# Admission Control and Routing to Parallel Queues with Delayed Information via Marginal Productivity Indices

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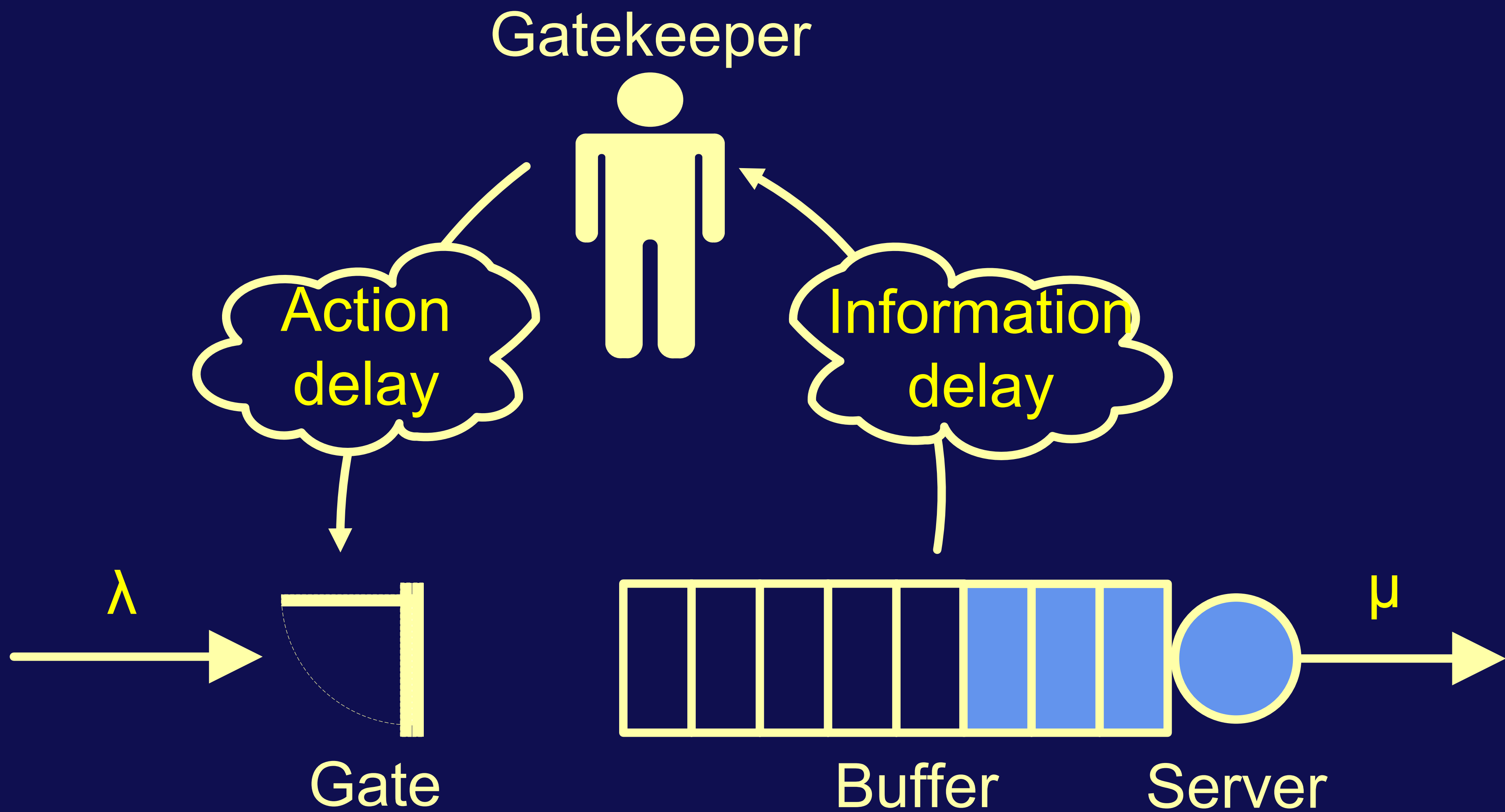
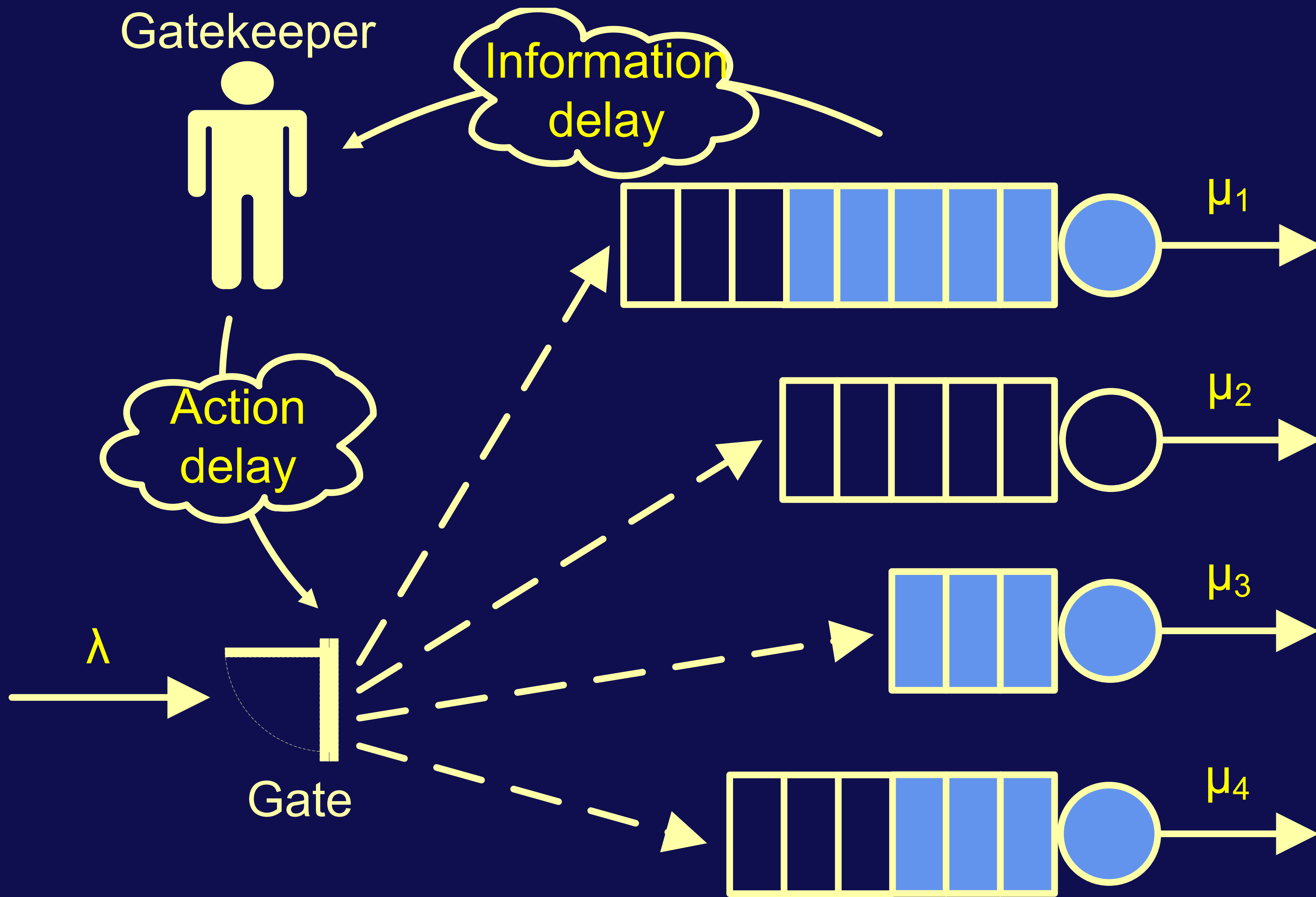
# Motivation

