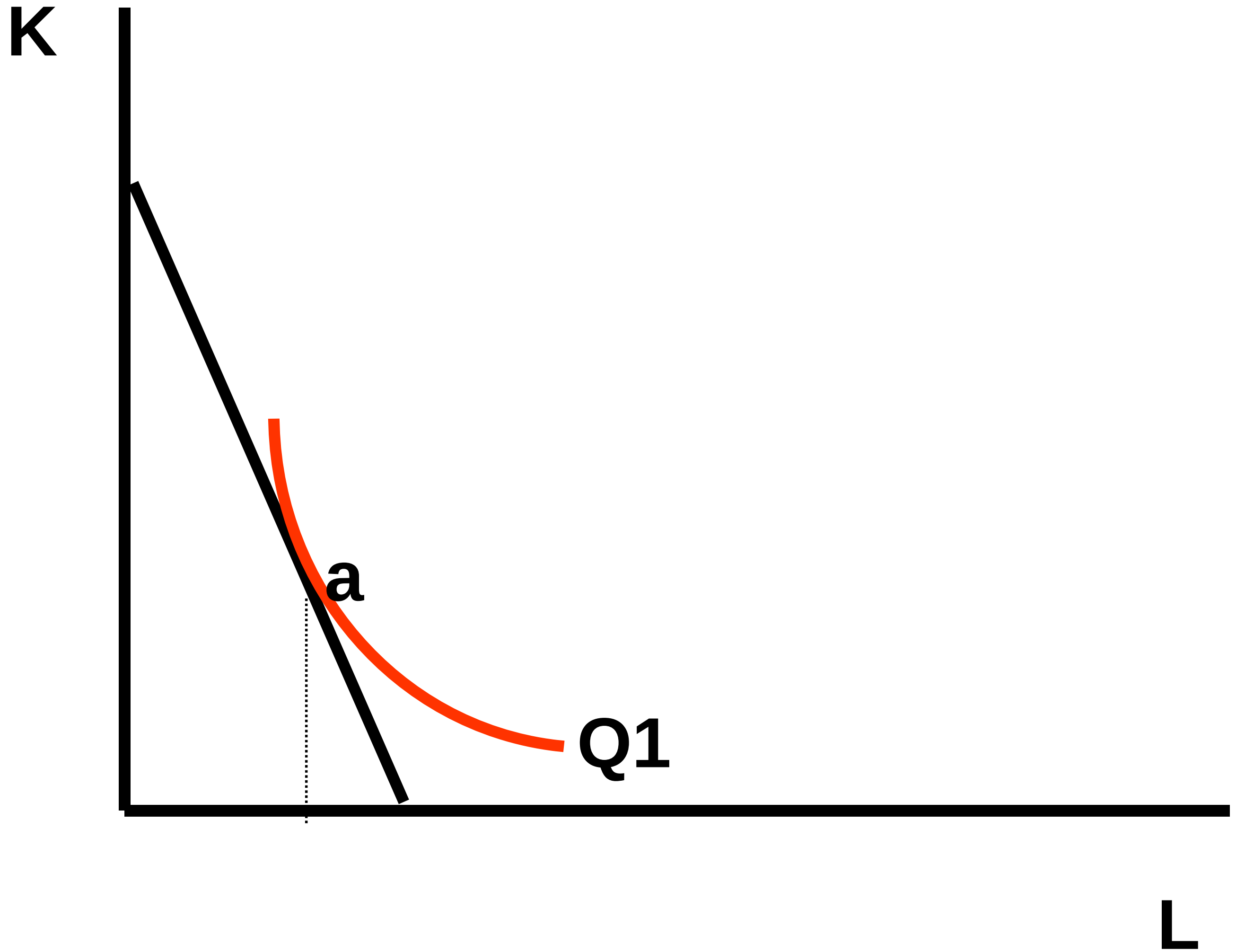


NB

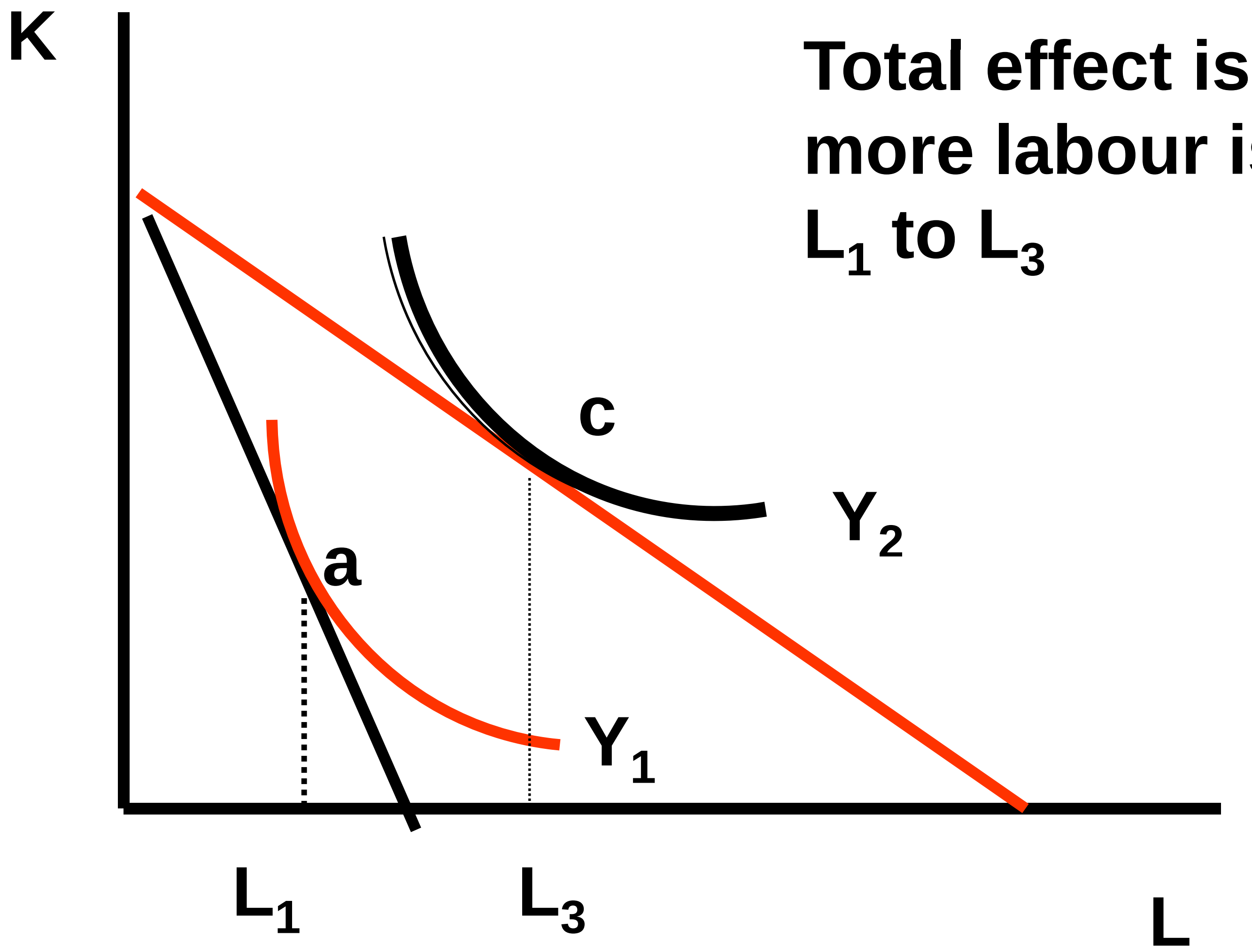
Total Price Effect

- Recall the income and substitution effect of a price change.
- We can also apply this technique to find out what happens when the price of a factor of production changes, e.g. wage falls
- Substitution and output (or scale) effects

