On the spatial dependence of extreme ocean storm seas

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Context

- Rational and consistent design and assessment of marine structures
  - Reduce bias and uncertainty in estimation of structural integrity
  - Quantify uncertainty as well as possible
- Non-stationary marginal, conditional and spatial extremes
  - Multiple locations, multiple variables, time-series
  - Multidimensional covariates
- Rational quantification of uncertainty
  - Data acquisition (simulator or measurement)
  - Data pre-processing (storm peak identification)
  - Threshold uncertainty
  - Model form (marginal measurement scale effect, spatial extremal dependence)
- Improved understanding and communication of risk
  - Incorporation within established engineering design practices
  - Knock-on effects of improved inference
Motivation: North Sea application

Storm peak $H_S$ from gridded NEXTRA winter storm hindcast for North Sea locations; directional variability in storm severity; “strips” of locations with different orientations; central location for directional model
Modelling extremal spatial dependence: why bother?

- Improved inference for the characteristics of extremes at one location exploiting data from multiple locations in a spatial neighbourhood.
- Improved estimation of risk for spatially-distributed structures (coastal defences, multiple installations) from spatially spread storm events.
- Can we estimate spatial extremes models usefully from typical metocean hindcast data?
- Can we see evidence for covariate effects in extremal spatial dependence for ocean storm severity?
Modelling extremal spatial dependence: mathematically

- Locations $j = 1, 2, \ldots, p$, continuous random variables and values $\{X_j\}, \{x_j\}$
- Spatial distribution of storm peak $H_S$

$$f(x_1, x_2, \ldots, x_p) = [f(x_1)f(x_2)\ldots f(x_p)]C(x_1, x_2, \ldots, x_p)$$

- $\{f(x_j)\}$ are marginal densities, $C(x_1, x_2, \ldots, x_p)$ is dependence “copula”
- Interested in estimating things like “the shape of an extreme storm”

$$f(x_1, x_2, \ldots, x_p|X_k = x_k > u_k) \text{ for large } u_k$$

- We know how to estimate extremes marginally, but what about extremal dependence?
- ⇒ study spatial extremes, i.e. sensible models for $C(x_1, x_2, \ldots, x_p)$
Modelling extremal spatial dependence: procedure

- Sample of peaks over threshold \( \{y\} \) at \( p \) locations, with covariates \( \{\theta\} \)
- Simple marginal gamma-GP model
- Sample transformed ("whitened") to standard Frechet scale per location
- Spatial extremes ("max-stable model") to estimate extremal spatial dependence
- Bayesian inference estimating joint distributions of parameters, uncertainties
Marginal: isolating storm peaks

$H_S \approx 4 \times \text{standard deviation of ocean surface time-series at specific location corresponding to a specified period (typically three hours)}$
Marginal: gamma-generalised Pareto

- Simple marginal gamma-GP model fitted using Bayesian inference
- GP $\xi$, $\sigma$, gamma $\alpha$, $\beta$, and threshold $\psi$ all functions of $\theta$
- Spline parameterisation for model parameters in terms of $\theta$
- $\psi$ for pre-specified threshold probability $\tau$

- Gibbs sampling when full conditionals available
- Otherwise Metropolis-Hastings (MH) within Gibbs, using suitable proposal mechanisms

- Sample of joint posterior of $\{\xi_\theta, \sigma_\theta, \alpha_\theta, \beta_\theta, \psi_\theta\}$ estimated

- Ross et al. [2017b], Frigessi et al. [2002], Behrens et al. [2004], MacDonald et al. [2011]
Marginal: transformation to standard Fréchet scale

Storm peak $H_S$ on direction for central location before and after standardisation to Fréchet scale
Extremes basics: marginal

- Block maxima $Y_k$ at location $k$ have distribution $F_{Y_k}$ which is “max-stable” in the sense that $F_{Y_k}^n(b'_{kn} + a'_{kn}y_k) = F_{Y_k}(y_k)$ for some sequences $\{a'_{kn} > 0\}$ and $\{b'_{kn}\}$