- Delays in information flow and action implementation
  - ▷ physical distance of nodes in networks
  - ▷ long-distance-controlled robots
  - ▷ advanced processing of observations
- May lead to important losses if ignored
- We deal with delays in:
  - ▷ admission control and routing to parallel queues
  - ▷ admission control to a single queue

# Admission Control to a Single Queue with Delays



# Admission Control and Routing with Delays



# Admission Control and Routing with Delays

- Even with $2$ queues it is hard to analyze
- Delay of one period and symmetric queues: JSEQ
  - ▷ "A large number of properties needs to be discovered and then tediously verified." (Kuri & Kumar, 1995)
- Delay of more than one period:
  - ▷ ". . . the approach quickly becomes very unwieldy. . . "
  - ▷ JSEQ is not optimal (both Kuri & Kumar, 1995)
  - ▷ "We were not able to derive significant results. . . " (Artiges, 1993)

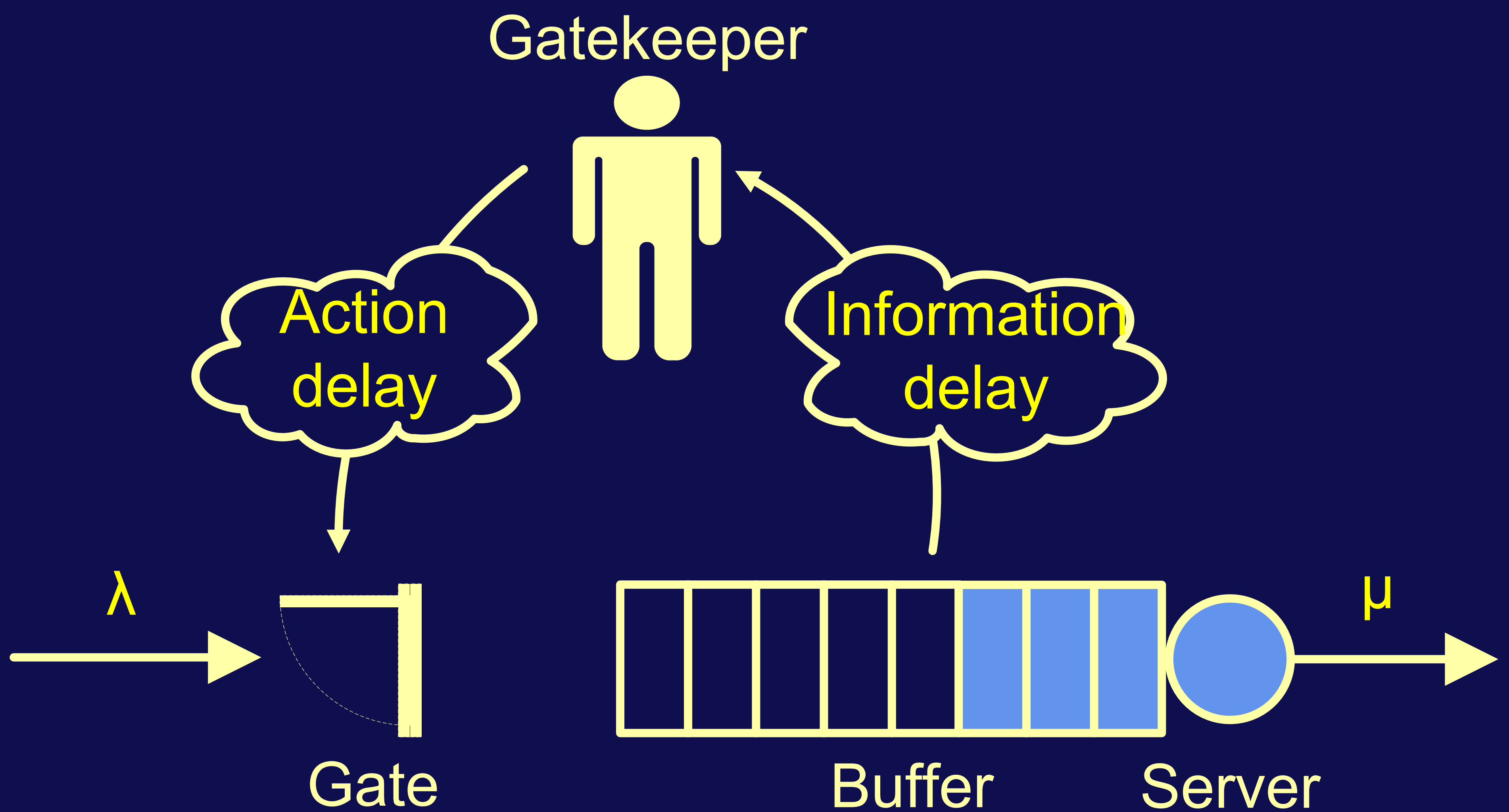
# Admission Control and Routing with Delays

- What if servers, buffers, holding costs and delays differ?
  - ▷ curse of dimensionality
- It is a joint decision of admitting to at most $1$ queue
  - ▷ there must be a queue where a job is worth admitting
  - ▷ if there are several such queues, route to the queue where admitting is most profitable
- Accomplished by an index policy
  - ▷ via marginal productivity index (MPI)
  - ▷ may be suboptimal due to ignored cross-dependence

# Outline

- Admission control to a single queue:
  - ▷ MDP model with no delay
  - ▷ MDP model with one period delay
  - ▷ exploiting special structure via bi-threshold policies
  - ▷ establishing existence of MPIs
  - ▷ obtaining a fast algorithm for MPI calculation
- MPI policy properties for
  - ▷ admission control and routing with one period delay
  - ▷ servers assignment problem with one period delay
- Discussion of generalizations

# Admission Control to a Single Queue with Delays



# Admission Control with No Delay

- Discrete time epochs $t = 0, 1, 2, \ldots$
- Bernoulli arrivals at rate $\lambda$ per period
- Geometric server at rate $\mu$ per period
- Buffer $+$ server room: $I$
- Holding costs at rate $C_i$ per period with $i$ jobs
  - ▷ convex, nondecreasing in $i$
- Loss costs at rate $\nu$ per rejected job

# MDP Model (No Delay)

- Action process $a(t) \in \mathcal{A} := \{0, 1\}$: closing the gate ($a(t) = 1$) or opening the gate ($a(t) = 0$)
- State process $X(t) \in \mathcal{I} := \{0, 1, \ldots, I\}$
  - ▷ state $I$ is uncontrollable
- At epoch $t$: $a(t)$ must be based on $X(t)$
- Transition probabilities $p_{ij}^a$
- One-period cost $C_i + \nu W_i^a$, where the work $W_i^a$ is

$$W_i^1 := \lambda \qquad W_i^0 := \begin{cases} \lambda & \text{if } i = I \\ 0 & \text{otherwise} \end{cases}$$

