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TRANSIENT PHENOMENA FOR MARKOV CHAINS AND APPLICATIONS

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Abstract

We consider a family of irreducible, ergodic and aperiodic Markov chains $X^{(\varepsilon)} = \{X_n^{(\varepsilon)},\ n \geqq 0\}$, depending on a parameter $\varepsilon > 0$, so that the local drifts have a critical behaviour (in terms of Pakes' lemma). The purpose is to analyse the steady-state distributions of these chains (in the sense of weak convergence), when $\varepsilon \downarrow 0$. Under assumptions involving at most the existence of moments of order $2 + \gamma$ for the jumps, we show that, whenever $X^{(0)}$ is not ergodic, it is possible to characterize accurately these limit distributions. Connections with the gamma and uniform distributions are revealed. An application to the well-known ALOHA network is given.

ERGODICITY; RANDOM ACCESS; STABILITY; WEAK CONVERGENCE

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1. Introduction

Let $X^{(\varepsilon)} = \{X_n^{(\varepsilon)},\ n \geqq 0\}$ be a sequence (with respect to ε) of irreducible aperiodic and ergodic Markov chains for $\varepsilon > 0$, with state space $\{0, 1, 2, \cdots\}$. We shall suppose that the transition probabilities of these chains have a property of convergence $p_{ij}^{(\varepsilon)} \to p_{ij}$ as $\varepsilon \to 0$, where $\|p_{ij}\|$ is the matrix of transition probabilities of an irreducible aperiodic Markov chain $X^{(0)}$. Let $\pi_j^{(\varepsilon)}$ be stationary probabilities for the matrix $\|p_{ij}^{(\varepsilon)}\|$. Then

$$(1) \qquad \pi_i^{(\varepsilon)} = \sum_{j \geqq 0} \pi_j^{(\varepsilon)} p_{ji}^{(\varepsilon)}, \qquad i \geqq 0.$$

Let $\zeta^{(\varepsilon)}$ denote a random variable with the distribution $\mathbb{P}[\zeta^{(\varepsilon)} = j] = \pi_j^{(\varepsilon)}$. In Sections 2 and 3, we consider the asymptotic behaviour of the distribution of $\zeta^{(\varepsilon)}$ as $\varepsilon \downarrow 0$ under some assumptions for $p_{ij}^{(\varepsilon)}$, which will be explained later. When $X^{(0)}$ is ergodic, the problem mentioned above concerns the *stability* (sometimes referred to as *transient phenomena*) of the ergodic distributions $X^{(\varepsilon)}$ with respect to the parameter ε, when $\varepsilon \downarrow 0$. However, the conditions discussed below are related, generally speaking, to another situation, the 'critical' one, when $X^{(0)}$ can be non-ergodic. It is worth remarking that the proposed approach is general and could also be used for non-Markov processes. *En passant,* we are able to distinguish

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between transience and null recurrence. Section 3 proposes an application of these results to a Markov chain arising from the analysis of the well-known basic ALOHA algorithm, which is employed to solve the contention in some distributed random access communication systems.

2. Transient phenomena theorems for Markov chains

This long section contains the main body of the paper. Although it may appear dense, its organization is simple. First, the four main theorems are stated. Then, the outlines of the proofs are given. The proofs consist mainly of lemmas. We use the following notation:

$$m_i^{(\varepsilon)} = \mathbb{E}[X_1^{(\varepsilon)} - i \mid X_0^{(\varepsilon)} = i], \qquad b_i^{(\varepsilon)} = \mathbb{E}[(X_1^{(\varepsilon)} - i)^2 \mid X_0^{(\varepsilon)} = i].$$

To provide a reasonable classification of the transient phenomena for the Markov chain $X^{(\varepsilon)}$, we suppose that the asymptotic behaviour of $m_i^{(\varepsilon)}$ and $b_i^{(\varepsilon)}$, as $i \to \infty$, $\varepsilon \downarrow 0$ is quite regular. In particular, we assume the existence of the following limits:

$$\lim_{i\to\infty} m_i^{(0)} = 0,$$

$$\lim_{i\to\infty} i m_i^{(0)} = -\mu, \quad -\infty \leqq \mu \leqq \infty,$$

$$\lim_{i\to\infty} b_i^{(0)} = b, \quad 0 < b < \infty.$$

It turns out that the parameters μ, b are essential for the ergodicity of $X^{(0)}$, and later on for the asymptotic behaviour of $\zeta^{(\varepsilon)}$ also. In particular, it was proved in [12], assuming mainly the boundedness of moments of order $2+\alpha$ for the increments, that $X^{(0)}$ is recurrent (not necessarily ergodic) if $-2\mu < b$ and transient if $-2\mu > b$ (note the minus sign). It is possible to improve this classification for the class of chains considered below. In particular, we show that, under mild conditions given in Theorem 4, $X^{(0)}$ is ergodic if $2\mu > b$, null recurrent if $b \geqq 2\mu > 0$ and transient if $-2\mu > b$. Since our principal goal is to analyse the stability of the family of Markov chains $X^{(\varepsilon)}$, $\varepsilon \geqq 0$, we shall consider the convergence of $\|p_{ij}^{(\varepsilon)}\|$ to $\|p_{ij}\|$ ensuring that the drifts $m_i^{(\varepsilon)}$ have a negative bias $-\varepsilon$ with respect to $m_i^{(0)}$, so that $\lim_{i\to\infty} m_i^{(\varepsilon)} = -\varepsilon$. Thus $X^{(\varepsilon)}$ is ergodic, for any $\varepsilon > 0$ (see [14], [16]). To be more exact, we shall enforce the following general conditions:

$$\lim_{i\to\infty,\,\varepsilon\downarrow 0} m_i^{(\varepsilon)} = 0,$$

(2) $$\lim_{i\to\infty,\,\varepsilon\downarrow 0} i(m_i^{(\varepsilon)} + \varepsilon) = -\mu, \quad -\infty \leqq \mu \leqq \infty,$$

(3) $$\lim_{i\to\infty,\,\varepsilon\downarrow 0} b_i^{(\varepsilon)} = b, \quad 0 < b < \infty, \quad \text{and} \quad \sup_{i\geqq 0,\,\varepsilon\geqq 0} b_i^{(\varepsilon)} < \infty.$$

A specific example of such Markov chains is the sequence X_n, defined by the equalities $X_{n+1} = (X_n + \xi_n)^+$, where the ξ_n are i.i.d. random variables with $\mathbb{E}[\xi_n] = -\varepsilon$. Transient phenomena for this type of random walk (in this case $\mu = 0$) and for some more general ones have been considered in [2], [3], [10], [15].

Theorem 1 (ergodicity and stability). If (2), (3) hold, $2\mu > b$ and

$$\sup_{i \geqq 0, \varepsilon \geqq 0} \mathbb{E}[|X_1^{(\varepsilon)} - i|^{2+\gamma} \mid X_0^{(\varepsilon)} = i] < C < \infty, \tag{4}$$

where γ is an arbitrary but strictly positive number, then the chain $X^{(0)}$ is ergodic and becomes stable as $\varepsilon \downarrow 0 : \zeta^{(\varepsilon)} \xrightarrow{\text{D}} \zeta^{(0)}$ as $\varepsilon \downarrow 0$, where, as usual, D indicates convergence in distribution. The case $\mu = \infty$ is covered by the statement of the theorem.

Theorem 2 (convergence to a Γ-distribution). If (2) and (3) hold and $-\infty < 2\mu < b$, then $X^{(0)}$ is non-ergodic. If, moreover, the series representing

$$b_i^{(\varepsilon)} = \sum_{u \geqq -i} p_{i,i+u}^{(\varepsilon)} u^2 \tag{5}$$

converges uniformly with respect to i and ε, then

$$2\varepsilon \zeta^{(\varepsilon)} \Rightarrow \Gamma_{1/b, 1-2\mu/b} \quad \text{as} \quad \varepsilon \downarrow 0,$$

where $\Rightarrow$ means convergence of distributions. It is worth mentioning that condition (5) will be satisfied if we assume merely the more stringent condition (4).

