LECTURE 3: MULTI-VARIABLE OPTIMIZATION

QUESTIONS AND PROBLEMS

True/False Questions

- If a $f(x_1, x_2)$ is a function of both x_1 and x_2 , then the partial derivative with respect to x_1 can be a function of *both* x_1 and x_2 .
- _____ The total differential is equal to the sum of all partial derivatives of a function.
- _____ It is not necessary to check the second order conditions for a function of two variables, because the first order conditions are sufficient to indicate whether the function is at a maximum or a minimum.
- _____ When a function has 3 variables, the first order conditions of maximization of that function consist of 3 equations, which must be solved as a system to obtain the maximum of that function.
- The total differential of a function is equal to the sum of all partial derivatives of that function multiplied by the changes in the values of the respective variables.

Short Questions

1. Consider the function $y = z^2 e^{2q}$. What are the first order partial derivatives of this function with respect to z and q?

2. A farm is using two major inputs: fertilizer (F) and pesticides (P). Its output of corn (C) is given by the function

$$C = 10 + 4 \log(1+F) + \log(1+2P)$$

What is the incremental change in output for this farm if it increases both its inputs, with fertilizer increase being 5 times that of the pesticide increase?

3. A firm's profit depends on the reliability of its product, R, and the number of units it sells, Q. The profit function is given by

$$\Pi = 10 \ Q - \frac{Q^2}{R}$$

What is the effect of increasing reliability on profits? What is the effect of increasing reliability on the marginal effect of output increases to the firm's profits? [Hint: the second question involves a cross-partial derivative.]

Problems

1. A firm produces two products, 1 and 2. Suppose its profit function is given by the equation

$$\Pi(q_1, q_2) = 8 + 4 q_1 q_2 - 4 q_1^2 - q_2^2$$

where q_1 and q_2 are the output levels of products 1 and 2, respectively.

- a. How does an increase in q_1 affect the firm's profits?
- b. How does an increase in q_2 affect the firm's profits?
- c. Suppose the value of q_1 is equal to 4. For what levels of output of product 2 does increasing the output of product 1 increase profits?

2. The amount demand for Lexus cars in the US depends on their price and on per capita income *(PCI)*, and is given by

$$Q_L = \alpha + \beta e^{PCI} - \gamma \frac{P_L}{PCI}$$

where α , β , and γ are positive constants (parameters).

- a. What is the effect of an increase in per capita income on the demand for Lexus cars?
- b. Is the effect of a price increase on the demand stronger or weaker if per capita income increases? Show your answers using calculus.

3. A firm's production costs are a function of the quantity, Q, it produces and the product's reliability, R. The cost function is given by

$$C = 10 R^3 Q^2$$

- a. What is the effect of increasing output on production costs?
- b. What is the effect of increasing reliability on costs? Is increasing reliability costly to the firm?

c. What is the effect of increasing reliability on the marginal cost of output? Is the cost of increasing reliability higher for firms that produce more units?

4. A firm produces two types of products, A and B. The total production cost of the firm as a function of the quantities of A and B it produces is given by:

$$C(Q_A, Q_B) = 3 Q_A^2 + Q_B^2 + Q_A Q_B$$

This cost function indicates that it is "more expensive" to produce product A than product B (at least if the firm is producing an equal amount of both) and that both products create cost externalities to the production of the other, i.e., the marginal cost if either is increasing in the output of the other.

Suppose the firm can sell a unit of A for a price $P_A = 4$ and a unit of B for a price $P_B = 2$, how much of each product should the firm produce in order to obtain the highest possible profits ? Recall that the firm's profits are equal to total revenue (from both products) minus total cost (of producing both products).

Note: You need only check the First Order conditions. The Second Order conditions are satisfied.

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If the price of product B is equal to 1 how high does the price of product A have to be before the firm produces an equal amount of both? Recall that the firm's profits are equal to total revenue (from both products) minus total cost (of producing both products). [Note: the price of product A must treated as an unknown parameter P_A when computing the profits of the firm.]