# MDP Model (One-Period Delay)

- $a(t)$ must be based on $\widetilde{X}(t) := (a(t-1), X(t-1))$
  - ▷ because $X(t)$ is not known at $t$
- Action space $\mathcal{A}$ as before
- Augmented states $\widetilde{\mathcal{I}} := (\mathcal{A} \times \{0, 1, \ldots, I-1\}) \cup \{(*, I)\}$
  - ▷ state $(*, I)$ appears by merging $(0, I)$ with $(1, I)$
- Transition probabilities $p^{a'}_{(a,i),(b,j)} := p^{a}_{ij} \cdot \mathbf{1}\{a' = b\}$
- One-period cost $C_{(a,i)} + \nu W_{(a,i)} := C_i + \nu W^{a}_{i}$
  - ▷ note the independence of the current-period action

# Objective

- Solving the $\nu$-wage problem: $\min_{\pi \in \Pi} f^{\pi}_{(a,i)} + \nu g^{\pi}_{(a,i)}$
  - ▷ choosing a non-anticipative control policy $\pi \in \Pi$
  - ▷ expected total discounted holding cost

$$f^{\pi}_{(a,i)} := \mathbb{E}^{\pi}_{(a,i)} \left[ \sum_{t=0}^{\infty} \beta^t C_{\widetilde{X}(t)} \right]$$

  - ▷ expected total discounted work (number of rejections)

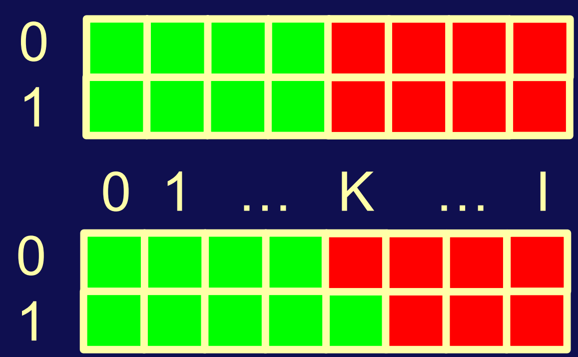
$$g^{\pi}_{(a,i)} := \mathbb{E}^{\pi}_{(a,i)} \left[ \sum_{t=0}^{\infty} \beta^t W_{\widetilde{X}(t)} \right]$$

# Exploiting Special Structure

- There is an optimal policy which is stationary, deterministic, independent of the initial state

- Represent such policies as active sets $\mathcal{S} \subseteq \widetilde{\mathcal{I}}$
  - ▷ the set of states in which it prescribes to shut the gate

- Bi-threshold policies are optimal (Altman & Nain, 1992)
  - ▷ $\widetilde{\mathcal{I}}_{K,K}$
  - ▷ $\widetilde{\mathcal{I}}_{K,K+1}$




- The family of all such active sets: $\mathcal{F}$

# Reduced Problem

- The $\nu$-wage problem can be solved by solving

$$\min_{\mathcal{S}\in\mathcal{F}} f^{\mathcal{S}}_{(a,i)} + \nu g^{\mathcal{S}}_{(a,i)}$$

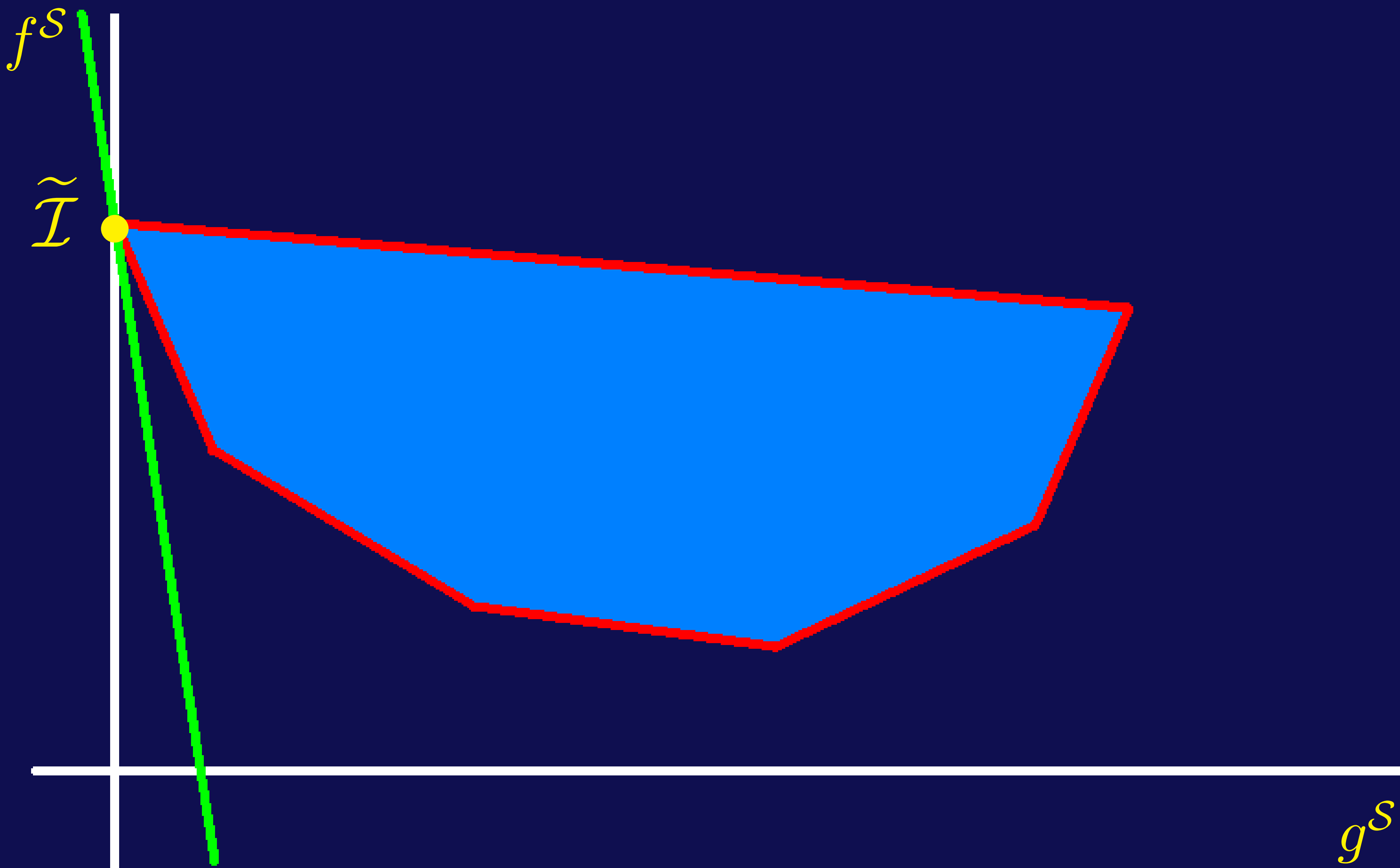
- Evaluating all $\mathcal{S} \in \mathcal{F}$ requires $\mathcal{O}(I^4)$ operations

- "Dual" approach in $\mathcal{O}(I^3)$: marginal productivity indices
  - ▷ but indexability (MPIs existence) must be proved
  - ▷ we do by verifying PCL-indexability (Niño-Mora, 2001)
  - ▷ we improve algorithm to $\mathcal{O}(I)$, as in no-delay case