Remark 1

(i) In fact, Theorems 1 and 2 will be proved under slightly more general assumptions. Indeed, in Theorem 1, (2) can be replaced by the inequalities $m_i^{(\varepsilon)} \leqq m_i^{(0)}$, for $i \geqq i_0$, where i_0 is some positive integer. In Theorem 2, (2) can be replaced by the representation

$$m_i^{(\varepsilon)} = -\varepsilon - \frac{\mu}{i} + o\left(\varepsilon + \frac{1}{i}\right) \quad \text{as} \quad i \to \infty, \quad \varepsilon \downarrow 0.$$

(ii) In the case $\mu = -\infty$, the limit distribution of $\zeta^{(\varepsilon)}$, after suitable normalization, will be *normal*. This result will be published later.

It remains to consider the critical case $2\mu = b$.

Theorem 3 (convergence to the uniform distribution). If (2), (3), (4) hold, $2\mu = b$, and

$$2i(m_i^{(\varepsilon)} + \varepsilon) + b_i^{(\varepsilon)} = o\left(\varepsilon + \frac{1}{i}\right) \tag{6}$$

then $X^{(0)}$ is null recurrent and $\log(\zeta^{(\varepsilon)})/\log(1/\varepsilon) \Rightarrow \text{U}[0, 1]$, as $\varepsilon \downarrow 0$, where U[0, 1] denotes the uniform distribution on [0, 1].

Remark 2

(i) The statement of Theorem 3 says that, roughly speaking, $\zeta^{(\varepsilon)}$ is distributed as $\varepsilon^{-\eta}$, where η is a random variable having the distribution U[0, 1].

(ii) If we take a different rate of convergence to 0 in (6), then the asymptotic behaviour of the distribution of $\log(\zeta^{(\varepsilon)})$ can be different too.

The next theorem gives a detailed account of the behaviour of $X^{(0)}$.

Theorem 4 (distinction between null recurrence and transience of $X^{(0)}$). Assuming $\varepsilon = 0$ and that (2), (3) and (4) hold, we have the following classification:

(i) If $2\mu > b$, then $X^{(0)}$ is ergodic (assumption (4) is not necessary here);

(ii) If $-b \leqq 2\mu < b$, then $X^{(0)}$ is null recurrent; this is also the case if $b = 2\mu$ and (6) holds;

(iii) If $b > -2\mu$, then $X^{(0)}$ is recurrent;

(iv) If $0 < b < -2\mu$, then $X^{(0)}$ is transient.

Proof of Theorem 1. The investigation of *stability* in this theorem relies essentially on the weak compactness of the family of distributions $\{\pi_i^{(\varepsilon)}, i \geqq 0\}$, for $\varepsilon \geqq 0$. In this connection, we prove two lemmas which yield the desired conclusions.

Lemma 1. If $\pi_0^{(\varepsilon)}$ does not tend to 0 as $\varepsilon \downarrow 0$, then the system (1) has a probabilistic solution.

Proof. On account of the underlying assumption, there exists a subsequence $\varepsilon_{k'} \downarrow 0$, such that $\pi_0^{(\varepsilon_{k'})} \to \pi_0 > 0$. It follows from Helly's theorem, see [13], that there also exists a subsequence ε_k, with $\varepsilon_{k'} \supseteq \varepsilon_k$, such that, for any $i \geqq 0$,

$$\lim_{\varepsilon_k \downarrow 0} \pi_i^{(\varepsilon_k)} = \pi_i > 0, \qquad \sum_{i \geqq 0} \pi_i \leqq 1.$$

We now have to show that the sequence $\{\pi_0^{(\varepsilon)}, i \geqq 0\}$ also satisfies system (1) when $\varepsilon = 0$. Setting $\varepsilon = \varepsilon_k$ in system (1), we obtain

$$\Big|\pi_i - \sum_{j \geqq 0} \pi_j p_{ji}\Big| = \Big|\pi_i - \pi_i^{(\varepsilon_k)} - \sum_{j \geqq 0} (\pi_j p_{ji} - \pi_j^{(\varepsilon_k)} p_{ji}^{(\varepsilon_k)})\Big|$$
$$\leqq |\pi_i - \pi_i^{(\varepsilon_k)}| + \sum_{j \leqq N} |\pi_j p_{ji} - \pi_j^{(\varepsilon_k)} p_{ji}^{(\varepsilon_k)}| + \sum_{j > N} (\pi_j p_{ji} + \pi_j^{(\varepsilon_k)} p_{ji}^{(\varepsilon_k)}).$$

The first and second terms on the right-hand side tend to 0 because $\pi_i^{(\varepsilon_k)} \to \pi_i$ and $p_{ji}^{(\varepsilon_k)} \to p_{ji}$, as $\varepsilon_k \downarrow 0$. Then, using the inequality $\pi_i < 1$ and Chebyshev's inequality, it follows that

$$\sum_{j > N} (\pi_j p_{ji} + \pi_j^{(\varepsilon_k)} p_{ji}^{(\varepsilon_k)}) \leqq \sum_{j > N} (p_{ji} + p_{ji}^{(\varepsilon_k)}) \leqq 2 \sum_{j > N} \frac{\sup b_i^{(\varepsilon)}}{(j-i)^2} \to 0,$$

as $N \to \infty$. Therefore $\pi_i = \sum_{j \geqq 0} \pi_j p_{ji}$ and the proof of Lemma 1 is concluded.

As mentioned previously, the main idea in studying *stability* in Theorem 1 is to show that, for $\varepsilon \geqq 0$, the family $\{\pi_i^{(\varepsilon)},\ i \geqq 0\}$ is weakly compact. This emerges at once from the next result.

Lemma 2. Suppose that $2\mu > b$. Choose γ (defined in the statement of Theorem 1) arbitrarily small and positive, satisfying

$$0 < \gamma < \frac{2\mu}{b} - 1. \tag{7}$$

Then, under the assumptions of Theorem 1, the moment of order γ of the stationary distributions $\{\pi_i^{(\varepsilon)},\ i \geqq 0\}$ is uniformly bounded with respect to $\varepsilon \geqq 0$:

$$\sup_{\varepsilon \geqq 0} \mathbb{E}[\zeta^{(\varepsilon)\gamma}] = \sup_{\varepsilon \geqq 0} \sum_{i \geqq 0} \pi_i^{(\varepsilon)} i^\gamma < \infty.$$

Proof. We shall prove that the series

$$\sum_{i \geqq 0} \pi_i^{(\varepsilon)} \mathbb{E}[X_{n+1}^{(\varepsilon)\gamma+2} - i^{\gamma+2} \mid X_n^{(\varepsilon)} = i]. \tag{8}$$

converges to a non-negative value. Define $W_n^{(\varepsilon)} = X_{n+1}^{(\varepsilon)} - X_n^{(\varepsilon)}$. We apply the well-known Taylor formula to the function $(1+y)^{\gamma+2}$. Then

$$(1+y)^{\gamma+2} = 1 + y(\gamma+2) + \frac{y^2}{2}(\gamma+1)(\gamma+2)(1+y\theta(y))^\gamma,$$

where $\theta(y)$ is a continuous function of y satisfying

$$0 < \theta(y) < 1, \qquad \theta(-1) = 1 - \left(\frac{2}{\gamma+2}\right)^{1/\gamma} < 1. \tag{9}$$