Total Price Effect

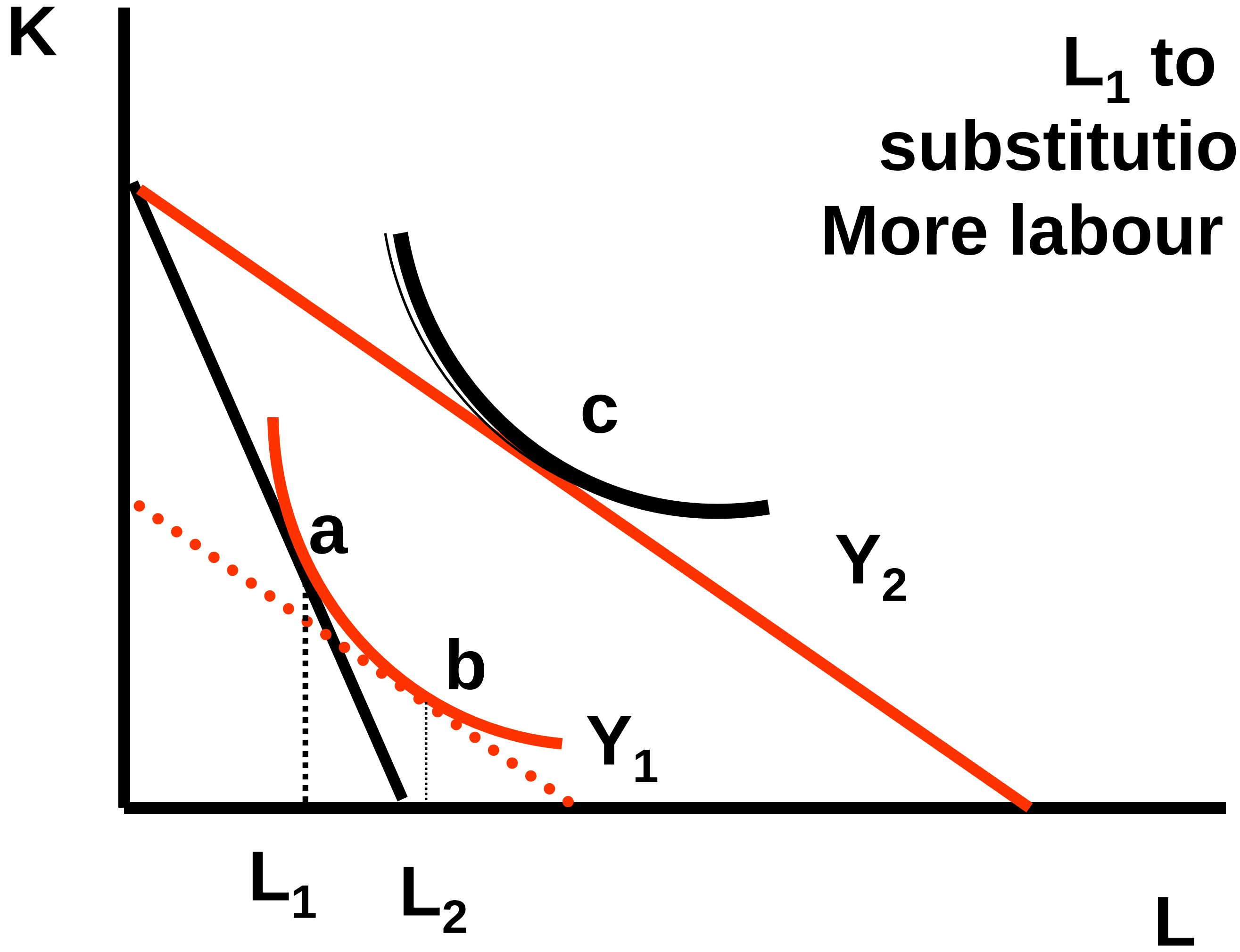


Total Price Effect



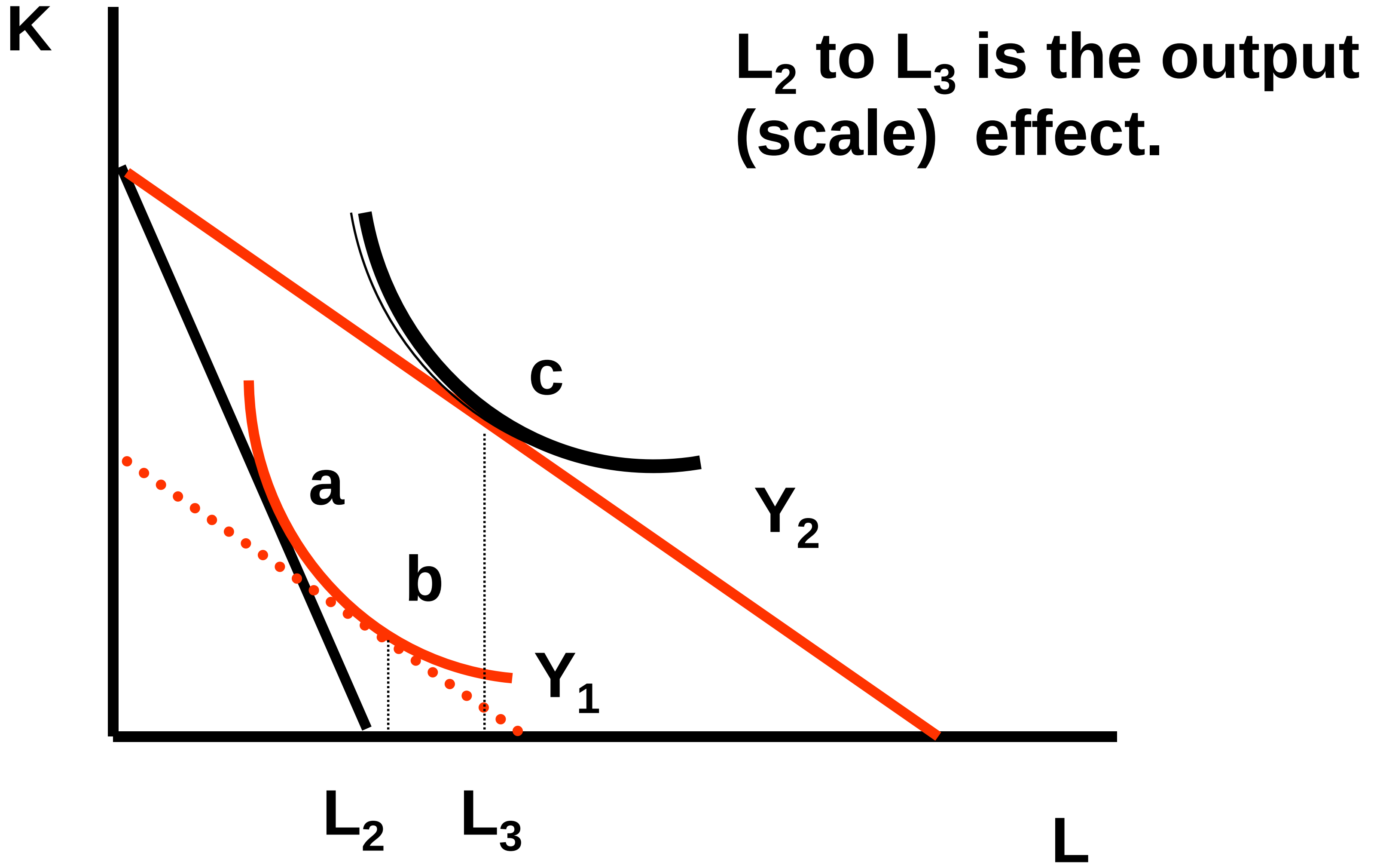
Total effect is a to c, more labour is used L_1 to L_3

Total Price Effect



L_1 to L_2 is the substitution effect. More labour is being used.

Total Price Effect



Total Price Effect

- **What about perfect substitutes and perfect complements? (Homework)**