# Indexability

- $\nu$-wage problem is indexable, if
  - ▷ the optimal active set decreases monotonically from $\widetilde{\mathcal{I}}$ to $\emptyset$ as $\nu$ increases from $-\infty$ to $\infty$
- Equivalently, there exist values $\nu_{(a,i)}$ such that
  - ▷ it is optimal to shut the gate at state $(a, i)$ if $\nu_{(a,i)} \geq \nu$
  - ▷ it is optimal to open the gate at state $(a, i)$ if $\nu_{(a,i)} \leq \nu$
- $\nu_{(a,i)}$ is the marginal productivity index
  - ▷ capturing the marginal productivity of work
  - ▷ how much is worth shutting w.r.t. opening the gate

# Indexability



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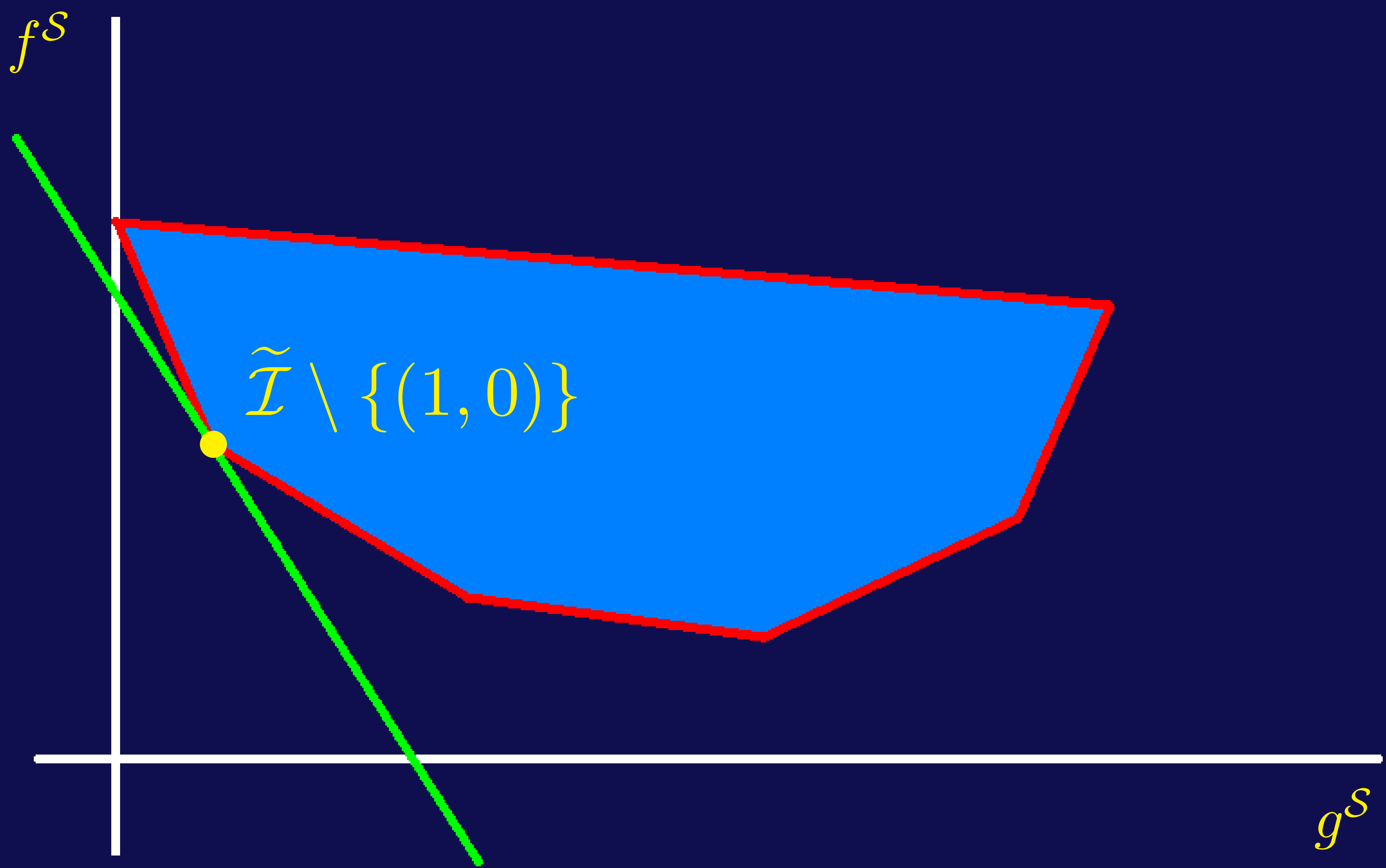

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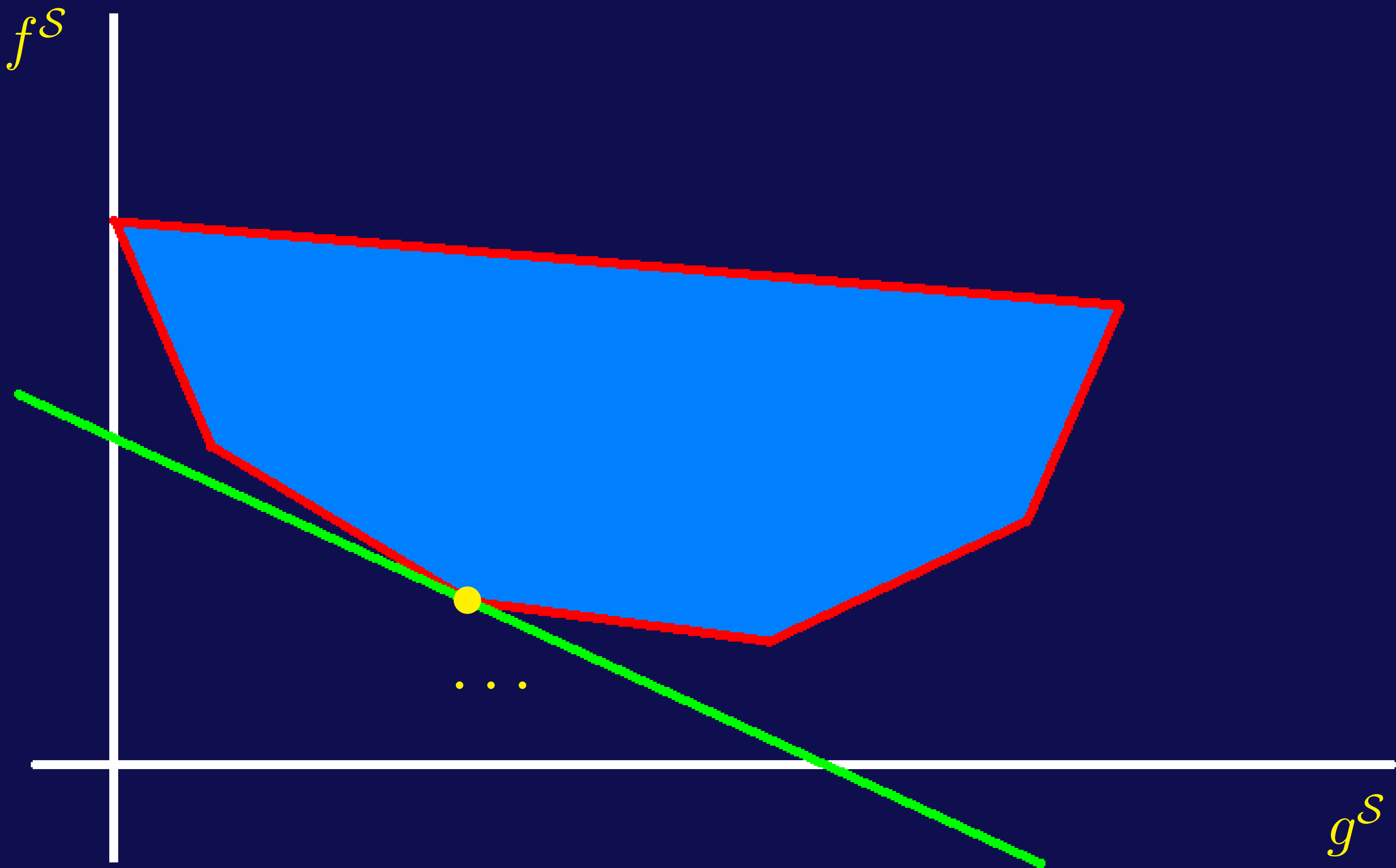

