A slot allocation model with queuing constraints based on the server-always-busy approximation

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1 Introduction

2 Queuing Model

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Airport congestion

- In many airports around the world, demand to use the airport infrastructure exceeds available capacity
- This leading to congestion-related delays or infeasible slot scheduling
- The expansion of infrastructure is not possible in the short to medium term and so congestion must be mitigated through the management of demand.
Slot Allocation

- Outside of the US, demand is managed through the allocation of slots under the IATA Worldwide Slot Guidelines.
- A slot is a time interval during which an aircraft can use an airport infrastructure for the purposes of landing or take-off.
- A *coordinator* proposes an initial allocation for slots to airlines based on their requests.
- The quality of a schedule is often measured through the schedule displacement.

<table>
<thead>
<tr>
<th>Requested Slot</th>
<th>Assigned Slot</th>
<th>Schedule Displacement</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time</td>
<td></td>
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</table>
Airport Capacity

- A key parameter in the problem of allocating slots is the *declared capacity* which limits the number of slots that can be allocated over time.
- The declared capacity should take into consideration physical constraints of an airport:
  - Runway capacity
  - Terminal capacity (number of passengers)
  - Available apron stands

Key Issue

Capacity and allocation of slots should also take into account *operational delays* which arise from *queueing*.
Queuing and Slot Allocation

- [JO15] developed a scheduling algorithm which incorporated queuing constraints:

  - Schedule Optimisation
    - Minimise approximate queue lengths subject to constraints on maximum displacement

  - Evaluate Queueing Delays
    - Calculate expected queue lengths via stochastic model

  - Are the queueing delays acceptable?
    - YES
      - Finish
    - NO
      - Adjust Model
        - Relax constraints on maximum displacement

Issues:
- Bounding expectation does not effectively mitigate against possibility of large queues
- Reducing queue lengths by controlling maximum displacement may lead to more displacement than necessary
Aims

- Development of slot allocation model:
  - Directly incorporate stochastic queuing dynamics into slot allocation optimization model
  - Constrain *probability* of large queues rather than expectation

Warning - under development

We do not implement IATA WSG
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Set-up

- Airport runway usage often modeled as a queueing system [JOW17]
- We suppose that arriving and departing aircraft can be managed separately (e.g. separate runways) and consider queuing delays for arriving aircraft
- We concentrate on the management of aircraft arrivals
- The term *arrival* refers to both the arrival of an aircraft, and the arrival of a customer in the abstract queueing model
• Assume time is discretized into time periods \( \{1, \ldots, T\} \)
• We model arriving aircraft at an airport as the following queueing system:
  \[
  \text{GI}/E_k/1
  \]
• We assume that arrival distributions are known
• Service rate is \( \mu_t \) in time period \( t \), that is service time in period \( t \in T \) is assumed to be distributed as \( \text{Erlang}(k, \mu_t) \)
Probability of long queues

• Let $X_t$ be the length of the queue after time period $t$
• We would like to construct a schedule with low probability of long queues, that is such that

$$\mathbb{P}(X_t \leq n) \geq p \quad \text{for each } t = 1, \ldots, T$$

where $n$ and $0 < p < 1$ are specified tolerances.
Arrival Distribution Approximation

Let

\[ A_t = \text{number of arrivals by end of period } t \]
\[ F_m^t = \text{probability that } m \text{ arrives by the end of time period } t \]

Then,

\[ \mathbb{E} [A_t] = \sum_m F_m^t \]
\[ \text{Var} [A_t] = \sum_m F_m^t (1 - F_m^t) \]

Assuming independent and numerous arrivals, we can approximate \( A_t \) by the central limit theorem:

\[ \tilde{A}_t \sim \text{Normal} \left( \mathbb{E} [A_t], \text{Var} [A_t] \right) \]
Departure Distribution Approximation

If the server is always busy, then the number of Erlang service stages completed is Poisson distributed:

\[ D_t \sim \text{Poisson} \left( k \sum_{s=1}^{t} \mu_s \right) \]

For the purpose of tractability, we will use the following Normal approximation which is valid for Poisson distributions with a large mean:

\[ \tilde{D}_t \sim \text{Normal} \left( k \sum_{s=1}^{t} \mu_s, k \sum_{s=1}^{t} \mu_s \right) \]
SAB Approximation