- Only limiting distribution for $F_{Y_k}$ is generalised extreme value (GEV)

$$F_{Y_k}(y_k) = \exp[-\exp\{(y_k - \eta)/\tau\}] \text{ for } \xi = 0$$

$$= \exp[-\{1 + \xi(y_k - \eta)/\tau\}_{+}^{-1/\xi}] \text{ otherwise}$$
Extremes basics: spatial

- Similarly, $F_Y$ for block maxima $Y$ at $p$ locations “max-stable” when 
  \[ F^n_Y(b'_1n + a'_1ny_1, b'_2n + a'_2ny_2, \ldots, b'_pn + a'_pny_p) = F_Y(y_1, y_2, \ldots, y_p) \]

- Transform to unit Fréchet $Z_k = \{1 + \xi(Y_k - \eta)/\tau\}^{1/\xi}$, $F_{Z_k}(z_k) = \exp(-1/z_k)$, for $z_k > 0$. Then
  \[ F_Z(z_1, z_2, \ldots, z_p) = F_Z(nz_1, nz_2, \ldots, nz_p)^n \]

- Only choices of $F_Z$ exhibiting this “homogeneity” correspond to finite-dimensional distributions from max-stable processes (MSPs), and are hence valid for spatial extreme value modelling.
Spatial: basic theory

- Max-stable process (MSP): a means of extending the GEV for modelling maxima at one location, to multivariate extreme value distributions for modelling of component-wise maxima observed on a lattice.
- On unit Fréchet scale, only choices of \( F_Z \) exhibiting “homogeneity” are valid for spatial extreme value modelling.
- Convenience: “exponent measure” \( V_Z \)
  \[
  F_Z(z_1, z_2, \ldots, z_p) = \exp\{ -V_Z(z_1, z_2, \ldots, z_p) \}
  \]
- Convenience: “extremal coefficient” \( \theta_p \)
  \[
  F_Z(z, z, \ldots, z) = \exp(-V_Z(z, z, \ldots, z)) \\
  = \exp\left(-z^{-1}V_Z(1, 1, \ldots, 1)\right) \text{ from the homogeneity property} \\
  = \exp\left(-\theta_p/z\right)
  \]
Spatial : data

Fréchet scale observations of the spatial distribution of storm peak $H_S$ over the North Sea spatial grid for 8 typical events (a)-(h). The spatial maximum for each event is given as a white disc, and the spatial minimum as a black disc (with white outline). The white → yellow → red → black colour scheme indicates the spatial variation of relative magnitude of storm peak $H_S$.
Spatial : data

Fréchet scale scatter plots of storm peak $H_S$ for different pairs of locations. Panel (a) for the central location and its nearest neighbour to the West along the approximate West-East transect with angle $\phi = 4.6$; panel (b) for the end locations of the same transect. Panel (c) for the central location and its nearest neighbour to the North along the approximate North-South transect with angle $\phi = -72.2$; panel (d) for the end locations of the same transect.
Spatial : $V_Z$ for Smith, Schlather and Brown-Resnick processes

- **Smith**: For two locations $s_k, s_l$ in $\mathcal{S}$, $V_{kl}$ for Smith process given by

  $$V_{kl}(z_k, z_l; h(\Sigma)) = \frac{1}{z_k} \Phi\left(\frac{m(h)}{2} + \frac{\log(z_l/z_k)}{m(h)}\right) + \frac{1}{z_l} \Phi\left(\frac{m(h)}{2} + \frac{\log(z_k/z_l)}{m(h)}\right)$$

- $h = s_l - s_k$, $m(h)$ is Mahalanobis distance $(h'\Sigma^{-1}h)^{1/2}$ between $s_k$ and $s_l$
- $\Sigma$ is $2 \times 2$ covariance matrix (2-D space) to be estimated. $\Sigma$ scalar in 1-D
- $V_{kl}(1, 1; h(\Sigma)) = 2\Phi(m(h)/2)$ by construction