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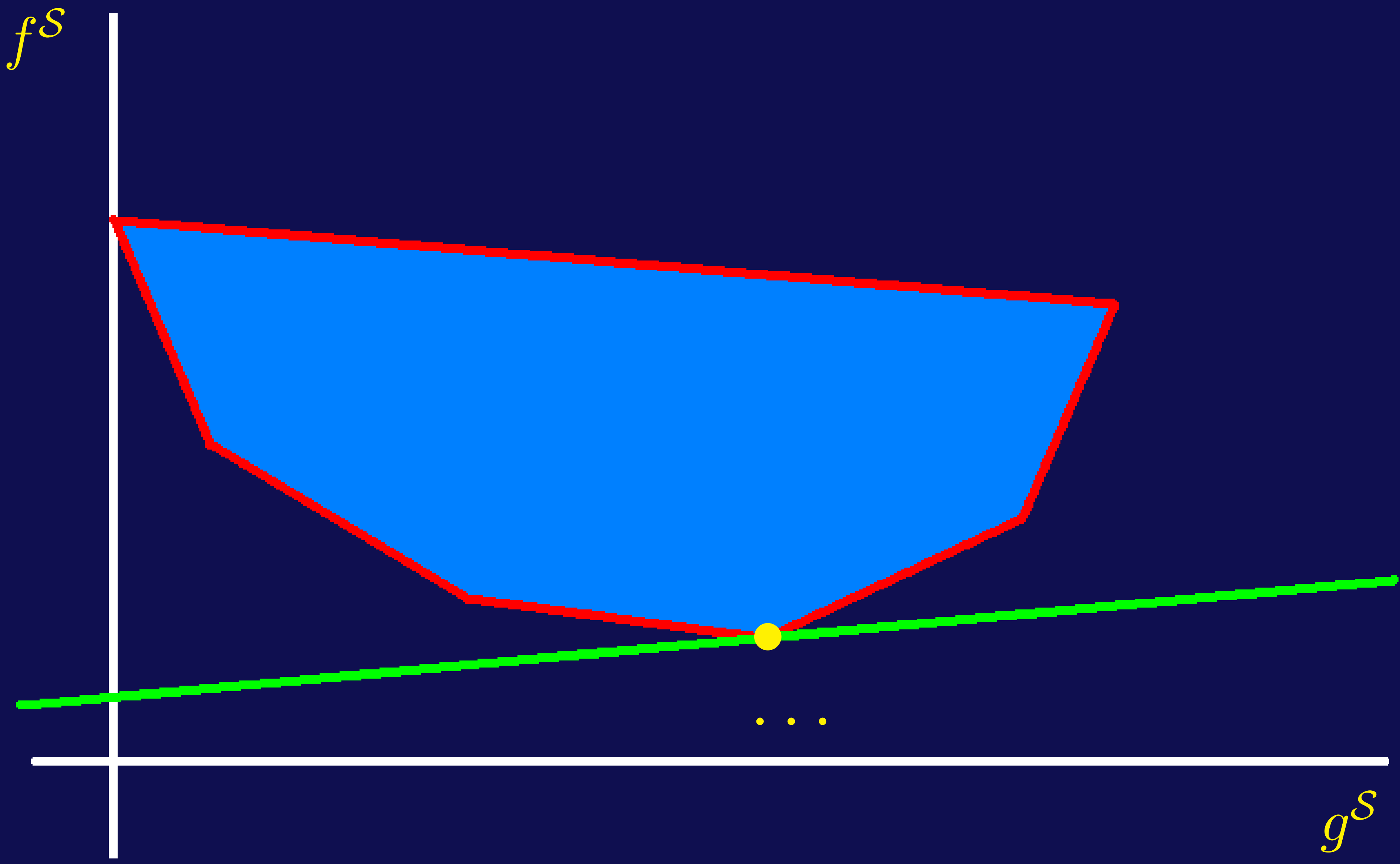

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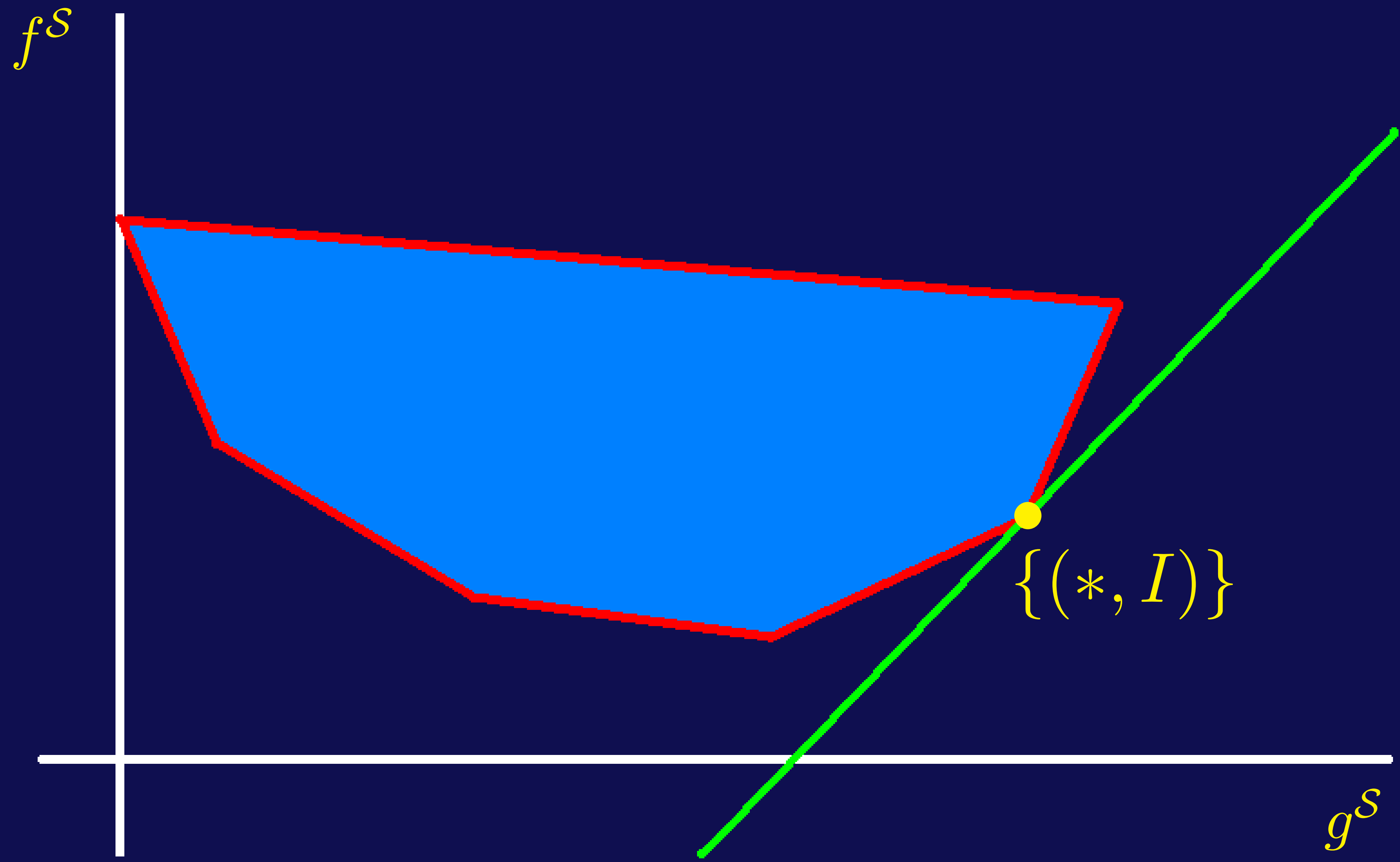