- Under the assumption that there is a small probability that the queue is ever empty, $X_t$ can be approximated by $kA_t - D_t$
- This is called the Server Always Busy (SAB) approximation
- For tractability, we use our Normal approximations in the SAB approximation

$$\tilde{X}_t = k\tilde{A}_t - \tilde{D}_t$$

- The constraint on long queue probabilities can now be written as follows:

$$\mathbb{P}(X_t \leq n) \geq p$$

$$\Leftrightarrow \Phi \left( \frac{n + 0.5 - \mathbb{E}[\tilde{X}_t]}{\sqrt{\text{Var}[\tilde{X}_t]}} \right) \geq p$$
Deterioration of SAB approximation

- The SAB approximation becomes invalid in periods of low demand:
Conditional Queuing Constraints

• The SAB approximation will grossly underestimate queue lengths in periods of low demand
• We therefore use stronger constraints conditioned on the queue length being zero:

\[ P(X_v > n \mid X_u = 0) \leq p \quad \text{for all } 1 \leq u \leq v \leq T \]

• There are \( O(T^2) \) conditional constraints rather than \( O(T) \) constraints
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- Define the following set of binary decision variables:

\[ x_m^t = \begin{cases} 
1 & \text{if request } m \text{ allocated slot } t \\
0 & \text{otherwise} 
\end{cases} \]

- And the following set of known parameters:

\[ \mu_s = \text{Erlang rate parameter for time period } s \] 
for service time distribution

\[ q_{uv}^t = \text{probability that request scheduled for time } t \]
arrives between beginning of \( u \) and end of \( v \)
• The parameters $q_{uv}^t$ can be calculated from specified distributions.

• The arrival time distribution of a flight with time slot $t$ is assumed to be

  $\text{Normal}(t - 0.5, \sigma)$

truncated to $[0, T]$. 

Arrival Distributions
Auxiliary Variables

- We introduce the auxiliary decision variables $F_{uv}^t$ to represent probability that movement $m$ arrives between beginning of movement $u$ and end of period $v$:

$$F_{uv}^m = \sum_{s=1}^{T} q_{uv}^s x_m^s \quad \text{for each } m \in \mathcal{M}, (u, v) \in \mathcal{T}^2 \text{ (with } u < v)$$

- Using SAB approximation, the mean and variance of queue lengths at end of time $v$ assuming empty at beginning of time $u$ are:

$$\mu_{X_{uv}} = k \sum_{m \in \mathcal{M}} F_{uv}^t - \sum_{s=u}^{v} \mu_s \quad \text{for } (u, v) \in \mathcal{T}^2 \text{ (with } u < v)$$

$$\Sigma_{X_{uv}} = k^2 \sum_{m \in \mathcal{M}} F_{uv}^t (1 - F_{uv}^t) + \sum_{s=u}^{v} \mu_s \quad \text{for } (u, v) \in \mathcal{T}^2 \text{ (with } u < v)$$
Standard Deviation and Queueing Constraints

- Standard Deviation can be defined with following non-convex quadratic constraints:
  \[ \sum_{uv} X \leq (\sigma^X_t)^2 \quad \text{for each} \quad (u, v) \in T^2 \quad \text{(with} \quad u < v) \]

- The queueing constraints are now formulated as follows:
  \[ \Phi \left( \frac{(n + 0.5) - \mu^X_{uv}}{\sigma^X_{uv}} \right) \geq p \]
  \[ \iff \mu^X_{uv} + \sigma^X_t \Phi^{-1}(1 - p) \leq n + 0.5 \quad \text{for each} \quad (u, v) \in T^2 \quad \text{(with} \quad u < v) \]
# Full Model - Notation