Hence,

$$\begin{aligned} \mathbb{E}[X_{n+1}^{(\varepsilon)\gamma+2} - i^{\gamma+2} \mid X_n^{(\varepsilon)} = i] &= i^{\gamma+2}\mathbb{E}\left[\left(1 + \frac{W_n^{(\varepsilon)}}{i}\right)^{\gamma+2} - 1 \,\middle|\, X_n^{(\varepsilon)} = i\right] \\ &= (\gamma+2) i^\gamma \mathbb{E}\left[iW_n^{(\varepsilon)} + \frac{(\gamma+1)}{2} W_n^{(\varepsilon)2}\left(1 + \frac{\theta_{ni} W_n^{(\varepsilon)}}{i}\right)^\gamma \,\middle|\, X_n^{(\varepsilon)} = i\right], \end{aligned} \tag{10}$$

where θ_{ni} is a function of $W_n^{(\varepsilon)}$ satisfying $0 < \theta_{ni} < 1$. Upon applying the elementary inequality $|1+a|^\gamma \leqq 1 + |a|^\gamma$, $0 \leqq \gamma \leqq 1$, we obtain, since the random variable $1 + (\theta_{ni} W_n^{(\varepsilon)}/i)$ is always positive,

$$\mathbb{E}\left[W_n^{(\varepsilon)2}\left(1 + \frac{\theta_{ni} W_n^{(\varepsilon)}}{i}\right)^\gamma \,\middle|\, X_n^{(\varepsilon)} = i\right] \leqq b_i^{(\varepsilon)} + i^{-\gamma}\mathbb{E}[|W_n^{(\varepsilon)}|^{\gamma+2} \mid X_n^{(\varepsilon)} = i]. \tag{11}$$

On account of the assumption (4) in Theorem 1, the expectation on the right-hand side of (11) is uniformly bounded with respect to i and ε. Thus, combining (10) and

(11), we obtain

$$\text{(12)} \qquad \mathbb{E}[X_{n+1}^{(\varepsilon)\gamma+2} - i^{\gamma+2} \mid X_n^{(\varepsilon)} = i] \leqq (\gamma+2)i^\gamma\left[im_i^{(\varepsilon)} + \frac{(\gamma+1)}{2} b_i^{(\varepsilon)} + Ci^{-\gamma}\right],$$

where C is the constant introduced in (4). Now using (7), together with the condition $m_i^{(\varepsilon)} \leqq m_i^{(0)} \approx -\mu/i$, for $i \geqq i_0$, it follows that, for some sufficiently large $i_* \leqq i$ and small ε_*, (12) can be rewritten as

$$\text{(13)} \qquad \mathbb{E}[X_{n+1}^{(\varepsilon)\gamma+2} - i^{\gamma+2} \mid X_n^{(\varepsilon)} = i] \leqq Hi^\gamma < 0, \qquad \text{for } i \geqq i_*,\ \varepsilon \leqq \varepsilon_*,$$

where H is a strictly negative constant.

The inequality (13) does play a basic role in our proof. It entails, in particular, that the partial sums S_i of series (8) are decreasing for $i \geqq i_*$, $\varepsilon \leqq \varepsilon_*$. Hence, the limit of these partial sums does exist and is either finite or equal to $-\infty$. In fact, we shall prove the following stronger statement: the sum of the series (8) is non-negative. Indeed, suppose that this sum is negative. Then, there exists a number $i_1 \geqq i_*$, such that $S_{i_1} < 0$. Consider the chain $X_n^{(\varepsilon)}$, with $X_0^{(\varepsilon)} = 0$. Under the assumptions of Theorem 1, we have $\mathbb{E}[X_n^{(\varepsilon)\gamma+2}] < \infty$, for any $n \geqq 0$. Upon now setting $\pi_{ni}^{(\varepsilon)} = \mathbb{P}[X_n^{(\varepsilon)} = i]$, it follows from the ergodicity that $\lim_{n\to\infty} \pi_{ni}^{(\varepsilon)} = \pi_i^{(\varepsilon)}$, for any $i \geqq 0$. This, in turn, implies

$$\sum_{i \leqq i_1} \pi_{ni}^{(\varepsilon)} \mathbb{E}[X_n^{(\varepsilon)\gamma+2} - i^{\gamma+2} \mid X_n^{(\varepsilon)} = i] \to S_{i_1} < 0, \quad \text{as} \quad n \to \infty.$$

Therefore, we have, for n sufficiently large,

$$\mathbb{E}[X_{n+1}^{(\varepsilon)\gamma+2} - X_n^{(\varepsilon)\gamma+2};\ X_n^{(\varepsilon)} \leqq i_1] = -\delta < 0.$$

It follows directly from (13) that

$$\sum_{i > i_1} \pi_{ni}^{(\varepsilon)} \mathbb{E}[X_{n+1}^{(\varepsilon)\gamma+2} - i^{\gamma+2} \mid X_n^{(\varepsilon)} = i] = \mathbb{E}[X_{n+1}^{(\varepsilon)\gamma+2} - X_n^{(\varepsilon)\gamma+2};\ X_n^{(\varepsilon)} > i_1] < 0, \text{ for any } n \geqq 0.$$

Hence, we obtain

$$\mathbb{E}[X_{n+1}^{(\varepsilon)\gamma+2}] \leqq \mathbb{E}[X_n^{(\varepsilon)\gamma+2}] - \delta, \text{ for } n \geqq n_0 \text{ large enough.}$$

Since $a = \mathbb{E}[X_{n_0}^{(\varepsilon)\gamma+2}] < \infty$, we finally get

$$0 < \mathbb{E}[X_{n_0+(a/\delta)+1}^{(\varepsilon)\gamma+2}] \leqq \mathbb{E}[X_{n_0}^{(\varepsilon)\gamma+2}] - \delta\left(\frac{a}{\delta} + 1\right) = -\delta < 0,$$

which is a contradiction. Therefore the sum of series (8) is non-negative. Hence,

$$\sum_{i \geqq i_*} \pi_i^{(\varepsilon)} \mathbb{E}[X_{n+1}^{(\varepsilon)\gamma+2} - i^{\gamma+2} \mid X_n^{(\varepsilon)} = i] \geqq -\sum_{i < i_*} \pi_i^{(\varepsilon)} \mathbb{E}[X_{n+1}^{(\varepsilon)\gamma+2} - i^{\gamma+2} \mid X_n^{(\varepsilon)} = i] \overset{\text{def}}{=} A(i_*).$$

Using (13), it follows that, for $\varepsilon \leqq \varepsilon_*$,

$$\sum_{i \geqq i_*} \pi_i^{(\varepsilon)} i^\gamma \leqq \frac{A(i_*)}{H} < \infty.$$

The proof of Lemma 2 is concluded.

Remark 3. It is indeed easy to verify that, if $2\mu > b$, the Markov chain $X^{(0)}$ is ergodic. Indeed, upon introducing the process $Y_n = [X_n^{(0)}]^2$, which is again an irreducible Markov chain on the positive integers, we obtain

$$\mathbb{E}[Y_{n+1} - Y_n \mid Y_n = i^2] = 2im_i^{(0)} + b_i^{(0)} < -\varepsilon,$$

for $i \geqq I_0$ sufficiently large and the conclusion follows from Pakes' lemma [14].

We are now in a position to prove Theorem 1. According to Lemma 2, $\{\pi_i^{(\varepsilon)}, i \geqq 0\}$ is a weakly compact family of distributions and the weak convergence now becomes straightforward. Let $\{\pi_i, i \geqq 0\}$ be a limit point such that (as $\varepsilon \downarrow 0$) $\pi_i = \lim \pi_i^{(\varepsilon_k)}$, $\varepsilon_k \to 0$. The proof of Lemma 1 entails that $\{\pi_i, i \geqq 0\}$ satisfies system (1), corresponding to $\varepsilon = 0$, and, from weak compactness, that $\sum_{i \geqq 0} \pi_i = 1$. Moreover, it is known that system (1) has a unique solution (see [9]) with the property $\sum_{i \geqq 0} \pi_i = 1$. Thus $\pi_i = \pi_i^{(0)}$. This completes the proof.

Proof of Theorem 2. First, we give a brief description of the main ideas used below. Lemma 3 is simply an intermediate and very intuitive result, which tells us that ergodicity does not depend on a finite number of transition probabilities. In Lemma 4, it will be proved that $2\varepsilon\zeta^{(\varepsilon)}$ is a weakly compact family of random variables. Next, from system (1), we obtain an equation for the generating function of the limit distribution of $2\varepsilon\zeta^{(\varepsilon)}$. Then, we shall overcome the main difficulty: the convergence of $\pi_0^{(\varepsilon)}$ to 0 as $\varepsilon \downarrow 0$ will be shown, allowing us to neglect some terms in the equation for the generating function. Finally, a change of variables yields the convergence of the Laplace transform of the random variable $2\varepsilon\zeta^{(\varepsilon)}$ to the Laplace transform of a Γ-distribution.

