TIGHTNESS AND CONTINUITY OF A FAMILY OF INVARIANT MEASURES FOR MARKOV CHAINS DEPENDING ON A PARAMETER

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§ 1. Introduction

Let S be a metric space with metric ρ and σ-algebra $\mathscr{B}(S)$ of Borel sets. We consider a family of time-homogeneous Markov chains $\{X_n^{(\delta)}\}_{n=0}^{\infty}$ (indexed by δ, $\delta \in \mathbf{R}$) taking values in S. We denote by $P^{(\delta)}(x, B)$ the transition probability for the chain $\{X_n^{(\delta)}\}$, i.e.,

$$P^{(\delta)}(x, B) \equiv \mathbf{P}\{X_{n+1}^{(\delta)} \in B \mid X_n^{(\delta)} = x\}, \quad x \in S,\ B \in \mathscr{B}(S).$$

We assume that for every $\delta \neq 0$ each of the chains $\{X_n^{(\delta)}\}$ has at least one invariant measure $\pi^{(\delta)}$, i.e., a measure $\pi^{(\delta)}$ satisfying the equation

$$\pi^{(\delta)}(\cdot) = \int_S P^{(\delta)}(x, \cdot)\pi^{(\delta)}(dx), \quad \pi^{(\delta)}(S) = 1.$$

We shall assume as well that the chain $\{X_n^{(0)}\}$ admits at most one invariant measure. A rather broad conditions for the existence of an invariant measure can be found in [1]. Note that the chains $\{X_n^{(\delta)}\}$ are not supposed irreducible.

In § 2 we give some conditions under which the invariant measure $\pi^{(\delta)}$ depends continuously (in the sense of weak convergence) on the parameter as $\delta \to 0$. This problem was studied in various degree of generality [2, 3; 4, Chapter 8, § 3] (see also the references therein). In § 2 it is also shown how the problem of continuous dependence for invariant measures can sometimes be reduced to the problem of tightness for this family of invariant measures. So, in § 3–§ 6 we clarify conditions for tightness of the family $\{\pi^{(\delta)}\}$, i.e., conditions under which for every positive ε there exists a compact $K \subset S$ such that $\pi^{(\delta)}(K) \geq 1 - \varepsilon$ for each δ.

One of the most convenient methods for proving tightness for a family of invariant measures is (from our point of view) the method of test (Lyapunov) functions. In this connection, we make some remarks about test functions.

Let $\{X_n\}$ be a Markov chain taking values in $\mathbf{R}^+$. Denote by ξ_x the random variable whose distribution corresponds to the distribution of the increment of the chain $\{X_n\}$ at state x, i.e., $\mathbf{P}\{x + \xi_x \in B\} = P(x, B)$, $B \in \mathscr{B}(\mathbf{R}^+)$. We usually (for example, in studying ergodicity conditions for a Markov chain) want to find a nonnegative measurable function $V : \mathbf{R}^+ \to \mathbf{R}^+$ such that, for some positive N and ε, the inequality

$$m(x) \equiv \mathbf{E}\{V(x + \xi_x) - V(x)\} \leq -\varepsilon < 0 \tag{1.1}$$

holds if $x \geq N$, and

$$\sup_{x \geq 0} m(x) < \infty. \tag{1.2}$$

We introduce the following notations for the iterated logarithms:

$$\log^{(0)} x \equiv 1, \quad \log^{(1)} x \equiv \log x, \quad \log^{(k+1)} x \equiv \log(\log^{(k)} x), \quad k \geq 1, \tag{1.3}$$

and their products:

$$L^{(k)}(x) \equiv \prod_{i=0}^{k} \log^{(i)} x, \quad k \geq 0. \tag{1.4}$$

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It turns out that, in studying ergodicity conditions for Markov chains, the most interesting test functions are functions increasing not faster than $x^2 L^{(k)}(x)$ for some $k \geq 0$ (see, for instance, [5–7]; some prompting arguments can be found in [8]). We illustrate this by the following example.

Consider the test function $V(x) = x^3$. Suppose that the increments of the chain $\{X_n\}$ are asymptotically nonvanishing; more precisely, suppose that $\underline{\lim}_{x\to\infty} \mathbf{E}(\xi_x)^2 = b > 0$. Assume that the mean drift of the test function V at x is negative for x large enough:

$$\mathbf{E}\{V(x+\xi_x) - V(x)\} = 3x^2\mathbf{E}\xi_x + 3x\mathbf{E}(\xi_x)^2 + \mathbf{E}(\xi_x)^3 < 0, \quad x > N.$$

Since $\xi_x \geq -x$; therefore, $\mathbf{E}(\xi_x)^3 \geq -x\mathbf{E}(\xi_x)^2$, and consequently for $x > N$

$$3x^2\mathbf{E}\xi_x + 2x\mathbf{E}(\xi_x)^2 \leq 0.$$

Hence,

$$2x\mathbf{E}\xi_x + \mathbf{E}(\xi_x)^2 \leq -\mathbf{E}(\xi_x)^2/3,$$

and, in view of the assumption $\underline{\lim}_{x\to\infty} \mathbf{E}(\xi_x)^2 = b > 0$, we have

$$\mathbf{E}\{(x+\xi_x)^2 - x^2\} = 2x\mathbf{E}\xi_x + \mathbf{E}(\xi_x)^2 \leq -b/6 < 0$$

for x large enough. Thus, the mean drift of the test function x^2 is also strictly negative for x large enough.

The example shows us that, roughly speaking, there is no necessity for considering the test function x^3; we may consider the more slowly increasing test function x^2. The same arguments can be repeated for every function $x^2 L^{(k)}(x)(\log^{(k+1)} x)^2$. It gives us grounds to restrict ourselves to studying test functions increasing not faster than $x^2 L^{(k)}(x)$ for some $k \geq 0$.

Suppose that (1.1) and (1.2) hold for every chain $\{X_n^{(\delta)}\}$. We are interested in the following question: which additional condition will provide tightness for the family of invariant measures $\{\pi^{(\delta)}\}$? In [3; 4, Chapter 8, § 4] it is assumed, in particular, that, for some positive γ,

$$\sup_{x\geq 0, \delta\in\mathbf{R}} \mathbf{E}\big|V\big(x+\xi_x^{(\delta)}\big) - V(x)\big|^{1+\gamma} < \infty. \tag{1.5}$$

This condition is rather natural if the function $V(x)$ satisfies the Lipschitz condition, i.e.,

$$\sup_{x,y>0} \frac{|V(x+y) - V(x)|}{y} < \infty.$$

Observe, however, that this condition is often violated. For instance, it does not hold for the functions $x^{1+\alpha}$, $\alpha > 0$. In general, (1.5) fails for such functions. For example, if the test function $V(x)$ is x^2, then

$$V(x+\xi_x) - V(x) = 2x\xi_x + (\xi_x)^2 \approx 2x\xi_x$$

for x large enough. It is easy to see that usually the family of random variables $\{x\xi_x, x \geq 0\}$ is not uniformly integrable.

In this connection, it is more natural to assume the uniform integrability of the family $\big\{V\big(\big|\xi_x^{(\delta)}\big|\big), x \geq 0, \delta \in \mathbf{R}\big\}$ instead of (1.5). It is the the approach that we take in § 3–§ 6.

§ 2. A Criterion for Continuity of Invariant Measures for Markov Chains on a Metric Space

Let $\big\{X_n^{(\delta)}\big\}_{n=0}^{\infty}$ be a Markov chain taking values in a metric space (S, ρ) with the transition probabilities depending on the parameter $\delta \in \mathbf{R}$. The metric space S is assumed neither separable nor complete. We suppose that for every $\delta \neq 0$ the chain $\big\{X_n^{(\delta)}\big\}$ has at least one invariant measure $\pi^{(\delta)}$, whereas the chain $\big\{X_n^{(0)}\big\}$ has at most one invariant measure. The following criterion for continuity (in the sense of weak convergence) of distribution $\pi^{(\delta)}$ as $\delta \to 0$ is valid:

Theorem 1. *Suppose that the transition probabilities $P^{(\delta)}(x,\cdot)$ depend continuously on the parameters x and δ, $\delta \to 0$, i.e., for every $x \in S$ the following weak convergence holds:*

$$P^{(\delta)}(y,\cdot) \Rightarrow P^{(0)}(x,\cdot) \quad \text{as } y \to x \text{ and } \delta \to 0. \tag{2.1}$$

Let the family of measures $\{\pi^{(\delta)}, \delta \neq 0\}$ on S be tight. Then there exists an invariant measure $\pi^{(0)}$ for the chain $\{X_n^{(0)}\}$, and the weak convergence $\pi^{(\delta)} \Rightarrow \pi^{(0)}$ takes place as $\delta \to 0$.