<table>
<thead>
<tr>
<th>Sets</th>
<th></th>
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</thead>
<tbody>
<tr>
<td>$\mathcal{M}$</td>
<td>set of slot requests</td>
</tr>
<tr>
<td>$T = {1, \ldots, T}$</td>
<td>set of coordination time intervals</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Parameters</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$k$</td>
<td>Erlang shape parameter for service time distribution</td>
</tr>
<tr>
<td>$\mu_s$</td>
<td>Erlang rate parameter for time period $s$ for service time distribution</td>
</tr>
<tr>
<td>$t_m$</td>
<td>requested time for movement $m$</td>
</tr>
<tr>
<td>$q_{uv}^t$</td>
<td>probability that request scheduled for time $s$ arrives between beginning of $u$ and end of $v$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Decision variables</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_{mt}$</td>
<td>indicates whether movement $m$ is assigned slot $t$</td>
</tr>
<tr>
<td>$F_{uv}^t$</td>
<td>probability request $m$ arrives in interval $[u-1, v]$</td>
</tr>
<tr>
<td>$\mu_{uv}^X$</td>
<td>expected queue length at end of time period $v$ given empty queue at beginning of time period $u$</td>
</tr>
<tr>
<td>$\Sigma_{uv}^X$</td>
<td>variance of queue length at end of time period $v$ given empty queue at beginning of time period $u$</td>
</tr>
<tr>
<td>$\sigma_{uv}^X$</td>
<td>standard deviation of queue length at end of time period $v$ given empty queue at beginning of time period $u$</td>
</tr>
</tbody>
</table>
Full Model - Formulation

minimize \( \sum_{m \in \mathcal{M}} |t - t_m| x^t_m \) \hspace{1cm} \text{(minimize total displacement)}

subject to \( \sum_{t=1}^{T} x^t_m = 1 \) \hspace{1cm} \text{(allocate a slot to every arrival)}

\[ F^m_{uv} = \sum_{s=1}^{T} q^s_{uv} x^s_m \] \hspace{1cm} \text{for each } m \in \mathcal{M}, \ (u, v) \in \mathcal{T}^2 \ (\text{with } u < v)

\[ \mu^X_{uv} = k \sum_{m \in \mathcal{M}} F^t_{uv} - \sum_{s=u}^{v} \mu_s \] \hspace{1cm} \text{for each } (u, v) \in \mathcal{T}^2 \ (\text{with } u < v)

\[ \Sigma^X_{uv} = k^2 \sum_{m \in \mathcal{M}} F^t_{uv} (1 - F^t_{uv}) + \sum_{s=u}^{v} \mu_s \] \hspace{1cm} \text{for each } (u, v) \in \mathcal{T}^2 \ (\text{with } u < v)

\[ \Sigma^X_{uv} \leq (\sigma^X_t)^2 \] \hspace{1cm} \text{for each } (u, v) \in \mathcal{T}^2 \ (\text{with } u < v)

\[ \mu^X_{uv} + \sigma^X_t \Phi^{-1}(1 - p) \leq n + 0.5 \] \hspace{1cm} \text{for each } (u, v) \in \mathcal{T}^2 \ (\text{with } u < v) \hspace{1cm} \text{(queue length constraint)}

\[ \sigma^X_{uv} \geq 0 \] \hspace{1cm} \text{for each } (u, v) \in \mathcal{T}^2 \ (\text{with } u < v)

\[ x^t_m \in \{0, 1\} \] \hspace{1cm} \text{for each } m \in \mathcal{M}, \ t \in \mathcal{T}
Remarks

- Due to the non-convex quadratic constraints, this model is a mixed integer non-linear program (MINLP).
- Model would be linear if we did not use $\sigma_{uv}^X$ decision variables.
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Set-up

- Model solved using SCIP solver [Ach09]
- $T = 12$, $\mu = 4.5$, $\sigma = 0.5$, $k = 3$, $p = 0.8$
- Randomly generated requests:

![Aggregate Demand Graph]
• We solve model for \( n = 12, \ldots, 3 \):
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Conclusions

- Proposed new slot scheduling model which incorporates stochastic queueing dynamics
- Model is MINLP and based on “Server always busy” approximation of queues
- Demonstrated model is tractible for small instances
Future Work

- Implement model with arrivals and departures
- Develop more efficient exact and heuristic solution algorithms
- Compare use of model to one which uses deterministic queueing dynamics ([JO15])
- Test model on real request data and investigate trade-off between displacement and operational delay
- Investigate whether model can be incorporated into a model which implements full IATA scheme (e.g. through decomposition)
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