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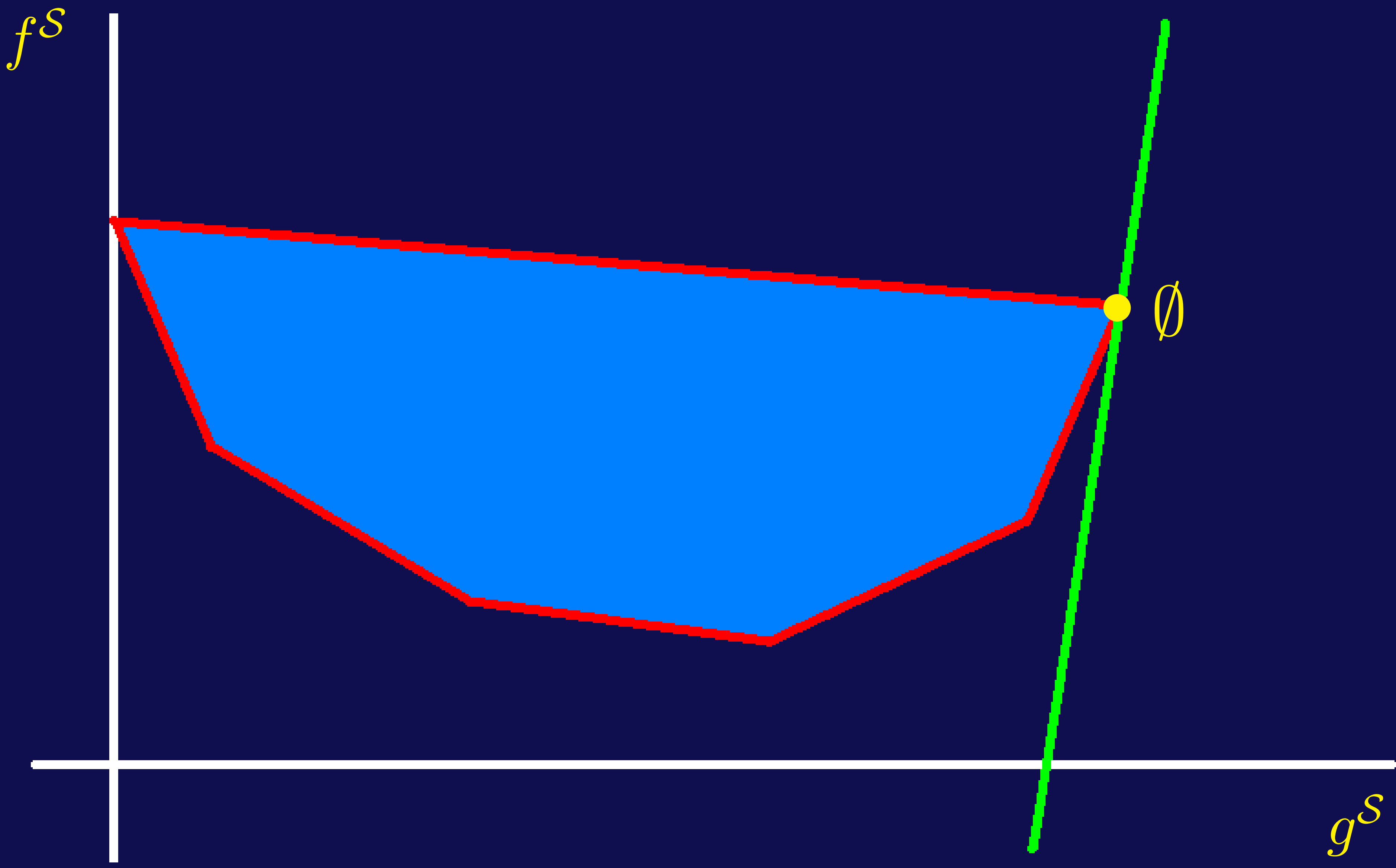


# PCL-Indexability

- A sufficient condition for indexability
- $\nu$-wage problem is PCL($\mathcal{F}$)-indexable, if

(i) $w_{(a,i)}^{\mathcal{S}} > 0$ for each $\mathcal{S} \in \mathcal{F}$ and $(a,i) \in \widetilde{\mathcal{I}}$
(ii) there is an optimal $\mathcal{S} \in \mathcal{F}$ for every rejection cost $\nu$

- ▷ we establish (i) by proving $\Delta_1 g_{(1,i)}^{\mathcal{S}} := g_{(1,i)}^{\mathcal{S}} - g_{(0,i)}^{\mathcal{S}} > 0$
  - because $w_{(a,i)}^{\mathcal{S}}$'s are expected values of $\Delta_1 g_{(1,i)}^{\mathcal{S}}$'s
- ▷ Altman & Nain (1992) established (ii)

- Niño-Mora (ValueTools 2007): $\mathcal{O}(I^3)$ MPI algorithm
  - ▷ here, we simplify it to $\mathcal{O}(I)$ under linear holding costs

# Fast Index Algorithm (FI)

{Input $I, \lambda, \mu, c, \beta$}
{Output $\{\nu_{(a,i)}\}_{(a,i)\in\widetilde{\mathcal{I}}}$}
{Initialization}
$\zeta := \lambda(1-\mu); \quad \eta := \mu(1-\lambda); \quad \varepsilon := 1-\zeta-\eta;$
$A_0 := 0; \quad A_0' := \beta\zeta; \quad B := \beta\mu/(1-\beta+\beta\mu); \quad B' := \beta\zeta B + \beta(\mu-\eta); \quad C := c/(1-\beta+\beta\mu); \quad D_0 := 0;$
$\nu_{(1,0)} := \beta\zeta C/\lambda;$

$$\nu_{(0,0)} := \frac{\beta\zeta C}{\lambda}\cdot\frac{(1-\beta+\beta\mu)(1+\beta\lambda+\lambda\mu)+\beta\zeta(\mu+\beta\mu+\beta\zeta)}{(1-\beta+\beta\mu)(1+\beta\zeta)+\beta\zeta(\beta\zeta-B')};$$