The following lemma will be useful below. It gives a condition equivalent to (2.1). Denote by $C(S)$ the class of bounded continuous real-valued functions on S.

Lemma 1. *Condition* (2.1) *is satisfied if and only if the following two conditions hold*:
(a) *For each function $f \in C(S)$ and a compact set $K \subset S$*

$$\int\limits_S f(z)P^{(\delta)}(x,dz) \to \int\limits_S f(z)P^{(0)}(x,dz) \tag{2.2}$$

as $\delta \to 0$ uniformly in $x \in K$, and
(b) *The transition probability $P^{(0)}(x,\cdot)$ is weak Feller* (*see, for example,* [1]), *i.e., for each $x \in S$*

$$P^{(0)}(y,\cdot) \Rightarrow P^{(0)}(x,\cdot) \quad \text{as } y \to x. \tag{2.3}$$

PROOF. Suppose that (2.1) holds whereas (2.2) fails. Then there exists a function $f \in C(S)$, a subsequence δ_k, $\delta_k \to 0$ as $k \to \infty$, and a sequence of elements $x^{(\delta_k)} \in K$ such that, for each k, the inequality

$$F^{(\delta_k)} \equiv \int_S f(z)[P^{(\delta_k)}(x^{(\delta_k)},dz) - P^{(0)}(x^{(\delta_k)},dz)] \geq \varepsilon \tag{2.4}$$

holds for some positive ε. Since the set K is compact, without loss of generality we can assume that the sequence of the elements $x^{(\delta_k)}$ has a limit in K: $x^{(\delta_k)} \to x_0$, $x_0 \in K$, $\delta_k \to 0$. The following estimate is valid:

$$F^{(\delta_k)} \leq \left| \int_S f(z)[P^{(\delta_k)}(x^{(\delta_k)},dz) - P^{(0)}(x_0,dz)] \right| + \left| \int_S f(z)[P^{(0)}(x_0,dz) - P^{(0)}(x^{(\delta_k)},dz)] \right|.$$

By (2.1), each term on the right-hand side of the inequality tends to zero as $\delta_k \to 0$. This contradicts (2.4).

Now, let (2.2) and (2.3) be satisfied. Prove that for every element $x \in S$ and every sequence $y_n \in S$ tending to x,

$$\rho(y_n,x) \to 0 \quad \text{as } n \to \infty, \tag{2.5}$$

the following weak convergence takes place:

$$P^{(\delta)}(y_n,\cdot) \Rightarrow P^{(0)}(x,\cdot) \quad \text{as } n \to \infty \text{ and } \delta \to 0. \tag{2.6}$$

Assume $f \in C(S)$. By (2.5) and (2.3), the following convergence takes place:

$$\int\limits_S f(z)P^{(0)}(y_n,dz) \to \int\limits_S f(z)P^{(0)}(x,dz) \quad \text{as } n \to \infty. \tag{2.7}$$

Since, in view of (2.5), the set $\{x, y_n, n = 1,2,\dots\} \subset S$ is compact; therefore, by condition (2.2),

$$\int\limits_S f(z)P^{(\delta)}(y_n,dz) \to \int\limits_S f(z)P^{(0)}(y_n,dz) \tag{2.8}$$

as $\delta \to 0$ uniformly in n. It follows from (2.7) and (2.8) that

$$\int_S f(z) P^{(\delta)}(y_n, dz) \to \int_S f(z) P^{(0)}(x, dz)$$

as $n \to \infty$ and $\delta \to 0$. So, the (2.6) is proved.

Proof of Theorem 1. Since the family of invariant measures $\{\pi^{(\delta)}, \delta \neq 0\}$ is tight, by Prochorov's Theorem (see, for example, [9, p. 37]) the family is relatively compact. In particular, the family of the limit points (in a sense of weak convergence) is nonempty. Let $\tilde{\pi}$ be an arbitrary limit point, i.e.,

$$\pi^{(\delta_k)} \Rightarrow \tilde{\pi} \tag{2.9}$$

for some subsequence $\delta_k \to 0$, where $\tilde{\pi}(S) = 1$. Prove that $\tilde{\pi}$ is an invariant measure for the chain $\{X_n^{(0)}\}$.

Take $f \in C(S)$. Denote $\psi^{(\delta)}(x) \equiv \mathbf{E}\{f(X_1^{(\delta)}) \mid X_0^{(\delta)} = x\}$. From (2.1) it follows, in particular, that

$$\psi^{(0)}(x) \in C(S). \tag{2.10}$$

If the chain $\{X_n^{(\delta)}\}$ is in equilibrium (i.e., if $X_n^{(\delta)}$ has the distribution $\pi^{(\delta)}$ for every n), then the equality $\mathbf{E} f(X_1^{(\delta)}) = \mathbf{E} f(X_0^{(\delta)})$ holds. Taking the conditional expectation, we obtain

$$\int_S f(x) \pi^{(\delta)}(dx) = \mathbf{E} f(X_1^{(\delta)}) = \mathbf{E}\{\mathbf{E}\{f(X_1^{(\delta)}) \mid X_0^{(\delta)}\}\} = \int_S \psi^{(\delta)}(x) \pi^{(\delta)}(dx)$$

for every $\delta \neq 0$. Let K be an arbitrary compact set in S. Using the last equality, we deduce the following estimate:

$$\begin{aligned}
&\left| \int_S f(x) \tilde{\pi}(dx) - \int_S \psi^{(0)}(x) \tilde{\pi}(dx) \right| \\
&\qquad = \left| \int_S f(x)[\tilde{\pi}(dx) - \pi^{(\delta_k)}(dx)] - \left[\int_S \psi^{(0)}(x) \tilde{\pi}(dx) - \int_S \psi^{(\delta_k)}(x) \pi^{(\delta_k)}(dx) \right] \right| \\
&\qquad \le \left| \int_S f(x)[\tilde{\pi}(dx) - \pi^{(\delta_k)}(dx)] \right| + \left| \int_K [\psi^{(\delta_k)}(x) - \psi^{(0)}(x)] \pi^{(\delta_k)}(dx) \right| \\
&\qquad + \left| \int_{S \setminus K} [\psi^{(\delta_k)}(x) - \psi^{(0)}(x)] \pi^{(\delta_k)}(dx) \right| + \left| \int_S \psi^{(0)}(x)[\pi^{(\delta_k)}(dx) - \tilde{\pi}(dx)] \right| \equiv I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

We now estimate each of the summands. We have $I_1 \to 0$ as $\delta_k \to 0$, in view of weak convergence (2.9). Furthermore, $I_2 \to 0$ as $\delta_k \to 0$ for each fixed compact set K, because of Lemma 1. Given positive ε, there exists a compact set $K \subset S$ such that $|I_3| \le \varepsilon$ for each $\delta \neq 0$, in view of tightness of the family of measures $\{\pi^{(\delta)}, \delta \neq 0\}$ and boundedness of the function $\psi^{(\delta)}(x)$:

$$\psi^{(\delta)}(x) \le \sup_{x \in S} |f(x)| < \infty.$$

Finally, $I_4 \to 0$ as $\delta_k \to 0$, in view of weak convergence (2.9) and boundedness and continuity of the function $\psi^{(0)}(x)$ (see (2.10)).

Thus, combining the estimates, we obtain the equality

$$\int_S f(x) \tilde{\pi}(dx) = \int_S \psi^{(0)}(x) \tilde{\pi}(dx)$$

for each bounded continuous function $f(x)$. Therefore, $\mathbf{E}f\big(X_0^{(0)}\big) = \mathbf{E}f\big(X_1^{(0)}\big)$ if the random element $X_0^{(0)}$ has the distribution $\tilde{\pi}$. This implies (see, for example, [9, p. 9]), that the random elements $X_0^{(0)}$ and $X_1^{(0)}$ have the same distribution, i.e., $\tilde{\pi}$ is indeed an invariant measure for the chain $\big\{X_n^{(0)}\big\}$. So, the chain $\big\{X_n^{(0)}\big\}$ admits an invariant measure $\pi^{(0)}$, $\pi^{(0)} \equiv \tilde{\pi}$. By the hypothesis of the theorem, the chain $\big\{X_n^{(0)}\big\}$ admits at most one invariant measure. Hence, each limit point of the family of measures $\{\pi^{(\delta)}, \delta \neq 0\}$ is equal to $\pi^{(0)}$, which completes the proof.