Lemma 3. If the irreducible aperiodic transition matrices $\|p_{ij}\|$ and $\|p_{ij}^*\|$ differ only by a finite number of transition probabilities, then the existence of an invariant probability measure for $\|p_{ij}^*\|$ follows from the existence of an invariant measure for $\|p_{ij}\|$.

Proof. We consider the situation where $p_{ij} \neq p_{ij}^*$, only if $i = i_0$ and $j \leqq J$, where i_0 and J are fixed. It is known (see [9]) that an irreducible aperiodic Markov chain X is ergodic iff there exists an invariant probabilistic measure for the matrix $\|p_{ij}\|$. On the other hand, X is ergodic iff it is positive recurrent. Let $r_{ij} = \min\{n > 0 : X_n = j \mid X_0 = i\}$ be the length of the path from the state i to the state j. Positive recurrence means that $\mathbb{E}[r_{ii}] < \infty$. Besides (see [9]) we have $\pi_i = \lim \mathbb{P}\{X_n = i\} = 1/\mathbb{E}[r_{ii}]$. Consider the chains $\{X_n\}$, with $X_0 = i_0$ and $\{X_n^*\}$, with $X_0^* = i_0$. It is easy to see that the value of $\mathbb{E}[r_{ji_0}]$, for $j \neq i_0$, does not depend on the values of p_{i_0j}. Moreover, it is well known that, since $\mathbb{E}[r_{i_0i_0}]<\infty$, $\mathbb{E}[r_{ji_0}]<\infty$, for any finite j. Hence, the formula

$$\mathbb{E}[r^*_{i_0i_0}]=p^*_{i_0i_0}+\sum_{j\leqq J, j\neq i_0} p^*_{i_0j}(1+\mathbb{E}[r_{ji_0}])+\sum_{j>J, j\neq i_0} p_{i_0j}(1+\mathbb{E}[r_{ji_0}])<\infty,$$

holds with and without the asterisks, thus showing that $\mathbb{E}[r_{j_0i_0}]$ and $\mathbb{E}[r^*_{j_0i_0}]$ are finite simultaneously. This completes the proof of the lemma.

According to (2) and (3), let us define

$$m_i^{(\varepsilon)}=-\varepsilon-\frac{\mu}{i}+\Delta(i,\varepsilon)\left(\varepsilon+\frac{1}{i}\right), \tag{14}$$

where $\Delta(i,\varepsilon)\to 0$ as $i\to\infty$, $\varepsilon\downarrow 0$. The ergodicity of $X_n^{(\varepsilon)}$ follows at once from the inequality $m_i^{(\varepsilon)}\leqq -\varepsilon/2$, for i large enough (see [14]).

Lemma 4. The family of random variables $2\varepsilon\zeta^{(\varepsilon)}$ is a weakly compact family and $\sup_{0<\varepsilon\leqq\varepsilon_0} 2\varepsilon\mathbb{E}[\zeta^{(\varepsilon)}]<\infty$, for sufficiently small ε_0.

Proof. Consider the series

$$\sum_{i\geqq 0}\pi_i^{(\varepsilon)}\mathbb{E}[X_{n+1}^{(\varepsilon)2}-i^2 \mid X_n^{(\varepsilon)}=i]. \tag{15}$$

The mathematical expectation in (15) is

$$\begin{aligned}\mathbb{E}[X_{n+1}^{(\varepsilon)2}-i^2 \mid X_n^{(\varepsilon)}=i]&=\mathbb{E}[2iW_n^{(\varepsilon)}+W_n^{(\varepsilon)2} \mid X_n^{(\varepsilon)}=i]\\ &=2im_i^{(\varepsilon)}+b_i^{(\varepsilon)}\leqq -\frac{\varepsilon}{2}i<0,\end{aligned}$$

for i sufficiently large. Following the argument of Lemma 2, we can assert that series (15) is summable and its sum is non-negative. Hence,

$$\sum_{i\geqq 0}\pi_i^{(\varepsilon)}[-2i\varepsilon(1-\Delta(i,\varepsilon))-2\mu+2\Delta(i,\varepsilon)+b_i^{(\varepsilon)}]\geqq 0,$$

or, equivalently,

$$\begin{aligned}2\varepsilon\sum_{i\geqq 0}\pi_i^{(\varepsilon)}i(1-\Delta(i,\varepsilon))&\leqq\sum_{i\geqq 0}\pi_i^{(\varepsilon)}(-2\mu+2\Delta(i,\varepsilon)+b_i^{(\varepsilon)})\\ &\leqq\sup_{i,\varepsilon}|-2\mu+2\Delta(i,\varepsilon)+b_i^{(\varepsilon)}|=A_*<\infty.\end{aligned}$$

There exist i_0 and ε_0 such that $1-\Delta(i,\varepsilon)>1/2$, for $i\geqq i_0$, $\varepsilon\leqq\varepsilon_0$. Therefore

$$\varepsilon\sum_{i\geqq i_0} i\pi_i^{(\varepsilon)}\leqq A_*+2\varepsilon\sum_{i<i_0}\pi_i^{(\varepsilon)}i\,|1-\Delta(i,\varepsilon)|\leqq A_{**}<\infty, \quad \text{for } \varepsilon\leqq\varepsilon_0.$$

The proof of Lemma 4 is concluded.

Denoting by $u_i^{(\varepsilon)}(z)$ the generating function of the distribution $\{p_{i,i+k}^{(\varepsilon)}\}_{k\geqq -i}$, we have $u_i^{(\varepsilon)}(z)=\sum_{k\geqq -i}p_{i,i+k}^{(\varepsilon)}z^k$, $|z|\leqq 1$. Turning back to the proof of Theorem 2, we now derive from system (1) an equation for the generating function of the limit distribution associated to $\zeta^{(\varepsilon)}$.

(16) $$\pi^{(\varepsilon)}(z)=\sum_{i\geqq 0}\pi_i^{(\varepsilon)}z^i u_i^{(\varepsilon)}(z).$$

Unless otherwise stated, from now on, z will always denote a real number with $0\leqq z\leqq 1$.

Lemma 5. If (2) and (3) hold, together with $2\mu<b$, we have $\pi_0^{(\varepsilon)}=\mathbb{P}[\zeta^{(\varepsilon)}=0]\to 0$, as $\varepsilon\downarrow 0$.

Proof. Lemma 1 implies that it suffices to check the non-existence of a probabilistic solution of system (1) corresponding to $\varepsilon=0$. Suppose in fact that system (1), corresponding to $\varepsilon=0$, has a probabilistic solution. For the sake of brevity, we omit the superscript 0. Substituting $\varepsilon=0$ in (16) and differentiating with respect to z, we obtain

(17) $$\sum_{i\geqq 0}\pi_i z^{i-1}f_i(z)=0,$$

where $f_i(z)=i(1-u_i(z))-zu_i'(z)$. By Taylor's formula, we can write

(18) $$f_i(z)=-u_i'(1)+(1-z)[(1+i)u_i'(\theta_i)+\theta_i u_i''(\theta_i)],$$

with $z<\theta_i<1$, $\forall i\geqq 0$. Using (18) in (17) yields

(19) $$\sum_{i\geqq 0}\pi_i z^{i-1}u_i'(1)=(1-z)\sum_{i\geqq 0}\pi_i z^{i-1}[(i+1)u_i'(\theta_i)+\theta_i u_i''(\theta_i)].$$

On the other hand, using Taylor's formula again in (16), we get $\pi(z)=\sum_{i\geqq 0}\pi_i z^i[1+(z-1)u_i'(\beta_i)]$, which in turn implies

(20) $$\sum_{i\geqq 0}\pi_i z^i u_i'(\beta_i)=0,\qquad 0<z<1,$$

where $z<\beta_i<1$, $\forall i\geqq 0$. Since

$$u_i'(\beta_i)=u_i'(1)+(\beta_i-1)u_i''(\gamma_i),\quad \beta_i<\gamma_i<1,$$

we have

(21) $$0=\sum_{i\geqq 0}\pi_i z^i u_i'(1)+\sum_{i\geqq 0}\pi_i\left(\frac{z}{\gamma_i}\right)^i(\beta_i-1)\gamma_i^i u_i''(\gamma_i).$$

The function $u_i''(z)$ has a power series expansion with positive coefficients and, from the basic assumptions made in Theorem 2, $z^{i+2}u_i''(z)<u_i''(1)<\infty$, $\forall i\geqq 0$. Hence, the second term on the right-hand side of (21) is a uniformly convergent series of continuous functions for $0\leqq z\leqq 1$, which tend to 0 as $z\to 1$. Consequently

(22) $$\sum_{i\geqq 0}\pi_i u_i'(1)=0.$$

Note that (22) is in general not valid for Markov chains without further assumptions.

