LECTURE 4: CONSTRAINED OPTIMIZATION

QUESTIONS AND PROBLEMS

True/False Questions

____ The Lagrangian method is one way to solve constrained maximization problems.

____ The substitution method is a way to avoid using calculus when solving constrained maximization problems.

____ The Lagrange multiplier (Lagrangian) method is a way to solve minimization problems that are subject to a constraint.

____ The value of the Lagrange multiplier measures how the objective function of an economic agent changes as the constraint is relaxed (by a bit).

____ The substitution and the Lagrange multiplier methods are guaranteed to give identical answers.

____ At the optimum of a constrained maximization problem solved using the Lagrange multiplier method, the value of the Lagrange multiplier is equal to zero.

____ When taking no constraint into consideration, a firm's optimal choices of output levels for its two products are 4 and 5, respectively. If for that firm the sum of its two products output levels is constrained to be less than 10, then we must solve the firm's constrained optimization problem to make sure that 4 and 5 are still the optimal output levels.

Short Questions

1. What is the economic interpretation of the Lagrange multiplier?

2. How does the substitution method work?

Problems

1. An accounting firm uses partners and staff to produce an audit. The quality of the audit (as measured by reduction in litigation liability and the likelihood of audit errors) is a function of the composition of the audit team. In particular,

$$r = P^{0.2} S^{0.4}$$

where r is the audit quality, P is the partner-hours devoted to the audit, and S is the staff-hours devoted to the audit. Notice that both partners and staff are essential for audit quality, and that audit quality increases in the amount of either, but at a decreasing rate.

The cost of a partner-hour is 50 while the cost of a staff-hour is 5. The budget for this audit is 5000. How many partner and staff hours will the accounting firm choose to maximize quality subject to this budget constraint?

2. Consider a two product firm with a profit function

$$\Pi(q_1, q_2) = -50 + 4 q_1 - 2 q_1^2 + 3 q_2 - q_2^2 - q_1 q_2$$

where q_1 and q_2 are the output levels of products 1 and 2, respectively. The manufacturing of the two products uses a scarce resource: one unit of product 1 uses one unit of the resource; one unit of product 2 uses two units of the resource. The firm has K units of this scarce resource.

a. Write down the firm's constraint that involves the use of this scarce product.

b. Assume that the constraint is binding, i.e., that K is sufficiently small that all of the scarce resource will be used. Solve the constrained maximization problem of the firm using the substitution method.

c. What is the profit of the firm at the optimal values of q_1 and q_2? [Hint: the answer will be a function of K.]

d. What is the marginal value of the scarce resource to the firm (in terms of increased profit)?

e. For what value of K would the resource no longer be scarce, i.e., how high must K be for the firm to choose not to use all of this resource?

3. A firm can raise up to 10 million dollars in the financial markets to develop and market a new product. The profits of the firm are given (in million of dollars) by the equation

$$\Pi = 200\, D^{0.6}\, M^{0.2} - D - M$$

where D is the amount of money spent on development (in millions) and M is the amount of money spent on marketing (in millions). The constraint that the firm can raise up to 10 million for D and M implies that

$$D + M \leq 10$$

Assume that this constraints binds (which is indeed the case), i.e, take as given that .

$$D + M = 10$$

a. What is the Lagrangian expression for this constrained maximization problem?

b. What are the associated First Order Conditions of maximization of the Lagrangian? [Note: there is no need to proceed to the solution of this particular problem, as it is algebraically too tedious.]

c. What would the way the profit function look like if I used the constraint to substitute away one of the two variables?

4. A person's satisfaction (or utility) from playing tennis and golf is given by

$$U = \log(G) + 3\log(T)$$

where G is the number of hours spent playing golf and T is the number of hours spent playing tennis. This person has 10 hours per week to devote to these sports. However, each hour of playing tennis typically entails one hour of waiting for an empty court, thus using up twice the time of actual play. As a consequence, for example, if he spent 2 hours playing golf and 4 hours playing tennis, he would have used up the full 10 hours of his available time.

a. What equation describes this person time constraint?

b. What is the Lagrangian expression of this constrained maximization problem?

c. Use this Lagrangian expression to find out what is the satisfaction (or utility) maximizing choice of time to play golf and tennis.

5. An upstart firm has a total expense budget of *B*. This budget can be spent on (i) advertizing the firm's product and (ii) R&D that reduces the production costs. Denote the amount spent on advertizing by *A* and the amount spent on R&D by *RD*. The expense budget, *B*, is determined by the firm's financial backers, and is not in the control of the firm. The firm only controls the values of *A* and *RD*.

Following the decision on advertising and R&D, the firm produces the product at a unit cost of $c = 5 - RD$ and sells it at a unit price of 10. Note that the production costs are **not** financed out of the expense budget *B*. They are incurred by the firm, but are financed out of current revenue. The quantity that the firm sells equals $Q = 5 A$, i.e., the more the firm advertizes, the more units of the product it will be able to sell.

a. Write the firm's profit function in terms of *A* and *RD* (i.e., in terms of the decision variables of the firm). [Note that costs consist of total production costs, the advertising expenditure and the R&D expenditure.]

b. What allocation of funds to advertizing and R&D maximizes the firm's profits? You can solve this problem either with the substitution or the Lagrangian method.