The situation sometimes arises when the Markov chain $\big\{X_n^{(\delta)}\big\}$ takes values, in some subset $S^{(\delta)} \subset S$ rather than in the whole S; more precisely, the transition probabilities $P^{(\delta)}(x,\cdot)$ of the chain $\big\{X_n^{(\delta)}\big\}$ are defined only for $x \in S^{(\delta)}$. In such situation the following theorem holds:

Theorem 2. *Suppose that for every sequence of elements $y^{(\delta)}$, $y^{(\delta)} \in S^{(\delta)}$, having a limit $x \in S$, the following convergence takes place:*

$$P^{(\delta)}(y^{(\delta)},\cdot) \Rightarrow P^{(0)}(x,\cdot) \quad \text{as } \delta \to 0. \tag{2.11}$$

Suppose that (2.3) *holds, and let $\{\pi^{(\delta)}, \delta \neq 0\}$ be a tight family on S. Then there exists an invariant measure $\pi^{(0)}$ for the chain $\big\{X_n^{(0)}\big\}$ and the weak convergence $\pi^{(\delta)} \Rightarrow \pi^{(0)}$ holds as $\delta \to 0$.*

Proof. Consider a new Markov chain $\big\{\widetilde{X}_n^{(\delta)}\big\}$ with transition probabilities defined by

$$\widetilde{P}^{(\delta)}(y,\cdot) = \begin{cases} P^{(\delta)}(y,\cdot) & \text{if } y \in S^{(\delta)}, \\ P^{(0)}(y,\cdot) & \text{if } y \notin S^{(\delta)}. \end{cases}$$

The sequence of the Markov chains $\big\{\widetilde{X}_n^{(\delta)}\big\}$ satisfies conditions of Theorem 1 and $\pi^{(\delta)}$ is an invariant measure for the chain $\big\{\widetilde{X}_n^{(\delta)}\big\}$. Theorem 2 now follows from Theorem 1.

Theorems 1 and 2 mostly reduce the problem of continuity for invariant measures to the problem of tightness for the family of invariant measures. In the next sections some sufficient conditions (in fact rather close those necessary) for tightness of the family of invariant measures are given in terms of mean drifts of Markov chains (§ 5) and in terms of test functions (§ 3 and § 6).

§ 3. Tightness of a Family of Invariant Measures for Markov Chains on a Metric Space

In this section we consider a family of Markov chains $\big\{X_n^{(\delta)}, \delta \in \Delta\big\}$ (indexed by δ) with values in a general metric space (S,ρ). The parameter set Δ is arbitrary. In this and the subsequent sections we assume that for each $\delta \in \Delta$ there exists an invariant measure $\pi^{(\delta)}$ (not unique in general) for the chain $\big\{X_n^{(\delta)}\big\}$.

Suppose that there exists a nonnegative test function $V : S \to \mathbf{R}^+$ and positive numbers $c, \varepsilon, T > 0$ such that

$$\mathbf{E}\big\{V\big(X_1^{(\delta)}\big) - V(x) \mid X_0^{(\delta)} = x\big\} \le -\varepsilon < 0 \tag{3.1}$$

if $V(x) \ge T$, and

$$\mathbf{E}\big\{V\big(X_1^{(\delta)}\big) - V(x) \mid X_0^{(\delta)} = x\big\} \le c < \infty \tag{3.2}$$

for every $x \in S$ and $\delta \in \Delta$. Denote by $\eta_x^{(\delta)}$ the random variable with the probability law

$$\mathbf{P}\big\{\eta_x^{(\delta)} \in B\big\} = \mathbf{P}\big\{V\big(X_1^{(\delta)}\big) - V(x) \in B \mid X_0^{(\delta)} = x\big\}, \quad B \in \mathscr{B}(\mathbf{R}),$$

where $\mathscr{B}(\mathbf{R})$ is a σ-algebra of Borel sets in $\mathbf{R}$. In the sequel we need the following assertion about the mean drift of a test function.

Theorem 3 [5]. *Let $\{Y_n\}_{n=0}^{\infty}$ be a Markov chain with values in the metric space (S,ρ), and let $V(x)$ be a nonnegative function, $V: S \to \mathbf{R}^+$. Suppose that the chain $\{Y_n\}$ admits an invariant measure π. If*

$$\int\limits_S \max(0, \mathbf{E}\{V(Y_1) - V(x) \mid Y_0 = x\})\pi(dx) < \infty,$$

then the following inequality holds:

$$\int\limits_S \mathbf{E}\{V(Y_1) - V(x) \mid Y_0 = x\}\pi(dx) \ge 0. \tag{3.3}$$

In particular, if

$$\sup_{x\in S} \mathbf{E}\{V(Y_1) - V(x) \mid Y_0 = x\} < \infty,$$

then (3.3) *holds.*

Generally speaking, conditions (3.1) and (3.2) do not imply tightness of the family of invariant measures $\{\pi^{(\delta)}, \delta \in \Delta\}$. In the following theorem we give a condition under which some conclusions about the tightness of the family $\{\pi^{(\delta)}, \delta \in \Delta\}$ can be came to.

Theorem 4. *Suppose that* (3.1) *and* (3.2) *hold. Let the family of random variables $\{\max(0, \eta_x^{(\delta)})\}$ be integrable uniformly in $x \in S$ and $\delta \in \Delta$, i.e.,*

$$\sup_{x,\delta} \mathbf{E}\{\eta_x^{(\delta)}; \eta_x^{(\delta)} \ge N\} \to 0 \quad \text{as } N \to \infty.$$

Then $\{\pi^{(\delta)}(V^{-1}(\cdot)), \delta \in \Delta\}$ is a tight family of measures on $\mathbf{R}^+$.

Proof. The uniform integrability condition for the family of the random variables $\{\max(0, \eta_x^{(\delta)})\}$ is equivalent to the existence of a differentiable nonnegative concave function $Q(t): \mathbf{R}^+ \to \mathbf{R}^+$ such that $Q(t) \uparrow \infty$ as $t \to \infty$ and

$$Q'(t) \le \tilde{c}/t, \quad \tilde{c} < \infty, \tag{3.4}$$

$$\sup_{x\in S, \delta\in\Delta} \mathbf{E}\{\eta_x^{(\delta)} Q(\eta_x^{(\delta)}); \eta_x^{(\delta)} \ge 0\} \le \hat{c} < \infty. \tag{3.5}$$

Consider the test function W given by $W(x) = V(x)Q(V(x))$, $x \in S$. Estimate the mean drift of the function W at x from above. Define the event A by $A = \{V(X_1^{(\delta)}) \ge V(X_0^{(\delta)})\}$, $\overline{A} = \Omega \setminus A$. We have

$$\begin{aligned}
&\mathbf{E}\{W(X_1^{(\delta)}) - W(x) \mid X_0^{(\delta)} = x\} \\
&\quad= \mathbf{E}\{[V(X_1^{(\delta)})Q(V(X_1^{(\delta)})) - V(x)Q(V(x))]\mathbf{I}_{\overline{A}} \mid X_0^{(\delta)} = x\} \\
&\qquad+ \mathbf{E}\{[V(X_1^{(\delta)}) - V(x)]Q(V(X_1^{(\delta)}))\mathbf{I}_A | X_0^{(\delta)} = x\} \\
&\qquad\quad+ \mathbf{E}\{V(x)[Q(V(X_1^{(\delta)})) - Q(V(x))]\mathbf{I}_A \mid X_0^{(\delta)} = x\} = I_1 + I_2 + I_3.
\end{aligned} \tag{3.6}$$

Since the function Q is increasing, the term I_1 admits the following estimate

$$I_1 \le \mathbf{E}\{[V(X_1^{(\delta)}) - V(x)]Q(V(x))\mathbf{I}_{\overline{A}} \mid X_0^{(\delta)} = x\}. \tag{3.7}$$

The function Q is concave; thus, the term I_2 can be estimated as follows:

$$\begin{aligned}
I_2 &\le \mathbf{E}\{[V(X_1^{(\delta)}) - V(x)]Q(V(x))\mathbf{I}_A | X_0^{(\delta)} = x\} \\
&\quad+ \mathbf{E}\{[V(X_1^{(\delta)}) - V(x)]Q(V(X_1^{(\delta)}) - V(x))\mathbf{I}_A \mid X_0^{(\delta)} = x\} \\
&= Q(V(x))\mathbf{E}\{[V(X_1^{(\delta)}) - V(x)]\mathbf{I}_A \mid X_0^{(\delta)} = x\} + \mathbf{E}\{\eta_x^{(\delta)} Q(\eta_x^{(\delta)}); \eta_x^{(\delta)} \ge 0\}.
\end{aligned} \tag{3.8}$$