Now, using (22), we rewrite (19) in the form

$$\sum_{i\geqq 0}\pi_i\left(\frac{z^{i-1}-1}{1-z}\right)u_i'(1)=\sum_{i\geqq 0}[(i+1)u_i'(\theta_i)+\theta_i u_i''(\theta_i)]\pi_i z^{i-1},$$

or, equivalently,

$$\sum_{i\geqq 0}\pi_i\left[\left(\frac{1-z^{i-1}}{1-z}\right)u_i'(1)+(i+1)u_i'(1)z^{i-1}+\theta_i u_i''(\theta_i)z^{i-1}\right] \tag{23}$$
$$=\sum_{i\geqq 0}\pi_i(i+1)z^{i-1}[u_i'(1)-u_i'(\theta_i)].$$

We want to prove that the right-hand side of (23) tends to 0 as $z\to 1$. Observe first that

$$\frac{(1-z^i)(u_i'(1))}{1-z}\leqq i\,|u_i'(1)|\approx|\mu|\quad\text{as}\quad i\to\infty,\quad 0\leqq z\leqq 1.$$

As already remarked, $z^i u_i''(z)$ are positive continuous functions, which, by (3) and (5), are uniformly bounded, for all $i\geqq 0$, $0\leqq z\leqq 1$. Thus, the three series in the left-hand side of (23) are uniformly convergent and, consequently, their sums are continuous functions of z on $]0,1]$. Now, by using Taylor's formula, the right-hand side of (23) can be rewritten as

$$g(z)\equiv\sum_{i\geqq 0}\pi_i(i+1)z^{i-1}(1-\theta_i)u_i''(\gamma_i),\quad 0<z<\theta_i<\gamma_i\leqq 1,\quad \forall i\geqq 0.$$

Hence, $g(z)$ represents the sum of a convergent series of continuous positive terms. From what has just been said, $g(z)$ is continuous and we are in a position to apply Dini's theorem, which asserts that the series with sum $g(z)$ is *uniformly convergent,* whence, since $\theta_i\to 1$ as $z\to 1$, $g(1)=0$. Using (22) and letting $z\to 1$ in (23), we obtain the basic equality

$$\sum_{i\geqq 0}a_i\pi_i=0, \tag{24}$$

where $a_i=b_i+2im_i$. Hence, $a_i\approx b-2\mu>0$, for $i\geqq i_0$, i_0 being chosen large enough. The positivity of the numbers $\{a_i,\ i<i_0\}$ depends only on a finite number of transition probabilities $\{p_{ij},\ 0\leqq i\leqq i_0\}$. It then follows by Lemma 3 that we can always assume $a_i>0$ for every $i\geqq 0$. But this contradicts (24). The proof of Lemma 5 (and at the same time of the first part of Theorem 2) is concluded.

To get the weak convergence result stated in Theorem 2, we proceed to the derivation of a functional differential equation, which a suitable Laplace transform does satisfy. To this end, substitute $z=\exp(-2\varepsilon t)$, $t\in\mathbb{R}^+$, and introduce the notation

$$\pi^{(\varepsilon)}(z)=\mathbb{E}[\exp(-2\varepsilon t\zeta^{(\varepsilon)})]\equiv\beta_{2\varepsilon\zeta^{(\varepsilon)}}(t)\equiv\beta^{(\varepsilon)}(t)\quad\text{and}\quad\beta^{(\varepsilon)\prime}(t)\equiv\frac{d\beta^{(\varepsilon)}(t)}{dt}.$$

The function $\beta^{(\varepsilon)}(t)$ is the Laplace transform of the random variable $2\varepsilon\zeta^{(\varepsilon)}$. Moreover,

$$-2\varepsilon z \frac{d}{dz}\pi^{(\varepsilon)}(z) = \frac{d\beta^{(\varepsilon)}(t)}{dt}.$$

Now differentiating (16) with respect to z, as in (17), leads to

$$\sum_{i \geqq 0} \pi_i^{(\varepsilon)} z^{i-1} i(1 - u_i^{(\varepsilon)}(z)) = \sum_{i \geqq 0} \pi_i^{(\varepsilon)} u_i^{(\varepsilon)\prime}(z) z^i. \tag{25}$$

Upon repeatedly using Taylor's formula, we obtain

$$1 - u_i^{(\varepsilon)}(z) = (1-z)\left[m_i^{(\varepsilon)} + \frac{(z-1)}{2} u_i^{(\varepsilon)\prime\prime}(\theta_i)\right], \quad u_i^{(\varepsilon)\prime}(z) = m_i^{(\varepsilon)} + (z-1)u_i^{(\varepsilon)\prime\prime}(\gamma_i), \tag{26}$$

where $z < \gamma_i$, $\theta_i < 1$. Hence, from (16), we get

$$\pi^{(\varepsilon)}(z) = \sum_{i \geqq 0} \pi_i^{(\varepsilon)} z^i \left[1 + (z-1)m_i^{(\varepsilon)} + \frac{(z-1)^2}{2} u_i^{(\varepsilon)\prime\prime}(\theta_i)\right],$$

or

$$\sum_{i \geqq 0} \pi_i^{(\varepsilon)} z^i m_i^{(\varepsilon)} = \frac{1-z}{2} \sum_{i \geqq 0} \pi_i^{(\varepsilon)} z^i u_i^{(\varepsilon)\prime\prime}(\theta_i). \tag{27}$$

Replacing (26) and (27) in (25), and dividing by $(1-z)$, we easily obtain the main equality

$$\sum_{i \geqq 0} \pi_i^{(\varepsilon)} z^{i-1} i\left[m_i^{(\varepsilon)} + \frac{(z-1)}{2} u_i^{(\varepsilon)\prime\prime}(\theta_i)\right] = \sum_{i \geqq 0} \pi_i^{(\varepsilon)} z^i \left[\frac{1}{2} u_i^{(\varepsilon)\prime\prime}(\theta_i) - u_i^{(\varepsilon)\prime\prime}(\gamma_i)\right]. \tag{28}$$

In Equation (28), θ_i and γ_i are in fact (continuous) functions of z. By using (14) and $z = 1 - 2\varepsilon t + o(\varepsilon t)$, we rewrite (28) in a form which explicitly reveals the functions $\beta^{(\varepsilon)}(t)$ and $\beta^{(\varepsilon)\prime}(t)$, as follows:

$$\begin{aligned} &\beta^{(\varepsilon)\prime}(t)[1 + bt + o(1)] + \sum_{i \geqq 0} i\pi_i^{(\varepsilon)} z^{i-1}\left[\Delta(i, \varepsilon)\left(\varepsilon + \frac{1}{i}\right) + \frac{(z-1)}{2}(u_i^{(\varepsilon)\prime\prime}(\theta_i) - b)\right] \\ &\quad = \beta^{(\varepsilon)}(t)[2\mu - b + o(1)] + \sum_{i \geqq 0} \pi_i^{(\varepsilon)} z^i \left[\frac{1}{2} u_i^{(\varepsilon)\prime\prime}(\theta_i) - u_i^{(\varepsilon)\prime\prime}(\gamma_i) + \frac{b}{2}\right]. \end{aligned} \tag{29}$$