- **Schlather** similar likelihood, parameterised in terms of $\Sigma$ only

- **Brown-Resnick** identical likelihood, parameterised in terms of $\Sigma$ and scalar Hurst parameter $H$ (estimated up front)
Spatial : Constructive representation

- MSP is maximum of multiple copies \( \{W_i\} \) \((i \geq 1)\) of random function \( W \)
- Each \( W_i \) weighted using Poisson process \( \{\rho_i\} \) \((i \geq 1)\).
- The MSP \( Z(s) \) for \( s \) in spatial domain \( S \) is \( Z(s) = \mu^{-1} \max_i \{W_i^+(s)/\rho_i\} \)
- \( W_i^+ = \max\{W_i(s), 0\} \)
- \( \mu = E(W^+(s)) = 1 \) by construction typically
- \( \rho_i = \epsilon_i \) for \((i = 1)\), \( \rho_i = \epsilon_i + \rho_{i-1} \) for \((i > 1)\), and \( \epsilon_i \sim \text{Exp}(1) \)
- Different choices of \( W(s) \) give different MSPs.

- **Smith** : \( W_i(s; s_i, \Sigma) = \varphi(s - s_i; \Sigma)/f_S(s_i) \), with \( s_i \) sampled from density \( f_S(s_i) \) on \( S \), with \( \varphi \) representing standard Gaussian density
- **Schlather, Brown-Resnick** : Similar
Spatial : illustrations

Illustrative realisations of Smith (a,e), Schlather (b,f), and Brown-Resnick (c,d,g,h) processes for different parameter choices. The first row corresponds to parameter settings \((\Sigma_{11}, \Sigma_{22}, \Sigma_{12}) = (300, 300, 0)\) for all processes, and the second row to \((30, 20, 15)\). For Brown-Resnick processes (c,g), Hurst parameter \(H = 0.95\). For Brown-Resnick processes (d,h), \(H = 0.65\). Each panel can be considered to show a possible spatial realisation of storm peak \(H_5\), similar to those shown earlier.
Theory gives us models for pairs of locations. Cannot write down full joint likelihood $\ell(\Sigma; \{y_j\})$. Approximate with “composite” likelihood $\ell_C(\Sigma; \{y_j\})$

$$\ell(\Sigma; \{y_j\}) \approx \ell_C(\Sigma; \{y_j\}) = \sum_{\{k,l\} \in \mathcal{N}} w_{kl} \log f_{kl}(y_k, y_l; h(\Sigma))$$

Theory applies for block maxima $Z$, but we have peaks over threshold $Y$. For $y_k, y_l > u$ for large $u$, approximate

$$\Pr [Y_k \leq y_k, Y_l] \approx \Pr [Z_k \leq y_k, Z_l]$$

Need $f_{kl}(y_k, y_l; h(\Sigma))$ for non-exceedances of $u$ also, so make “censored” likelihood approximation
Spatial: estimation

- Estimate joint distribution of $\Omega = [\Sigma_{11}, \Sigma_{22}, \Sigma_{12}]$ (2-D space, or $\Omega = \Sigma$ in 1-D)
- MCMC using Metropolis-Hastings
  - Current state $\Omega_{r-1}$, marginal posterior $f_M(\beta_M)$, original sample $D$ of storm peak $H_S$.
  - Draw a set of marginal parameters $\beta_{Mr}$ from $f_M$, independently per location.
  - Use $\beta_{Mr}$ to transform $D$ to standard Fréchet scale, independently per location, obtaining sample $D_{Fr}$.
  - Execute “adaptive” MCMC step from state $\Sigma_{r-1}$ with sample $D_{Fr}$ as input, obtain $\Sigma_r$.
- Adaptive MCMC candidates generated using $\Omega^c_r = \Omega_{r-1} + \gamma \epsilon_1 + (1 - \gamma) \epsilon_2$
  - $\gamma \in [0, 1]$, $\epsilon_1 \sim N(0, \delta^2_1 I_3/3)$, $\epsilon_2 \sim N(0, \delta^2_2 S_{\Omega_{r-1}}/3)$
  - $S_{\Omega_{r-1}}$ estimate of variance of $\Omega_{r-1}$ using samples to trajectory to date
  - Roberts and Rosenthal [2009]
Spatial : $\hat{\sigma}(\phi)$ for Smith