Again using the concavity of the function Q, we obtain

$$\mathbf{E}\big\{\big[Q\big(V\big(X_1^{(\delta)}\big)\big) - Q(V(x))\big]\mathbf{I}_A \mid X_0^{(\delta)} = x\big\}$$
$$\le Q'(V(x))\mathbf{E}\big\{\big[V\big(X_1^{(\delta)}\big) - V(x)\big]\mathbf{I}_A \mid X_0^{(\delta)} = x\big\} \le Q'(V(x))c_1, \quad c_1 < \infty,$$

by (3.5). Hence, in view of (3.4), the following estimate for I_3 holds:

$$I_3 \le c_1 Q'(V(x))V(x) \le \tilde{c}c_1. \tag{3.9}$$

Substituting (3.7)–(3.9) into (3.6) and using (3.5), (3.1) and (3.2), we obtain the inequalities

$$\mathbf{E}\big\{W\big(X_1^{(\delta)}\big) - W(x) \mid X_0^{(\delta)} = x\big\} \le Q(V(x))\mathbf{E}\big\{V\big(X_1^{(\delta)}\big) - V(x) \mid X_0^{(\delta)} = x\big\} + \hat{c} + \tilde{c}c_1$$
$$\le \begin{cases} -\varepsilon Q(V(x)) + \hat{c} + \tilde{c}c_1 & \text{if } V(x) \ge T, \\ cQ(T) + \hat{c} + \tilde{c}c_1 & \text{if } V(x) < T. \end{cases} \tag{3.10}$$

In particular, the mean drift of the test function W is bounded uniformly in $x \in S$ and $\delta \in \Delta$:

$$\mathbf{E}\big\{W\big(X_1^{(\delta)}\big) - W(x) \mid X_0^{(\delta)} = x\big\} \le cQ(T) + \hat{c} + \tilde{c}c_1 \equiv \bar{c} < \infty.$$

Hence, we can apply Theorem 3 and obtain the inequality

$$\int_S \mathbf{E}\big\{W\big(X_1^{(\delta)}\big) - W(x) \mid X_0^{(\delta)} = x\big\}\pi^{(\delta)}(dx) \ge 0.$$

Using (3.10), we deduce that

$$\int_{x:V(x)\ge T} (-\varepsilon Q(V(x)) + \hat{c} + \tilde{c}c_1)\pi^{(\delta)}(dx) + \int_{x:V(x)<T} \bar{c}\pi^{(\delta)}(dx) \ge 0,$$

whence the inequality follows:

$$\int_{x:V(x)\ge T} Q(V(x))\pi^{(\delta)}(dx) \le \frac{\bar{c}}{\varepsilon}$$

uniformly in $\delta \in \Delta$. Since $Q(t) \uparrow \infty$ as $t \to \infty$, the last inequality implies the statement of Theorem 4.

Remark 1. The theorem of [10] can be used instead of Theorem 3.

As mentioned in the Introduction, the uniform integrability condition of Theorem 4 is far from being valid rather often. We distinguish two most interesting cases when this condition is valid.

Corollary 1. *Suppose that (3.1) and (3.2) hold. In addition, let the family of random variables $\big\{r_x^{(\delta)}, x \in S, \delta \in \Delta\big\}$ with the probability law*

$$\mathbf{P}\big\{r_x^{(\delta)} < t\big\} = \mathbf{P}\big\{\rho\big(X_1^{(\delta)}, x\big) < t \mid X_0^{(\delta)} = x\big\}, \quad t \ge 0,$$

be integrable uniformly in $x \in S$ and $\delta \in \Delta$. Let the function V satisfy the Lipschitz condition, i.e., for some $c < \infty$, $|V(x) - V(y)| \le c\rho(x,y)$ for all x, $y \in S$. Then $\{\pi^{(\delta)}(V^{-1}(\cdot)), \delta \in \Delta\}$ is a tight family of measures on $\mathbf{R}^+$.

In Corollary 2 we consider a multi-dimensional Markov chain, i.e., $S = (\mathbf{R}^+)^d$. Given a vector $x = (x_1, \dots, x_d)$ and an integer k, $1 \le k \le d$, we put $P_k(x) = x_k$.

Corollary 2. *Suppose that the kth coordinate of the process $X_n^{(\delta)} \equiv \big(X_{n,1}^{(\delta)}, \dots, X_{n,d}^{(\delta)}\big)$*

(a) *has negative mean drift, i.e., for some $T > 0$,*

$$\mathbf{E}\big\{X_{n,k}^{(\delta)} - P_k(x) \mid X_{n-1}^{(\delta)} = x\big\} \le -\varepsilon < 0$$

if $P_k(x) \ge T$, and

(b) *is integrable uniformly in $\delta \in \Delta$, i.e.,*

$$\sup_{x,\delta} \mathbf{E}\big\{\big(X_{n,k}^{(\delta)} - P_k(x)\big)\mathbf{I}_{\{X_{n,k}^{(\delta)} - P_k(x) \ge N\}} \mid X_{n-1}^{(\delta)} = x\big\} \to 0 \quad \text{as } N \to \infty.$$

Then $\big\{\pi^{(\delta)}\big(P_k^{-1}(\cdot)\big), \delta \in \Delta\big\}$ is a tight family of measures on $\mathbf{R}^+$.

The scope of application of Theorem 4 is mainly restricted to the test functions satisfying the Lipschitz condition. If the Lipschitz condition is violated (for example, if $V(x) = x^2$, $x \in \mathbf{R}$), then we need more precise theorems. The next sections are devoted to such theorems.

§ 4. Relations Between the Mean Drift of Test Function and the Mean Values of the Increments for Real-Valued Markov Chain

In the present and subsequent sections, the main attention is paid to the test functions increasing faster than any linear function, and therefore not satisfying the Lipschitz condition. Of course, we need some restrictions (see (4.2), (4.3)) on the test function in question. These restrictions can in general be formulated only for the real state space $S = \mathbf{R}^+$. So, in § 4, § 5 and partially in § 6 we consider a real-valued Markov chains. Un § 6 we give some statements for the chains taking values in a general metric space.

Let $\{X_n\}_{n=0}^{\infty}$ be a Markov chain taking values in $\mathbf{R}^+$ with transition probabilities $P(x, \cdot)$. Denote by ξ_x the random variable corresponding to the increment of the chain $\{X_n\}$ at x, i.e., the random variable with the probability law

$$\mathbf{P}\{x + \xi_x \in B\} = P(x, B), \quad B \in \mathscr{B}(\mathbf{R}^+).$$

Let a nonnegative test function $V : \mathbf{R}^+ \to \mathbf{R}^+$ be such that, for some positive x_* and ε,

$$\mathbf{E}\{V(x + \xi_x) - V(x)\} \leq -\varepsilon < 0 \tag{4.1}$$

for all $x \geq x_*$. Here we are interested in the following question. Under condition (4.1), what is the "right" upper bound for the value of $\mathbf{E}\xi_x$? It turns out that the answer to this question depends on the rate of growth of the function V. Namely, the following two different cases are distinguished: (a) $V(x) = o(x^2)$ as $x \to \infty$, and (b) $V(x) \geq cx^2$ for some $c > 0$. In accordance with these cases we formulate Lemmas 2 and 3.

Lemma 2. *Suppose that condition* (4.1) *holds and let the increasing function V be convex. Then, for every $x \geq x_*$,*

$$\mathbf{E}\xi_x \leq -\varepsilon \frac{x}{V(2x)}.$$

Proof. The function V is convex; hence, for each point x, there exists a supporting line for the curve V. Denote by $d_V(x)$ the slope of the supporting line (if a supporting line is not unique, then we choose any of them). Since V is convex, the increments of the function V satisfy the inequality $V(y) - V(x) \geq d_V(x)(y - x)$ for all x, $y \geq 0$. This implies the following estimate for the mean drift of the test function V:

$$\mathbf{E}\{V(x + \xi_x) - V(x)\} \geq d_V(x)\mathbf{E}\xi_x.$$

From here and (4.1), we deduce the inequality

$$d_V(x)\mathbf{E}\xi_x \leq -\varepsilon$$

for all $x \geq x_*$. Using the convexity of V, we obtain

$$d_V(x) \leq \frac{V(2x) - V(x)}{2x - x} \leq \frac{V(2x)}{x}.$$

The last two inequalities imply the statement of the lemma.

The inequality of Lemma 2 is unimprovable (up to a constant multiple) only in the case of $V = o(x^2)$. If $V(x) \geq cx^2$ for some $c > 0$, then the inequality can be essentially improved by using only the first and the second moments of the increments of the chain as in the following lemma.