The next step consists in showing that, in (29), the two sums tend to 0 as $\varepsilon \to 0$. To that end, we first prove that, for any bounded function $\varphi(i, \varepsilon)$ satisfying the conditions

$$|\varphi(i, \varepsilon)| < K, \quad \forall i, \varepsilon, \quad \sup_{i \geqq i_0, \varepsilon \leqq \varepsilon_0} |\varphi(i, \varepsilon)| \to 0, \quad \text{as} \quad i_0 \to \infty, \quad \varepsilon_0 \to 0,$$

the following equality holds:

$$\lim_{\varepsilon \to 0} \sum_{i \geqq 0} \pi_i^{(\varepsilon)} z^i \varphi(i, \varepsilon) = 0. \tag{30}$$

This emerges directly from the decomposition

$$\sup_{\varepsilon \leqq \varepsilon_0} \left| \sum_{i \geqq 0} \pi_i^{(\varepsilon)} z^i \varphi(i, \varepsilon) \right| \leqq K \sum_{i \leqq i_0} \pi_i^{(\varepsilon)} + \sup_{i > i_0, \varepsilon \leqq \varepsilon_0} |\varphi(i, \varepsilon)|,$$

and from Lemma 5, by letting $i_0 \to \infty$, $\varepsilon_0 \to 0$. In a similar way, upon writing $1 - z = O(\varepsilon t)$, and using the decomposition which led to (30), we get, from Lemmas 4 and 5,

$$\lim_{\varepsilon \to 0} \sum_{i \geqq 0} (1 - z) i \pi_i^{(\varepsilon)} \varphi(i, \varepsilon) = 0. \tag{31}$$

Hence, in order to prove that the sums in (29) are $o(1)$, it suffices, according to (30) and (31), to show that

$$z^i [u_i^{(\varepsilon)''}(\theta_i) - b] \to 0, \quad \text{as} \quad i \to \infty, \quad \varepsilon \to 0, \tag{32}$$

uniformly in ε (or equivalent in z), for all $z \leqq \theta_i \leqq 1$. Since $u_i^{(\varepsilon)}(1) = b_i^{(\varepsilon)} - m_i^{(\varepsilon)}$, it follows from condition (3) that

$$\sup_{i, \varepsilon, z \leqq y \leqq 1} z^i u_i^{(\varepsilon)''}(y) \leqq \sup_{i, \varepsilon, y} y^i u_i^{(\varepsilon)''}(y) < \infty,$$

which in turn implies that (32) will hold if we can check

$$\left| \sum_{u \geqq 0} p_{iu}^{(\varepsilon)} (u - i)^2 (z^i - z^u) \right| \to 0, \quad \text{as} \quad i \to \infty, \ \varepsilon \to 0. \tag{33}$$

For any fixed i, we can write

$$\left| \sum_{u \geqq 0} p_{iu}^{(\varepsilon)} (u - i)^2 (z^i - z^u) \right| \leqq \sum_{|u - i| > J} p_{iu}^{(\varepsilon)} (u - i)^2 + \sum_{|u - i| < J} p_{iu}^{(\varepsilon)} (u - i)^2 |z^i - z^u|, \tag{34}$$

where J is an arbitrary positive number, and denote by T the first term on the right-hand side of (34). Then, under condition (5), given $\delta > 0$, it is possible to choose J, depending on δ but on neither ε nor i, such that

$$T < \delta. \tag{35}$$

On the other hand, by

$$z^i - z^u = (i - u) z^{i + \theta(u - i)} \log z, \quad 0 < \theta < 1,$$

we get the following rough bound, for the second term on the right-hand side of (34):

$$\sum_{|u - i| < J} p_{iu}^{(\varepsilon)} (u - i)^2 |z^i - z^u| < J^3 z^{i - J} |\log z|. \tag{36}$$

Now, fix J (function of δ) to satisfy (34) and (35) and then choose $z = 1 - O(J^{-3})$ to render the right-hand side of (36) smaller than any given positive δ_1: we have thus shown that, uniformly with respect to ε and i and whenever we take $z = 1 - O(J^{-3})$, the left-hand side of (33) becomes smaller than $\delta + \delta_1$, which was the assertion, since δ and δ_1 are arbitrary.

Hence, (32) holds and the sum on the right-hand side of (29) is $o(1)$, when $\varepsilon \to 0$. Quite similar arguments show that the sum on the left-hand side of (29) is also $o(1)$. Finally, from the preceding estimates, the equation we are seeking for the Laplace transform becomes

$$\beta^{(\varepsilon)'}(t)(1+bt+o(1)) = \beta^{(\varepsilon)}(t)(2\mu - b + o(1)) + o(1), \quad \text{as} \quad \varepsilon \downarrow 0.$$

It emerges from Lemma 4 that, for any $t > 0$, we have $\inf_{\varepsilon \geqq 0} \beta^{(\varepsilon)}(t) > 0$. Therefore the following equation holds, uniformly with respect to t taken from an arbitrary compact set on the positive real line,

$$\frac{\beta^{(\varepsilon)'}(t)}{\beta^{(\varepsilon)}(t)} = \frac{2\mu - b}{1+bt} + o(1),$$

or, equivalently,

$$\frac{d}{dt} \ln \beta^{(\varepsilon)}(t) = \frac{2\mu - b}{1+bt} + o(1),$$

whence $\lim_{\varepsilon \downarrow 0} \beta^{(\varepsilon)}(t) = (1+bt)^{2\mu/b-1}$. Thus, we get the stated convergence $\beta_{2\varepsilon\zeta^{(\varepsilon)}}(t) \to \beta(t)$ as $\varepsilon \downarrow 0$, where $\beta(t)$ is the Laplace transform of the distribution $\Gamma_{(1/b),1-(2\mu/b)}$. This completes the proof of Theorem 2.

Proof of Theorem 3. First we give a brief description of the main ideas used in the proof. It is proved in Lemma 8 that $\pi_0^{(\varepsilon)} \to 0$, which, together with Theorem 4, shows the stated null recurrence. Then, with the help of the Dirichlet series $\mathbb{E}[(1+\zeta^{(\varepsilon)})^\sigma]$ and of its derivatives with respect to σ, we obtain functional differential equations, which now cannot be solved in explicit form, but allow us to derive relations for the moments of the random variable $\log(1+\zeta^{(\varepsilon)})/\log(1/\varepsilon)$. More precisely, the following convergence will be proved:

$$\mathbb{E}\left[\left(\frac{\log(1+\zeta^{(\varepsilon)})}{\log(1/\varepsilon)}\right)^m\right] \to \frac{1}{m+1} = \mathbb{E}[\xi^m], \quad \varepsilon \downarrow 0, \quad m = 0, 1, 2, \cdots,$$

where ξ denotes a random variable with a uniform distribution on $[0, 1]$.

Lemma 6. Assume that, for $\varepsilon > 0$ and any positive measurable function f on the positive integers $\{0, 1, \cdots\}$, $\mathbb{E}[f(\zeta^{(\varepsilon)})]$ exists. Then the following relation holds:

$$O = \sum_{i \geqq 0} \pi_i^{(\varepsilon)} \mathbb{E}[f(X_1^{(\varepsilon)}) - f(i) \mid X_0^{(\varepsilon)} = i]. \tag{37}$$

Proof. Immediate by considering the ergodic Markov chain $X_n^{(\varepsilon)}$, with $X_0^{(\varepsilon)} \stackrel{d}{=} \zeta^{(\varepsilon)}$. Then $X_1^{(\varepsilon)} \stackrel{d}{=} X_0^{(\varepsilon)}$. This completes the proof of Lemma 6.

Lemma 7.
(i) For $\varepsilon > 0$ arbitrarily small, $\mathbb{E}[\zeta^{(\varepsilon)1+\gamma}] < \infty$.
(ii) $\sup_{\varepsilon \geqq 0} \mathbb{E}[(\varepsilon\zeta^{(\varepsilon)})^{1+\gamma}] \leqq M < \infty$.