For all transects with a given orientation $\phi$ estimated using 1-D (box-whisker) and 2-D (black) Smith processes. $\phi$ is quantified as the transect angle anticlockwise from a line of constant latitude. The first (second) row: marginal threshold non-exceedance probability 0.5 (0.8). The first (second) column: censoring threshold non-exceedance probability 0.5 (0.8). For 1-D estimates with a given $\phi$, box centres = median, box edges = 0.25 and 0.75 quantiles across all parallel transects; whisker edges = 0.025 and 0.975 quantiles. For 2-D estimates, the 0.025, 0.5 and 0.975 quantiles are shown as a function of $\phi$. Note that the colour coding of box-whisker plots corresponds to that of transect orientation.
Spatial: $\hat{\sigma}(\phi)$ for Schlather

Estimated extremal spatial dependence parameter $\hat{\sigma}(\phi)$ for all transects with a given orientation $\phi$ estimated using 1-D (box-whisker) and 2-D (black) Schlather processes. $\phi$ is quantified as the transect angle anticlockwise from a line of constant latitude. The first (second) row corresponds to a choice of marginal threshold with non-exceedance probability 0.5 (0.8). The first (second) column corresponds to a choice of censoring threshold with non-exceedance probability 0.5 (0.8).
Spatial : $\hat{\sigma}(\phi)$ for Brown-Resnick

Estimated extremal spatial dependence parameter $\hat{\sigma}(\phi)$ for all transects with a given orientation $\phi$ estimated using 1-D (box-whisker) and 2-D (black) Brown-Resnick processes with $H = 0.75$. $\phi$ is quantified as the transect angle anticlockwise from a line of constant latitude. The first (second) row corresponds to a choice of marginal threshold with non-exceedance probability 0.5 (0.8). The first (second) column corresponds to a choice of censoring threshold with non-exceedance probability 0.5 (0.8).
Spatial : extremal coefficient $\hat{\theta}(\phi)$

Estimated extremal coefficient $\hat{\theta}(\phi)$ for all transects with a given orientation $\phi$, estimated using 1-D Smith (black), Schlather (dark grey) and Brown-Resnick (light grey) processes. The first (second) row corresponds to marginal threshold with non-exceedance probability 0.5 (0.8). The first (second) column = censoring threshold with non-exceedance probability 0.5 (0.8).
Spatial : spatial dependence parameter $\hat{\sigma}(\phi, s)$ for individual transects

Smith process with marginal and censoring thresholds = non-exceedance probability of 0.8. (b)-(g): $\hat{\sigma}(\phi, s)$ for fixed orientation $\phi$ (given in the panel title) as a function of transect locator $s$. (a): transects with $s = 1$ for different orientations $\phi$. (b)-(g): abscissa values for transect locators are scaled to physical perpendicular distances between parallel transects.
Discussion

- Possible to estimate reasonable spatial extremes models for typical samples of hindcast data
- Consistent inferences from Smith, Schlather and Brown-Resnick models
- Evidence for directional and spatial anisotropy

- Only investigated “asymptotically dependent” models here, but see Kereszturi et al. [2016]
- Did not perform simultaneous marginal and dependence inference
- Essential that marginal modelling performed thoughtfully

- Fetch effects may be visible
- Other locations, basins
References