Lemma 3. *Suppose that* (4.1) *holds and let the increasing function $u(x) \equiv V(x)/x$ be convex, $u(0) = 0$. Then, for every $x \geq x_*$,*

$$2x\mathbf{E}\xi_x + \mathbf{E}(\xi_x)^2 \leq -\varepsilon \frac{x}{u(2x)}.$$

PROOF. As observed in the proof of Lemma 2, by convexity of u, the slope $d_u(x)$ of the supporting line for the curve u at x satisfies the inequalities

$$u(x+y)-u(x)\ge d_u(x)y,\quad y\ge -x, \tag{4.2}$$

$$\frac{u(x)}{x}\le d_u(x), \tag{4.3}$$

$$d_u(x)\le \frac{u(2x)-u(x)}{x}\le \frac{u(2x)}{x}. \tag{4.4}$$

The inequality (4.2) implies the following estimate for the increments of the function V, $y\ge -x$:

$$V(x+y)-V(x)=(x+y)u(x+y)-xu(x)\ge d_u(x)(xy+y^2)+u(x)y.$$

It follows from the last inequality and (4.1) that

$$d_u(x)[x\mathbf{E}\xi_x+\mathbf{E}(\xi_x)^2]+u(x)\mathbf{E}\xi_x\le \mathbf{E}\{V(x+\xi_x)-V(x)\}\le -\varepsilon \ \text{ for } x\ge x_*. \tag{4.5}$$

In particular, $\mathbf{E}\xi_x<0$ for all $x\ge x_*$. Substituting (4.3) into (4.5), we then obtain the inequality

$$d_u(x)[2x\mathbf{E}\xi_x+\mathbf{E}(\xi_x)^2]\le -\varepsilon$$

for all $x\ge x_*$. The statement of the lemma now follows from this estimate and (4.4).

The inequality of Lemma 3 is unimprovable only in the case of $V=o(x^2\log x)$ as $x\to\infty$. If $V(x)\ge cx^2\log x$ for some $c>0$, then the inequality mentioned can be improved by using only the first and the second moments of the chain increments as follows:

We use the notation (1.3) and (1.4) together with

$$L^{(i,k)}(x)=\prod_{j=i}^{k}\log^{(j)}x=\frac{L^{(k)}(x)}{L^{(i-1)}(x)},\quad i=1,\dots,k.$$

Fix $k\in\{1,2,\dots\}$. Let

$$V(x)=x^2L^{(k-1)}(x)u(\log^{(k)}x),\quad u:\mathbf{R}^+\to\mathbf{R}^+.$$

Without loss of generality, till the end of the present section we assume that the chains $\{X_n^{(\delta)}\}$ take values in the half-line $[A_k,\infty)$, where A_k is a solution of the equation $\log^{(k)}A_k=0$. In particular, $V(X_n^{(\delta)})\ge 0$ with probability one.

Lemma 4. *Suppose that condition* (4.1) *holds and let the nonnegative function $u(x)$ be convex, $u(0)=0$, and $u(\log^{(k)}x)\le c+x^{1/5}$. Moreover, let*

$$\sup_x \mathbf{E}\{V(|\xi_x|);\xi_x\le -N\}\to 0,\quad N\to\infty. \tag{4.6}$$

Then there exists $\hat{x}<\infty$ such that, for all $x\ge\hat{x}$,

$$2x\mathbf{E}\xi_x+\mathbf{E}(\xi_x)^2+\left[\frac{1}{\log x}+\frac{1}{L^{(2)}(x)}+\dots+\frac{1}{L^{(k)}(x)}\right]\left[x\mathbf{E}\xi_x+\frac{3}{2}\mathbf{E}(\xi_x)^2\right] \le -\frac{\varepsilon}{2}\frac{1}{L^{(k-1)}(x)u(2\log^{(k)}x)}.$$

If, in addition,

$$\sup_x \mathbf{E}(\xi_x)^2<\infty,$$

then, for x large enough,

$$2x\mathbf{E}\xi_x + \left[1 + \frac{1}{\log x} + \cdots + \frac{1}{L^{(k)}(x)}\right]\mathbf{E}(\xi_x)^2 \le -\frac{\varepsilon}{4}\frac{1}{L^{(k-1)}(x)u(2\log^{(k)} x)}.$$

PROOF. Denote $W(x) \equiv x^2 L^{(k-1)}(x)$, and let W^{-1} be the inverse of W. Put $m_x \equiv \mathbf{E}W(x+\xi_x)$. Since the function $tu(\log^{(k)} W^{-1}(t))$ is convex, we have

$$\mathbf{E}V(x+\xi_x) = \mathbf{E}W(x+\xi_x)u(\log^{(k)} W^{-1}(W(x+\xi_x))) \ge m_x u(\log^{(k)} W^{-1}(m_x)).$$

It follows from the last inequality and (4.1) that, in particular, for $x \ge x_*$

$$m_x \equiv \mathbf{E}W(x+\xi_x) \le W(x). \tag{4.7}$$

By convexity of u, the following inequality holds:

$$\begin{aligned} V(x+y) - V(x) &= W(x+y)(u(\log^{(k)}(x+y)) - u(\log^{(k)} x)) + [W(x+y) - W(x)]\, u(\log^{(k)} x) \\ &\ge W(x+y)d_u(\log^{(k)} x)(\log^{(k)}(x+y) - \log^{(k)} x) + [W(x+y) - W(x)]u(\log^{(k)} x). \end{aligned} \tag{4.8}$$

Since $d_u(t) \ge u(t)/t$,

$$u(\log^{(k)} x) \le d_u(\log^{(k)} x)\log^{(k)} x.$$

It follows from (4.7) and the last inequality in (4.8) that, for $x \ge x_*$,

$$\begin{aligned} \mathbf{E}\{V(x+\xi_x) - V(x)\} &\ge d_u(\log^{(k)} x)\mathbf{E}(x+\xi_x)^2 L^{(k)}(x+\xi_x) - d_u(\log^{(k)} x)\log^{(k)} x\mathbf{E}W(x+\xi_x) \\ &+ d_u(\log^{(k)} x)\log^{(k)} x\mathbf{E}\{W(x+\xi_x) - W(x)\} = d_u(\log^{(k)} x)[\mathbf{E}(x+\xi_x)^2 L^{(k)}(x+\xi_x) - x^2 L^{(k)}(x)]. \end{aligned}$$

Involving the inequality $d_u(t) \le (u(2t) - u(t))/t \le u(2t)/t$ and (4.1), we infer the following estimate for $x \ge x_*$:

$$\mathbf{E}(x+\xi_x)^2 L^{(k)}(x+\xi_x) - x^2 L^{(k)}(x) \le -\varepsilon\frac{\log^{(k)} x}{u(2\log^{(k)} x)}. \tag{4.9}$$

The first and the second derivatives of the function $x^2 L^{(k)}(x)$ are equal to

$$\frac{d}{dx}x^2 L^{(k)}(x) = 2xL^{(k)}(x) + xL^{(2,k)}(x) + \cdots + x\log^{(k)} x + x, \tag{4.10}$$

$$\frac{d^2}{dx^2}x^2 L^{(k)}(x) = 2L^{(k)}(x) + 3L^{(2,k)}(x) + \cdots + 3\log^{(k)} x + 3 + \psi(x), \tag{4.11}$$

where $\psi(x) \ge 0$ and $\psi(x) \le cL^{(3,k)}(x)/\log x$ for $x \ge A_k$. Using Taylor's formula, we deduce the inequality

$$\begin{aligned} &(x+y)^2 L^{(k)}(x+y) - x^2 L^{(k)}(x) \\ &\quad\ge y[2xL^{(k)}(x) + xL^{(2,k)}(x) + \cdots + x] + \frac{y^2}{2}[2L^{(k)}(x+\theta y) + 3L^{(2,k)}(x+\theta y) + \cdots + 3] \\ &\quad\ge xy[2L^{(k)}(x) + L^{(2,k)}(x) + \cdots + 1] + y^2\left[L^{(k)}(x-y^-) + \frac{3}{2}L^{(2,k)}(x-y^-) + \cdots + \frac{3}{2}\right], \end{aligned} \tag{4.12}$$

where $0 \le \theta \le 1$ and $y^- = \max(0, -y)$. Since $L^{(i,k)}(x - \sqrt{x}) = L^{(i,k)}(x) + O(x^{-1/4})$ and $L^{(i,k)}(x - y^-) \ge L^{(i,k)}(x) - L^{(i,k)}(A_k + y^-)$ (by concavity of $L^{(i,k)}$), (4.12) implies

$$\begin{aligned} \mathbf{E}\{(x+\xi_x)^2 L^{(k)}(x+\xi_x) - x^2 L^{(k)}(x)\} &\ge x\mathbf{E}\xi_x[2L^{(k)}(x) + L^{(2,k)}(x) + \cdots + 1] \\ &+ \mathbf{E}(\xi_x)^2\left[L^{(k)}(x) + \frac{3}{2}L^{(2,k)}(x) + \cdots + \frac{3}{2}\right] + o(x^{-1/4}) - \mathbf{E}\Big\{(\xi_x)^2\Big[L^{(k)}(A_k + \xi_x^-) \\ &+ \frac{3}{2}L^{(2,k)}(A_k + \xi_x^-) + \cdots + \frac{3}{2}\log^{(k)}(A_k + \xi_x^-)\Big]; \xi_x \le -\sqrt{x}\Big\} \\ &= L^{(k)}(x)(2x\mathbf{E}\xi_x + \mathbf{E}(\xi_x)^2) + L^{(k)}(x) \\ &\times\left[\frac{1}{\log x} + \frac{1}{L^{(2)}(x)} + \cdots + \frac{1}{L^{(k)}(x)}\right]\left[x\mathbf{E}\xi_x + \frac{3}{2}\mathbf{E}(\xi_x)^2\right] + o(x^{-1/4}) + o\left(\frac{\log^{(k)} x}{u(\log^{(k)} x)}\right) \end{aligned}$$

as $x \to \infty$ (the remainder terms are negligible in view of (4.6) and Chebychev's inequality). Substituting the last inequality into (4.9), we derive the first estimate of the lemma, for x large enough.