Proof. It follows directly from the arguments presented in Lemmas 3 and 4. Let α be a real number with $0 \leqq \alpha \leqq \gamma$. Using (4) and (6), the inequality (12) now takes the form

$$\mathbb{E}[X_{n+1}^{(\varepsilon)\alpha+2} - i^{\alpha+2} \mid X_n^{(\varepsilon)} = i] \leqq -(\alpha+2)\left[\varepsilon i^{\alpha+1} - i^{\alpha}\left(\frac{\alpha b_i^{(\varepsilon)}}{2} + o(\varepsilon + 1/i)\right) + C\right].$$

Thus

$$\sum_{i \geqq 0} \pi_i^{(\varepsilon)} \mathbb{E}[X_{n+1}^{(\varepsilon)\alpha+2} - i^{\alpha+2} \mid X_n^{(\varepsilon)} = i] \geqq 0,$$

which yields

$$\varepsilon \mathbb{E}[\zeta^{(\varepsilon)1+\alpha}] \leqq \frac{\alpha b}{2} \mathbb{E}[\zeta^{(\varepsilon)\alpha}(1 + o(1))] + C. \tag{38}$$

On the other hand, by Hölder's inequality, we have

$$\mathbb{E}[(\varepsilon\zeta^{(\varepsilon)})^{1+\alpha}] \geqq \mathbb{E}^{1+(1/\alpha)}[(\varepsilon\zeta^{(\varepsilon)})^{\alpha}].$$

Hence, setting $G = \mathbb{E}[\zeta^{(\varepsilon)\alpha}]$ and combining the last two inequalities, we get $G^{1+(1/\alpha)} \leqq DG + C$, where C is the constant introduced in (4) and D is a constant independent of ε. Thus $G < \infty$ and the two assertions of the lemma are obtained by multiplying (38) by ε^{α}, $0 \leqq \alpha \leqq \gamma$. This completes the proof of Lemma 7.

Lemma 8. If (2), (3), (6) hold, then $\pi_0^{(\varepsilon)} = \mathbb{P}\{\zeta^{(\varepsilon)} = 0\} \to 0$, as $\varepsilon \downarrow 0$.

Proof. Lemma 1 implies that it suffices again to check the non-existence of a probabilistic solution of system (1) corresponding to $\varepsilon = 0$. Suppose this is not the case. Then, it has been shown in Lemma 5, after using (22) and (23), that (with the notation of Lemma 5) $\sum_{i \geqq 0} \pi_i[2im_i + u_i''(1)] = 0$. But it follows from (6) that

$$2im_i + u_i''(1) = \frac{b}{2i} + o\left(\frac{1}{i}\right) \geqq 0, \quad i \geqq i_0,$$

where i_0 is taken large enough; the arguments of Lemma 5 can be repeated. This completes the proof of Lemma 8.

Lemma 9. For any $k = 1, 2, \cdots$,

$$2\varepsilon \mathbb{E}[(1 + \zeta^{(\varepsilon)}) \log^k (1 + \zeta^{(\varepsilon)})] = \sum_{i \geqq 0} \pi_i^{(\varepsilon)} k[b + o(1)] \log^{k-1}(1 + i), \tag{39}$$

where, as usual, $o(1)$ denotes a quantity tending to 0 when $\varepsilon \downarrow 0$ and $i \to \infty$. In particular, setting $k = 1$ in (39) yields

$$\lim_{\varepsilon \to 0} 2\varepsilon \mathbb{E}[(1 + \zeta^{(\varepsilon)}) \log (1 + \zeta^{(\varepsilon)})] = b. \tag{40}$$

Proof. The assumptions of Lemma 6 are satisfied by choosing

$$f(x)=(1+x)^\sigma \log^k(1+x),\ \mathrm{Re}(\sigma)<1+\gamma,\ k\geqq 0.$$

Using Taylor's expansion and Lemma 7, we get

$$(41)\quad 0=\sum_{i\geqq 0}\pi_i^{(\varepsilon)}\left[m_i^{(\varepsilon)}f'(i)+\frac{b_i^{(\varepsilon)}}{2}f''(i)+\mathbb{E}\left[\frac{W_0^{(\varepsilon)2}}{2}(f''(i+\theta_i W_0^{(\varepsilon)})-f''(i))\mid X_0^{(\varepsilon)}=i\right]\right],$$

where $0<\theta_i<1$ and

$$W_0^{(\varepsilon)}=X_1^{(\varepsilon)}-X_0^{(\varepsilon)},$$
$$f'(i)=(1+i)^{\sigma-1}\log^{k-1}(1+i)[\sigma\log(1+i)+k],$$
$$f''(i)=(1+i)^{\sigma-2}[\sigma(\sigma-1)\log^k(1+i)$$
$$+k(2\sigma-1)\log^{k-1}(1+i)+k(k-1)\log^{k-2}(1+i)].$$

But all the functions on the right-hand side of (41) are analytic with respect to σ. Thus, by using Lemma 7 and the principle of analytic continuation, (41) is indeed valid for $\mathrm{Re}(\sigma)<2+\gamma$. For our purpose, it suffices to take $\sigma=2$. With this choice of σ, we now have

$$f'(i)=(1+i)\log^{k-1}(1+i)[2\log(1+i)+k],$$
$$f''(i)=2\log^k(1+i)+3k\log^{k-1}(1+i)+k(k-1)\log^{k-2}(1+i).$$

It is easy to check that the first two terms on the right-hand side of (41) produce (39). It remains to estimate the last term in (41). To this end, we note first that, for any real $y\neq 0$,

$$y^2\mathbb{P}(|W_0^{(\varepsilon)}|>|y|)<C|y|^{-\alpha},\quad 0\leqq\alpha\leqq\gamma.$$

Then, taking $|y|=i^{(1-\beta)/3}$, $\beta\geqq 0$ arbitrarily small, we can write

$$|y^2[\log^k(1+i+\theta_i y)-\log^k(1+i)]|\leqq\frac{ky^3\log^{k-1}(1+i)}{1+i}[1+o(i^{-(2+\beta)/3})]$$
$$\leqq ki^{-\beta}\log^{k-1}(1+i)[1+o(i^{-(2+\beta)/3})]$$

The last two inequalities immediately yield

$$\mathbb{E}\left[\frac{W_0^{(\varepsilon)^2}}{2}(f''(i+\theta_i W_0^{(\varepsilon)})-f''(i))\mid X_0^{(\varepsilon)}=i\right]=o(i^{-\delta}),\quad\text{for some }\delta>0.$$

The proof of Lemma 9 is concluded.

Lemma 10. For any $k=1,2,\cdots$,

$$\lim_{\varepsilon\to 0}2\varepsilon\mathbb{E}\left[\frac{(1+\zeta^{(\varepsilon)})\log^{k+1}(1+\zeta^{(\varepsilon)})}{\log^k(1/\varepsilon)}\right]=b.$$

Proof. We proceed in two steps.

(i) The following estimate holds

$$\varepsilon\mathbb{E}[(1+\zeta^{(\varepsilon)})\log^{k+1}(1+\zeta^{(\varepsilon)});1+\zeta^{(\varepsilon)}\geqq\varepsilon^{-(x+1)}]\leqq\varepsilon\mathbb{E}\left[\frac{(1+\zeta^{(\varepsilon)})^{1+\alpha}}{(1+\zeta^{(\varepsilon)})^{\alpha-\beta}};\ 1+\zeta^{(\varepsilon)}\geqq\varepsilon^{-(x+1)}\right],$$

for some β, where $0<\beta<\alpha\leqq\gamma$ and x is a fixed strictly positive real number. Hence, it follows from Lemma 7 that

$$\varepsilon\mathbb{E}[(1+\zeta^{(\varepsilon)})\log^{k+1}(1+\xi^{(\varepsilon)});1+\zeta^{(\varepsilon)}\geqq\varepsilon^{-(x+1)}]\leqq M\varepsilon^{\delta},$$

where we can choose β such that

$$\delta=\alpha x-\beta(1+x)>0,\quad \text{i.e. } \beta<\frac{\alpha x}{1+x}.$$

On the other hand, we have

$$\begin{aligned}2\varepsilon\mathbb{E}\left[\frac{(1+\zeta^{(\varepsilon)})\log^{k+1}(1+\zeta^{(\varepsilon)})}{\log^k(1/\varepsilon)}\right]&=2\varepsilon\mathbb{E}\left[\frac{(1+\zeta^{(\varepsilon)})\log^{k+1}(1+\zeta^{(\varepsilon)})}{\log^k(1/\varepsilon)};1+\zeta^{(\varepsilon)}\leqq\varepsilon^{-(x+1)}\right]\\&\quad+2\varepsilon\mathbb{E}\left[\frac{(1+\zeta^{(\varepsilon)})\log^{k+1}(1+\zeta^{(\varepsilon)})}{\log^k(1/\varepsilon)};1+\zeta^{(\varepsilon)}>\varepsilon^{-(x+1)}\right]\\&\leqq 2\varepsilon\mathbb{E}[(1+\zeta^{(\varepsilon)})\log(1+\zeta^{(\varepsilon)})](1+x)^k+2M\varepsilon^{\delta}\log^{-k}(1/\varepsilon).\end{aligned}$$