{Loop}
for $K=1$ to $I-1$ do

$A_K := \beta\zeta/[1-\beta+\beta\zeta+\beta\eta(1-A_{K-1})]; \quad A_K' := \beta\zeta+\beta(\mu-\eta)A_K; \quad D_K := (c+\beta\eta D_{K-1})\,A_K/(\beta\zeta); \quad Z_K := A_K A_{K-1}'/A_K';$

$$f^0 := -\frac{\frac{\beta\zeta}{A_K}D_K + \beta\zeta(c+\beta\mu B D_{K-1}) + [c-\beta(\mu-\eta)\beta D_{K-1}]\,B'}{\frac{A_K'}{A_K}+\beta A_{K-1}'B'+\beta\zeta\beta\mu(1-BA_{K-1})};$$

$$f^1 := -\frac{\frac{\beta\zeta}{A_K}D_K + c\beta\zeta B A_{K-1} + [\beta\mu\beta\zeta+(1-\beta)\beta(\mu-\eta)]\,D_{K-1} + A_{K-1}'(c-\beta\zeta\beta C)}{\frac{A_K'}{A_K}+\beta A_{K-1}'B'+\beta\zeta\beta\mu(1-BA_{K-1})};$$

$$g^0 := \frac{\beta\lambda\,(1+B')}{\frac{A_K'}{A_K}+\beta A_{K-1}'B'+\beta\zeta\beta\mu(1-BA_{K-1})}; \quad g^1 := \frac{1+A_{K-1}'}{1+B'}g^0;$$

if $K>1$ then

$$\nu_{(0,K-1)} := \frac{[\beta(\mu-\eta)(D_{K-1}-c)+\beta\eta\beta\zeta D_{K-1}+\beta\zeta\beta\zeta C] - [\beta\eta Z_{K-1}+\beta\varepsilon]\,A_{K-1}'f^0 - \beta\zeta B' f^1}{\beta\lambda - [\beta\eta Z_{K-1}+\beta\varepsilon]\,A_{K-1}'g^0 - \beta\zeta B' g^1};$$

end {if};

$$\nu_{(1,K)} := \frac{[\beta(1-\mu)\beta\zeta C + \beta\mu\beta(\mu-\eta)D_{K-1}] - \beta\mu A_{K-1}'f^0 - \beta(1-\mu)B'f^1}{\beta\lambda - \beta\mu A_{K-1}'g^0 - \beta(1-\mu)B'g^1};$$

end {for};
{Termination}
$A_I := \beta\zeta/[1-\beta+\beta\zeta+\beta\eta(1-A_{I-1})]; \quad A_I' := \beta\zeta+\beta(\mu-\eta)A_I; \quad D_I := (c+\beta\eta D_{I-1})\,A_I/(\beta\zeta); \quad Z_I := A_I A_{I-1}'/A_I';$

$$f^0 := -\frac{\frac{\beta\zeta}{A_I}D_I - \beta(\mu-\eta)\beta\mu D_{I-1}}{\frac{A_I'}{A_I}+\beta\mu A_{I-1}'}; \quad g^0 := \frac{\lambda(1+\beta\mu)}{\frac{A_I'}{A_I}+\beta\mu A_{I-1}'};$$

$$\nu_{(0,I-1)} := \frac{[\beta(\mu-\eta)(D_{I-1}-c)+\beta\eta\beta\zeta D_{I-1}] - [\beta\eta Z_{I-1}+\beta\varepsilon]\,A_{I-1}'f^0}{\beta(1-\zeta)\lambda - [\beta\eta Z_{I-1}+\beta\varepsilon]\,A_{I-1}'g^0};$$

$$\nu_{(*,I)} := \frac{[\beta(\mu-\eta)+\beta\eta Z_I]\,D_{I-1}+cZ_I}{\lambda(1-Z_I)};$$

# Optimal Bi-Threshold Policy

- Can be obtained from MPIs (nondecreasing in $i$)
  - ▷ the optimal open-gate threshold is

$$K_0 := \min\{i \in \mathcal{I} : \nu_{(0,i)} \geq \nu\}$$

  - ▷ the optimal closed-gate threshold is

$$K_1 := \min\{i \in \mathcal{I} : \nu_{(1,i)} \geq \nu\}$$

  - ▷ if $\nu > \nu_{(*,I)}$, then the gate is open always
- FI can be used also for infinite buffer (never stops)
- FI also works under the time-average criterion ($\beta = 1$)

# MPI Properties

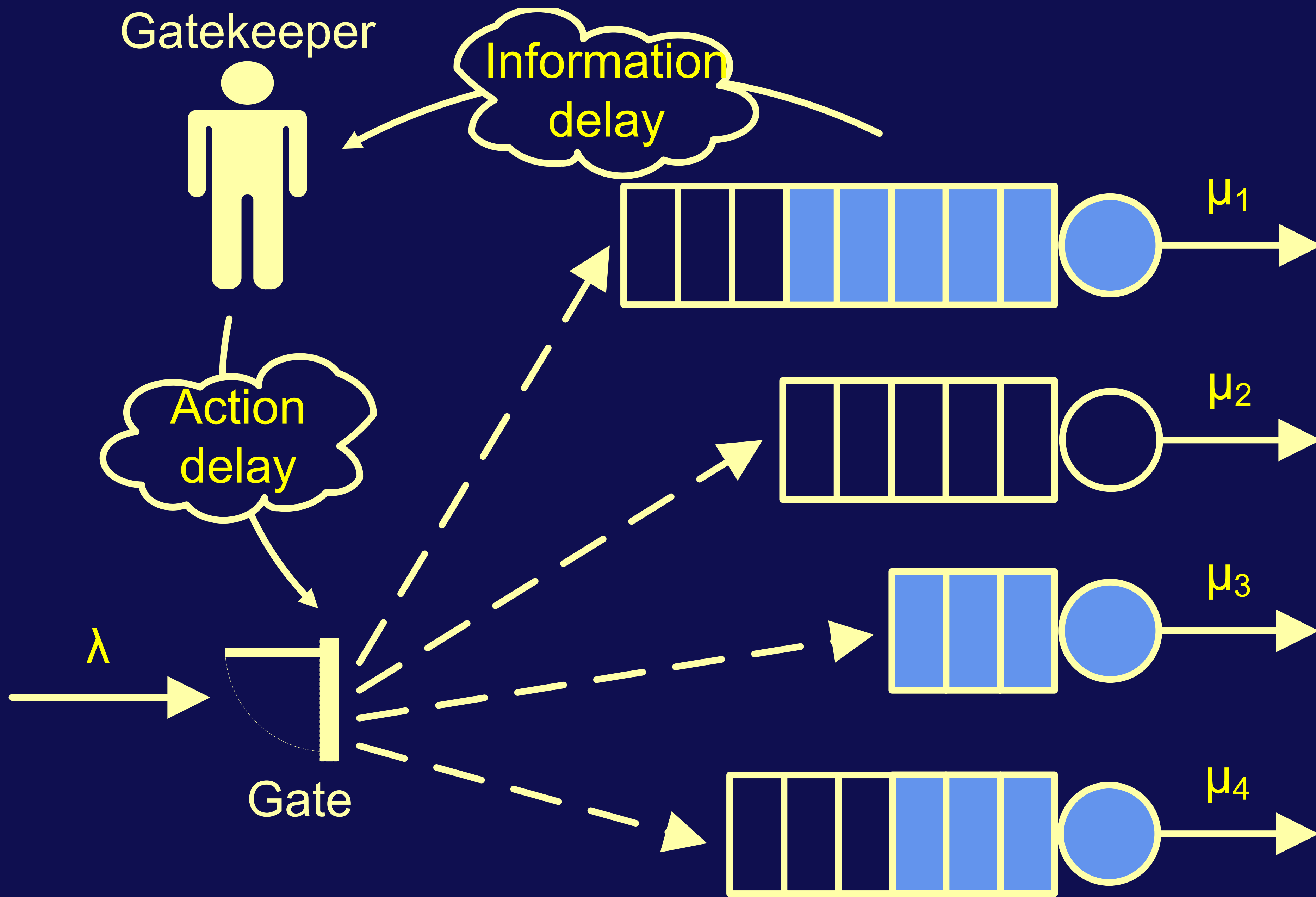
- Both $\nu_{(0,i)}$ and $\nu_{(1,i)}$ are nondecreasing in $i$, nondecreasing in $\lambda$, nonincreasing in $\mu$