Since the left-hand side of the first estimate in the lemma is equal to

$$(2x\mathbf{E}\xi_x + \mathbf{E}(\xi_x)^2)\left[1 + \frac{1}{2\log x} + \frac{1}{2L^{(2)}(x)} + \cdots + \frac{1}{2L^{(k)}(x)}\right] + \left[\frac{1}{\log x} + \frac{1}{L^{(2)}(x)} + \cdots + \frac{1}{L^{(k)}(x)}\right]\mathbf{E}(\xi_x)^2,$$

the second estimate follows from the first after multiplying by

$$\left[1 + \frac{1}{2\log x} + \frac{1}{2L^{(2)}(x)} + \cdots + \frac{1}{2L^{(k)}(x)}\right]^{-1}.$$

§ 5. A Tightness of a Family of Invariant Measures for Real-Valued Markov Chains in Terms of the Mean Values of the Chain Increments

Let $\{X_n^{(\delta)}\}_{n=0}^{\infty}$ be a Markov chain taking values in $\mathbf{R}^+$ with transition probabilities $P^{(\delta)}(x,\cdot)$; $\delta \in \Delta$, where Δ is an arbitrary parameter set. Denote by $\xi_x^{(\delta)}$ the random variable with the distribution law $\mathbf{P}\{x + \xi_x^{(\delta)} \in B\} = P^{(\delta)}(x, B)$. In Theorems 5 and 6 we clarify how fast the value of $\mathbf{E}\xi_x^{(\delta)}$ may increase (in x) for the family of invariant measures $\{\pi^{(\delta)}, \delta \in \Delta\}$ be tight.

Theorem 5. *Let a nonnegative concave function v be such that*

$$v(x)/x \to 0 \text{ as } x \to \infty. \tag{5.1}$$

Suppose that for some $\hat{x}$ the inequality

$$\mathbf{E}\xi_x^{(\delta)} \le -1/v(x) \tag{5.2}$$

holds for all $x \ge \hat{x}$ and $\delta \in \Delta$, and let the family of random variables $\{|\xi_x^{(\delta)}|v(|\xi_x^{(\delta)}|)\}$ be integrable uniformly in x and δ, i.e.,

$$\sup_{x,\delta} \mathbf{E}\{|\xi_x^{(\delta)}|v(|\xi_x^{(\delta)}|); |\xi_x^{(\delta)}| \ge N\} \to 0 \quad \text{as } N \to \infty. \tag{5.3}$$

Then $\{\pi^{(\delta)}, \delta \in \Delta\}$ is a tight family of invariant measures.

Theorem 5 (with $v(x) = \text{const}$) and Theorem 1 imply continuity theorems in [11, § 20] (see also [12, Chapter 4]).

Remark 2. Since the function v is nonnegative and concave, it is nondecreasing and the function $v(x)/x$ is nonincreasing. In particular the limit of the ratio $v(x)/x$ in (5.1) always exists (positive or equal to zero).

Proof of Theorem 5. Let a nonnegative increasing function $g : \mathbf{R}^+ \to \mathbf{R}^+$ be such that $g(x) \uparrow \infty$ and

$$\text{the function } Q(x) = v(x)g(x) \text{ is concave,} \tag{5.4}$$

$$\sup_{x,\delta} \mathbf{E}\{|\xi_x^{(\delta)}|Q(|\xi_x^{(\delta)}|)\} = c_1 < \infty. \tag{5.5}$$

The existence of g with these properties follows from (5.1) and (5.3).

Consider the test function

$$V(x) = \int_0^x Q(y)\,dy.$$

Estimate the increments of V. By Taylor's formula, we have

$$V(y) - V(x) = Q(x + \theta(y-x))(y-x) = Q(x)(y-x) + [Q(x+\theta(y-x)) - Q(x)](y-x)$$

for all x, $y \ge 0$ ($0 \le \theta = \theta(x,y) \le 1$). By concavity and growth (see (5.4)), the function Q satisfies the following property: $|Q(u) - Q(t)| \le Q(|u-t|)$ for all u, $t \ge 0$. Hence,

$$V(y) - V(x) \le Q(x)(y-x) + Q(|\theta(y-x)|)|y-x| \le Q(x)(y-x) + Q(|y-x|)|y-x|.$$

Whence and from (5.5) we infer that

$$\mathbf{E}\{V(x+\xi_x^{(\delta)}) - V(x)\} \le Q(x)\mathbf{E}\xi_x^{(\delta)} + c_1 = g(x)\big(v(x)\mathbf{E}\xi_x^{(\delta)}\big) + c_1 \le -g(x) + c_1, \quad c_1 < \infty,$$

for $x \ge \hat{x}$ and $\delta \in \Delta$, in view of (5.2). Since, by the construction, $g(x) \to \infty$ as $x \to \infty$; therefore, applying Theorem 3 (as in the proof of Theorem 4) completes the proof of Theorem 5.

Without paying attention to concavity of v, conditions (5.1) and (5.2) can be restated as

$$\limsup_{x\to\infty} x\mathbf{E}\xi_x^{(\delta)} = -\infty$$

uniformly in $\delta \in \Delta$. We further consider the case in which this condition generally fails; for example, when $\mathbf{E}\xi_x^{(\delta)} \sim q/x$ as $x \to \infty$ for some $q < 0$. It turns out that in this case the tightness of the family of invariant measures can still take place if the constant q is "negative enough". In this situation we have to involve the second moments of the chains increments as in the following theorem.

Theorem 6. *Let a nonnegative concave function v satisfy* (5.1). *Suppose that, for some $\hat{x} < \infty$, the following inequality holds:*

$$2x\mathbf{E}\xi_x^{(\delta)} + \mathbf{E}\big(\xi_x^{(\delta)}\big)^2 \le -1/v(\log x) \tag{5.6}$$

for all $x \ge \hat{x}$ and $\delta \in \Delta$. Let the family of random variables $\big\{\big(\xi_x^{(\delta)}\big)^2 v\big(\log \xi_x^{(\delta)}\big)\mathbf{I}\{\xi_x^{(\delta)} \ge 1\}\big\}$ be integrable uniformly in x and δ, i.e.,

$$\sup_{x,\delta} \mathbf{E}\big\{\big(\xi_x^{(\delta)}\big)^2 v\big(\log \xi_x^{(\delta)}\big); \xi_x^{(\delta)} \ge N\big\} \to 0 \quad \text{as } N \to \infty,$$

and let the second moments of the chains increments be bounded uniformly in x and δ, i.e.,

$$\sup_{x,\delta} \mathbf{E}\big(\xi_x^{(\delta)}\big)^2 < \infty.$$

Then $\{\pi^{(\delta)}, \delta \in \Delta\}$ is a tight family of invariant measures.

Theorems 6 and 1 imply the continuity theorems in [5, 13] (with $v(x) = \text{const} > 0$).

Proof. Without loss of generality, we may assume that the chains $\{X_n^{(\delta)}\}$ take values in $[1, \infty)$. It follows from the hypothesis of theorem that there exists a positive function g such that $g(x) \uparrow \infty$ as $x \to \infty$ and

$$\text{the function } \widetilde{Q}(x) = v(x)g(x) \text{ is concave,} \tag{5.7}$$

$$\sup_{x,\delta} \mathbf{E}\big\{\big(\xi_x^{(\delta)}\big)^2 \widetilde{Q}\big(\log\big(1+\xi_x^{(\delta)}\big)\big); \xi_x^{(\delta)} \ge 0\big\} = c_2 < \infty. \tag{5.8}$$

Put

$$\widehat{Q}(x) \equiv \int_0^x \widetilde{Q}(y)\,dy \quad \text{and} \quad Q(x) \equiv \int_0^x \widehat{Q}(y)\,dy.$$

Since the function $\widetilde{Q}$ is concave, its derivative $\widehat{Q}'(x) = \widetilde{Q}(x+1) - \widetilde{Q}(x)$ is nonincreasing, and therefore the function $\widehat{Q}$ is concave. In particular, for every $x \ge 0$,

$$Q'(x) = \widehat{Q}(x+1) - \widehat{Q}(x) \le \widehat{Q}(1). \tag{5.9}$$

The function Q is also concave. By definition, $\widetilde{Q}(x) \le Q(x) \le \widetilde{Q}(x+2)$; hence,

$$Q(x) \sim \widetilde{Q}(x) \quad \text{as } x \to \infty. \tag{5.10}$$

Consider the test function $V(x) = x^2 Q(\log x)$. Estimate the increments of this function from above. By Taylor's formula, for $y \ge -x+1$ ($0 \le \theta = \theta(x,y) \le 1$), we have the decomposition

$$
\begin{aligned}
V(x+y) - V(x) &= y[2xQ(\log x) + xQ'(s)|_{s=\log x}] \\
&\quad + \frac{y^2}{2}[2Q(\log(x+\theta y)) + (3Q'(s) + Q''(s))|_{s=\log(x+\theta y)}] \\
&\quad\quad \le (2xy + y^2)Q(\log x) + yxQ'(s)|_{s=\log x} + y^2\left(Q(\log(1+y^+)) + \frac{3}{2}\widehat{Q}(1)\right),
\end{aligned}
$$

where the inequality is valid in view of the growth and concavity of the function Q and (5.9) and because the second derivative $Q''(s)$; $y^+ = \max(0,y)$ is nonpositive. Hence, the following upper bound for the mean drift of the test function V at x can be derived:

$$
\begin{aligned}
\mathbf{E}\{V(x+\xi_x^{(\delta)}) - V(x)\} &\le \bigl[2x\mathbf{E}\xi_x^{(\delta)} + \mathbf{E}(\xi_x^{(\delta)})^2\bigr]Q(\log x) \\
&\quad + xQ'(s)|_{s=\log x}\mathbf{E}\xi_x^{(\delta)} + \mathbf{E}\bigl(\xi_x^{(\delta)}\bigr)^2 Q\bigl(\log\bigl(1+\xi_x^{(\delta)+}\bigr)\bigr) + \frac{3}{2}\widehat{Q}(1)\mathbf{E}\bigl(\xi_x^{(\delta)}\bigr)^2.
\end{aligned}
$$

Since $\mathbf{E}\xi_x^{(\delta)} < 0$ for $x \ge \hat{x}$ (by (5.6)), from (5.8) and the uniform boundedness of the second moments of the chain increments we obtain

$$\mathbf{E}\{V(x+\xi_x^{(\delta)}) - V(x)\} \le \bigl[2x\mathbf{E}\xi_x^{(\delta)} + \mathbf{E}\bigl(\xi_x^{(\delta)}\bigr)^2\bigr]Q(\log x) + {}_3.$$

for $x \ge \hat{x}$. From here and (5.6) and (5.10) we deduce the relations

$$\mathbf{E}\{V(x+\xi_x^{(\delta)}) - V(x)\} \le -\frac{Q(\log x)}{v(\log x)} + c_3 \sim -g(\log x), \quad x \to \infty,$$

because $g(\log x) \to \infty$ by construction. Applying Theorem 3 (as in the proof of Theorem 4) completes the proof.

Observe that conditions (5.1) and (5.6) can be restated as

$$\varlimsup_{x\to\infty}\bigl[2x\mathbf{E}\xi_x^{(\delta)} + \mathbf{E}\bigl(\xi_x^{(\delta)}\bigr)^2\bigr]\log x = -\infty$$

uniformly in $\delta \in \Delta$. It turns out that the tightness of the family of invariant measures can still take place even if this condition fails. The following result holds:

Theorem 7. *Let the nonnegative concave function v satisfy condition* (5.1). *Let $k \in \{1,2,\dots\}$ and, for some $\hat{x} < \infty$, the equality*

$$2x\mathbf{E}\xi_x^{(\delta)} + \left[1 + \frac{1}{\log x} + \frac{1}{L^{(2)}(x)} + \cdots + \frac{1}{L^{(k)}(x)}\right]\mathbf{E}\bigl(\xi_x^{(\delta)}\bigr)^2 \le -\frac{1}{L^{(k)}(x)v(\log^{(k+1)} x)} \tag{5.11}$$

hold for all $x \ge \hat{x}$ and $\delta \in \Delta$. Let

$$\sup_{x,\delta}\mathbf{E}\bigl\{\bigl(\xi_x^{(\delta)}\bigr)^2 L^{(k)}\bigl(\xi_x^{(\delta)}\bigr)v\bigl(\log^{(k+1)}\xi_x^{(\delta)}\bigr); \xi_x^{(\delta)} \ge N\bigr\} \to 0 \quad \text{as } N \to \infty$$

and let the second moments of the chains increments be uniformly bounded. Then the family of invariant measures $\{\pi^{(\delta)}, \delta \in \Delta\}$ is tight.

Proof. Without loss of generality, we may assume that the chains $\{X_n^{(\delta)}\}$ take values in $[A_{k+1},\infty)$, where A_{k+1} is a solution of the equation $\log^{(k+1)} A_{k+1} = 0$. By the hypothesis of the theorem, there exists a concave function $\widetilde{Q}$ such that $\widetilde{Q}(x)/v(x) \uparrow \infty$ as $x \to \infty$ and

$$\sup_{x,\delta}\mathbf{E}\bigl\{\bigl(\xi_x^{(\delta)}\bigr)^2 L^{(k)}\bigl(A_{k+1} + \xi_x^{(\delta)}\bigr)\widetilde{Q}\bigl(\log^{(k+1)}\bigl(A_{k+1} + \xi_x^{(\delta)}\bigr)\bigr); \xi_x^{(\delta)} \ge 0\bigr\} = c_4 < \infty. \tag{5.12}$$

Consider the test function $V(x) = x^2 L^{(k)}(x) Q(\log^{(k+1)} x)$, where the function Q is defined in the proof of Theorem 6. Taking (4.10) into account together with the fact that Q' is nonnegative, we obtain

$$\frac{d}{dx}V(x) \geq (2xL^{(k)}(x) + xL^{(2,k)}(x) + \cdots + x\log^{(k)} x + x)Q(\log^{(k+1)} x).$$

The function Q is concave; hence, $Q'' \leq 0$ and

$$\frac{d}{dx}Q(\log^{(k+1)} x) \leq \frac{\sup_s Q'(s)}{xL^{(k)}(x)} = \frac{c_5}{xL^{(k)}(x)}.$$

Together with (4.11), the preceding inequality gives the following upper bound for the second derivative of the function V:

$$\frac{d^2}{dx^2}V(x) \leq [2L^{(k)}(x) + 3L^{(2,k)}(x) + \cdots + 3\log^{(k)} x + 3]Q(\log^{(k+1)} x) + c_6.$$

By Taylor's formula,

$$\mathbf{E}\{V(x + \xi_x^{(\delta)}) - V(x)\} = V'(x)\mathbf{E}\xi_x^{(\delta)} + \frac{1}{2}\mathbf{E}\{V''(x + \theta\xi_x^{(\delta)})(\xi_x^{(\delta)})^2\}.$$

Since $\mathbf{E}\xi_x^{(\delta)} < 0$ for $x \geq \hat{x}$ (in view of (5.11)); using the estimates for V' and V'' and the boundedness of the second moments of chains increments, we obtain the following upper estimate for the mean drift of the test function V at x:

$$\begin{aligned}\mathbf{E}\{V(x + \xi_x^{(\delta)}) - V(x)\} &\leq x(2L^{(k)}(x) + L^{(2,k)}(x) + \cdots + \log^{(k)} x + 1)Q(\log^{(k+1)} x)\mathbf{E}\xi_x^{(\delta)} \\ &+\mathbf{E}(\xi_x^{(\delta)})^2 \left(L^{(k)}(\eta) + \frac{3}{2}[L^{(2,k)}(\eta) + \cdots + \log^{(k)}(\eta) + 1]\right) Q(\log^{(k+1)}(\eta))\big|_{\eta = x+\xi_x^{(\delta)+}} + c_7.\end{aligned}$$