- Interleaving values:
  - ▷ $\nu_{(0,i)} \leq \nu_{(1,i+1)} \leq \nu_{(0,i+1)}$
  - ▷ $\nu_{(1,i)} \leq \nu_{(0,i)} \leq \nu_{(1,i+1)}$

- Convergence
  - ▷ $\nu_{(1,i)} \longrightarrow \nu_{(0,i)}$ as $\lambda \longrightarrow 0$
  - ▷ $\nu_{(1,i)} \longrightarrow \nu_{(0,i-1)}$ as $\lambda(1-\mu) \longrightarrow 1$
  - ▷ $\nu_{(0,i)}, \nu_{(1,i)} \longrightarrow \beta c/(1-\beta)$ as $i \longrightarrow \infty$

- $\lambda\nu_{(1,0)} =$ the expected total discounted holding cost

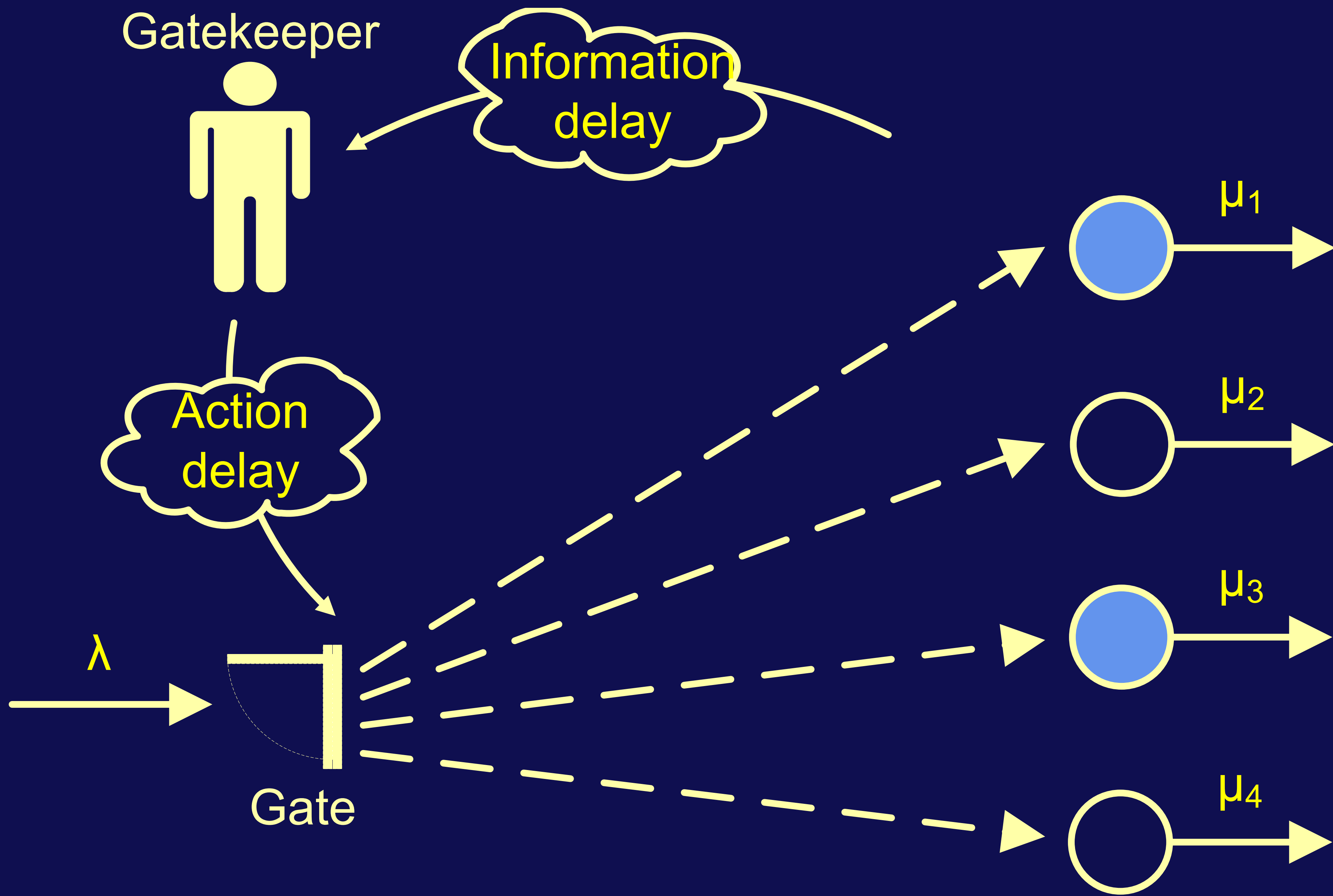
# Admission Control and Routing with Delay



# Admission Control and Routing with Delay

- MPI policy for $K$ queues:
  - ▷ Admit an arriving job iff $\nu > \nu_{\widetilde{X}_k(t)}$ for at least one queue $k$
  - ▷ If admitted, route to the queue with lowest MPI
- By MPI properties, a job is routed to a queue with
  - ▷ less waiting jobs
  - ▷ faster server
  - ▷ no job admitted in the previous period
  - ▷ lower holding costs
- JSEQ is recovered in case of two symmetric queues

# Servers Assignment Problem with Delay



# Servers Assignment Problem with Delay

- The MPI is $\nu_{(a,i)} = \frac{c\beta(1-\mu)}{1-\beta(1-\mu)}$
  - ▷ equal for all augmented states
  - ▷ equal to the MPI with no delay
  - ▷ equal under any arrival rate $\lambda$
- By MPI properties, a job is routed to a queue with
  - ▷ faster server
  - ▷ lower holding costs
- Jobs are routed always to the same queue

# Why MPI Policy is not Optimal?

- MPI policy is, in general, not optimal due to cross-dependence
  - ▷ we do not know after-routing arrival rates for each queue; moreover, they may be time-varying
  - ▷ computation of MPIs implicitly assumes that the threshold policy is the same in all periods
- MPI policy may be optimal in certain instances
- Mean behavior is nearly-optimal

# Summary of MPI Approach

- No news:
  - ▷ analysis of problems with delays is hard
- Good news:
  - ▷ yields tractable heuristics in heterogeneous problems
  - ▷ powerful to obtain an exact algorithm of the same complexity as in the no-delay case
  - ▷ some general patterns are extensible to other problems
  - ▷ promising for larger delays

# Thank you for your attention!