Since the functions $L^{(i,k)}(x)Q(\log^{(k+1)} x)$ are concave; therefore, for $y > 0$,

$$L^{(i,k)}(x+y)Q(\log^{(k+1)}(x+y)) \leq L^{(i,k)}(x)Q(\log^{(k+1)} x) + {}_8L^{(i,k)}(A_{k+1}+y)Q(\log^{(k+1)}(A_{k+1}+y))$$

and, hence,

$$\begin{aligned}\mathbf{E}\{V(x + \xi_x^{(\delta)}) - V(x)\} &\leq \left[2x\mathbf{E}\xi_x^{(\delta)} + \mathbf{E}(\xi_x^{(\delta)})^2\right]L^{(k)}(x)Q(\log^{(k+1)} x) \\ &+ \left[x\mathbf{E}\xi_x^{(\delta)} + \frac{3}{2}\mathbf{E}(\xi_x^{(\delta)})^2\right]\left[\frac{1}{\log x} + \cdots + \frac{1}{L^{(k)}(x)}\right] L^{(k)}(x)Q(\log^{(k+1)} x) + c_9\end{aligned}$$

in view of (5.12). By (5.11), $2x\mathbf{E}\xi_x^{(\delta)} + \mathbf{E}(\xi_x^{(\delta)})^2 < 0$ for $x \geq \hat{x}$; hence, from the last inequality we infer that

$$\begin{aligned}&\mathbf{E}\{V(x + \xi_x^{(\delta)}) - V(x)\} \\ &\leq \left[2x\mathbf{E}\xi_x^{(\delta)} + \left[1 + \frac{1}{\log x} + \cdots + \frac{1}{L^{(k)}(x)}\right]\mathbf{E}(\xi_x^{(\delta)})^2\right] L^{(k)}(x)Q(\log^{(k+1)} x) + c_9.\end{aligned}$$

Using (5.11), we deduce the following estimate for $x \geq \hat{x}$:

$$\mathbf{E}\{V(x + \xi_x^{(\delta)}) - V(x)\} \leq -\frac{Q(\log^{(k+1)} x)}{v(\log^{(k+1)} x)} + c_9 \to -\infty, \quad x \to \infty.$$

Applying Theorem 3 (as in the proof of Theorem 4) now completes the proof.

§ 6. A Tightness Conditions of a Family of Invariant Measures in Terms of Test Functions

In resolving the question of tightness conditions in terms of test (Lyapunov) functions, we will assume that there exists positive x_*, c, and ε such that

$$a^{(\delta)}(x) \equiv \mathbf{E}\bigl\{V\bigl(\xi_x^{(\delta)}\bigr) - V(x)\bigr\} \le -\varepsilon < 0 \tag{6.1}$$

if $x \ge x_*$, and $a^{(\delta)}(x) \le c$ for all $x \ge 0$ and $\delta \in \Delta$. Moreover, we will assume the following rather natural condition of uniform (in $x \ge 0$ and $\delta \in \Delta$) integrability of the family of random variables $\bigl\{V\bigl(\bigl|\xi_x^{(\delta)}\bigr|\bigr)\bigr\}$:

$$\sup_{x,\delta} \mathbf{E}\bigl\{V\bigl(\bigl|\xi_x^{(\delta)}\bigr|\bigr);\bigl|\xi_x^{(\delta)}\bigr| \ge N\bigr\} \to 0 \quad \text{as } N \to \infty \tag{6.2}$$

(a Lindeberg-type condition.) Condition (6.2) (see Theorems 8 and 9 below) is rather close to that necessary. We are interested in the following question: what additional restrictions (as minimal as possible) on the test function V will provide tightness for the family $\{\pi^{(\delta)}, \delta \in \Delta\}$? As in § 4, these restrictions depend on the rate of growth of the function V. In Theorem 8, we consider the case of $V(x) = o(x^2)$ as $x \to \infty$. In Theorem 9, we consider the case of $V(x) \ge cx^2$ for some $c > 0$.

Theorem 8. *Suppose that* (6.1) *and* (6.2) *hold. Let the increasing function V be convex and admit the asymptotic representation*

$$V(x) = xv(x)(1 + o(1)) \quad \textit{as } x \to \infty,$$

where the nonnegative function v is concave. Then the family of invariant measures $\{\pi^{(\delta)}, \delta \in \Delta\}$ is tight.

Proof. Observe that if v is concave, then $V(2x) \le 5V(x)$ for x large enough. In view of this observation, Theorem 8 follows from Lemma 2 and Theorem 5 (in the case $v(x)/x \to 0$) and from Lemma 3 and Theorem 6 (in the case $v(x) = cx$ for some $c > 0$).

Theorem 9. *Suppose that* (6.1) *and* (6.2) *hold. Let V admit the asymptotic representation*

$$V(x) = x^2 L^{(k)}(x) v(\log^{(k+1)} x)(1 + o(1)) \quad \textit{as } x \to \infty,$$

where $k \in \{0, 1, 2, \dots\}$ (we put $L^{(0)}(x) \equiv 1$), the nonnegative function v be concave, and $tv(\log(1 + t))$ be convex. Then the family of invariant measures $\{\pi^{(\delta)}, \delta \in \Delta\}$ is tight.

Proof. In the case $k = 0$ the theorem follows from Lemma 3 and Theorem 6. In the case $k \ge 1$ the theorem follows from Lemma 4 and Theorem 7.

Closing this section, we note how the above results can be applied to the Markov chains taking values in a general metric space S. Our proofs allow us to replace the real-valued Markov chain $\bigl\{X_n^{(\delta)}\bigr\}$ by a function of the Markov chain. Namely, let $\bigl\{Y_n^{(\delta)}\bigr\}$ be a Markov chain with values in a metric space S and let $f : S \to \mathbf{R}^+$ and $V : \mathbf{R}^+ \to \mathbf{R}^+$ be nonnegative functions. Assume that there exist positive x_* and ε such that, for each $\delta \in \Delta$,

$$a^{(\delta)}(t) \equiv \mathbf{E}\bigl\{V\bigl(f\bigl(Y_1^{(\delta)}\bigr)\bigr) - V(f(t)) \mid Y_0^{(\delta)} = t\bigr\} \le -\varepsilon$$

if $f(t) \ge x_*$, and

$$\sup_{t \in S, \delta \in \Delta} a^{(\delta)}(t) < \infty.$$

Denote by $\eta_t^{(\delta)}$ the random variable with the probability law

$$\mathbf{P}\bigl\{\eta_t^{(\delta)} \in B\bigr\} = \mathbf{P}\bigl\{f\bigl(Y_1^{(\delta)}\bigr) - f(t) \in B \mid Y_0^{(\delta)} = t\bigr\}, \quad B \in \mathscr{B}(\mathbf{R}^+).$$

Suppose that the family of random variables $\bigl\{V\bigl(\bigl|\eta_t^{(\delta)}\bigr|\bigr)\bigr\}$ is integrable uniformly in $t \in S$ and $\delta \in \Delta$. If V satisfies the hypothesis of Theorem 8 or 9, then $\{\pi^{(\delta)}(f^{-1}(\cdot)), \delta \in \Delta\}$ is a tight family of measures on $\mathbf{R}^+$.

This assertion can be applied to the multi-dimensional Markov chains in $(\mathbf{R}^+)^d$. Say, if one of the chain coordinates (which, of course, is usually not a Markov chain) satisfies the conditions like conditions in Theorem 8 or 9, then the family of measures corresponding to this coordinate is tight (as in Corollary 2).

References

1. *S. P. Meyn and R. L. Tweedie*, Markov Chains and Stochastic Stability, Springer, London (1993).
2. *A. F. Karr*, "Weak convergence of a sequence of Markov chains," Z. Wahrscheinlichkeistheorie und verw. Gebiete., **33**, 41–48 (1975).
3. *V. A. Malyshev and M. V. Menshikov*, "Ergodicity, continuity and analyticity of countable Markov chains," Trans. Moscow Math. Soc., **1**, 1–48 (1982).
4. *V. V. Kalashnikov, S. T. Rachev*, Mathematical Methods for Construction of Queueing Models, Wadsworth and Brooks/Cole, New York (1990).
5. *D. Korshunov*, "Transition phenomena for real-valued Markov chains," Siberian Adv. Math., **3**, 53–100 (1993).
6. *M. V. Menshikov, I. M. Asymont, R. Yasnogorodsky*, "Markov processes with asymptotically zero drift, [in Russian]," Problems Inform. Transmission, **31**, No. 3, 60–75 (1995).
7. *G. Fayolle, V. A. Malyshev, M. V. Menshikov*, Topics in the Constructive Theory of Countable Markov Chains, Cambridge Univ. Press, Cambridge (1995).
8. *J. Lamperti*, "Criteria for stochastic processes II: Passage-time moments," J. Math. Anal. Appl., **7**, 127–145 (1963).
9. *P. Billingsley*, Convergence of Probability Measures, John Wiley & Sons, New York (1968).
10. *L. V. Nazarov, S. N. Smirnov*, "Estimation of moments of stationary distribution of a Markov chain by the method of test functions, [in Russian]," Vestnik Moscov. Univ, Ser. 15, **4**, 43–48 ().
11. *A. A. Borovkov*, Stochastic Processes in Queueing Theory, Springer, New York (1976).
12. *A. A. Borovkov*, Asymptotic Methods in Queueing Theory, John Wiley & Sons, New York (1984).
13. *A. A. Borovkov, G. Fayolle, D. A. Korshunov*, "Transient phenomena for Markov chains and applications," Adv. Appl. Probab., **24**, 322–342 (1992).