Thus, by (40) in Lemma 9 and since x is arbitrary, we infer that

$$\limsup_{\varepsilon\to 0}2\varepsilon\mathbb{E}\left[\frac{(1+\zeta^{(\varepsilon)})\log^{k+1}(1+\zeta^{(\varepsilon)})}{\log^k(1/\varepsilon)}\right]\leqq b. \tag{42}$$

(ii) We have

$$\begin{aligned}\varepsilon\mathbb{E}\left[\frac{(1+\zeta^{(\varepsilon)})\log^{k+1}(1+\zeta^{(\varepsilon)})}{\log^k(1/\varepsilon)}\right]&\geqq\varepsilon\mathbb{E}[(1+\zeta^{(\varepsilon)})\log(1+\zeta^{(\varepsilon)})]\\&\quad+\varepsilon\mathbb{E}\left[(1+\zeta^{(\varepsilon)})\log(1+\zeta^{(\varepsilon)})\left(\frac{\log^k(1+\zeta^{(\varepsilon)})}{\log^k(1/\varepsilon)}-1\right);\ 1+\zeta^{(\varepsilon)}\leqq\varepsilon^{-1}\right].\end{aligned}$$

Define, for a moment, the random variable

$$H^{(\varepsilon)}\equiv\varepsilon(1+\zeta^{(\varepsilon)})\log(1+\zeta^{(\varepsilon)})\left(1-\frac{\log^k(1+\zeta^{(\varepsilon)})}{\log^k(1/\varepsilon)}\right).$$

For any $\alpha\geqq 0$, the modulus of the second term in the right-hand side of the above inequality can be rewritten as

$$\begin{aligned}&\mathbb{E}[H^{(\varepsilon)};1+\zeta^{(\varepsilon)}<\varepsilon^{-1+\alpha}]+\mathbb{E}[H^{(\varepsilon)};\ \varepsilon^{-1+\alpha}\leqq 1+\zeta^{(\varepsilon)}\leqq\varepsilon^{-1}]\\&\qquad\leqq\varepsilon^{\alpha}\log(1/\varepsilon)+\frac{b}{2}[1-(1-\alpha)^k]=\varepsilon^{\alpha}\log(1/\varepsilon)+\frac{bk\alpha}{2}(1+o(1)),\end{aligned}$$

which, from the arguments used in Lemma 9 and since α is arbitrary, tends to 0 when $\varepsilon \to 0$. Hence,

$$\liminf_{\varepsilon\to 0} 2\varepsilon \mathbb{E}\left[\frac{(1+\zeta^{(\varepsilon)}) \log^{k+1}(1+\zeta^{(\varepsilon)})}{\log^k(1/\varepsilon)}\right] \geqq b. \tag{43}$$

The proof of (40) and thus of Lemma 10 is completed by combining (42) and (43). Now, dividing (39) by k in Lemma 9 and using Lemma 10 yields

$$\mathbb{E}\left[\left(\frac{\log(1+\zeta^{(\varepsilon)})}{\log(1/\varepsilon)}\right)^k\right] \to \frac{1}{k+1} = \mathbb{E}[\xi^k], \quad \varepsilon \downarrow 0,\ k = 0, 1, 2, \cdots$$

This completes the proof of Theorem 3.

Proof of Theorem 4. Since we shall be dealing with $X^{(0)}$, we shall simply write $X = \{X_n, n \geqq 0\}$, omitting the superscript. The basic idea is to construct positive supermartingales, via Lyapounov functions of the form X^α, α being a real number (positive or negative), and then to apply the well-known Foster's criteria (see for instance [9]). Define $Z_n = X_n + 1$ and $W_n = X_{n+1} - X_n$. We have, by using Taylor's expansion,

$$\begin{aligned} Z_{n+1}^\alpha - Z_n^\alpha &= \alpha W_n Z_n^{\alpha-1} + \alpha(\alpha-1)\frac{W_n^2}{2}(Z_n + \theta W_n)^{\alpha-2} \\ &= \alpha Z_n^{\alpha-2}\left\{Z_n W_n + (\alpha-1)\frac{W_n^2}{2} + (\alpha-1)\frac{W_n^2}{2}\left[\left(1+\frac{\theta W_n}{Z_n}\right)^{\alpha-2} - 1\right]\right\}, \end{aligned} \tag{44}$$

where $\alpha < 2$ and θ is a random variable satisfying $0 < \theta < 1$. Taking conditional expectation in (44), we get

$$\mathbb{E}[Z_{n+1}^\alpha - Z_n^\alpha \mid Z_n = i] = \alpha i^{\alpha-2}\left[i m_i + \frac{\alpha-1}{2} b_i + \frac{\alpha-1}{2}\Delta_i\right], \tag{45}$$

with $i > 0$ and

$$\Delta_i = \mathbb{E}\left[W_n^2\left[\left(1+\frac{\theta W_n}{Z_n}\right)^{\alpha-2} - 1\right] \,\middle|\, Z_n = i\right].$$

The first step is to show that $\Delta_i = o(1)$, i.e $\Delta_i \to 0$, as $i \to \infty$. To this end, we write $\Delta_i = U_i + V_i$, where

$$U_i = \mathbb{E}\left[W_n^2\left[\left(1+\frac{\theta W_n}{Z_n}\right)^{\alpha-2} - 1\right] \,\middle|\, Z_n = i; |W_n| \leqq i\beta\right],$$

$$V_i = \mathbb{E}\left[W_n^2\left[\left(1+\frac{\theta W_n}{Z_n}\right)^{\alpha-2} - 1\right] \,\middle|\, Z_n = i; |W_n| > i^\beta\right],$$

and β is chosen so that $0 < 3\beta < 1$. From (4), we immediately get $\mathbb{E}[W_n^2 \mid Z_n = i; |W_n| \geqq i^\beta] \leqq C i^{-\beta\gamma}$, where C and γ are the constants introduced in (4). Remarking that Z_n is always strictly positive and $\alpha \leqq 2$, we have $(1+(\theta W_n/Z_n))^{\alpha-2}=O(1)$. Hence, $V_i = O(i^{-\beta\gamma})$. To estimate U_i, we note that, on $\{Z_n = i, |W_n| \leqq i^{\beta}\}$,

$$\left(1+\frac{\theta W_n}{Z_n}\right)^{\alpha-2} - 1 = O(i^{\beta-1}),$$

which yields $U_i = O(i^{3\beta-1})$. Since $0 < 3\beta < 1$, we obtain the estimate $\Delta_i = O(i^{-\rho})$, ρ being a strictly positive constant.

Proof of transience. Assume $b < -2\mu$. Choose $1+(2\mu/b) < \alpha < 0$: then, for i sufficiently large, the right-hand side of (45) becomes negative. This can be rephrased by saying that, outside some compact set, Z_n^{α} is a positive bounded supermartingale (α being negative, i^{α} tends to 0 when $i \to \infty$) and Foster's criterion for transience does apply directly.

The proof of recurrence, when $-2\mu < b$, is quite similar. Fixing now $1+(2\mu/b) > \alpha > 0$ and still using Equation (45), we see that Z_n^{α} is a positive unbounded supermartingale (since α is positive in this case) outside some compact set. This implies, by Foster's criterion, that X is recurrent. The distinction between positive and null recurrence follows directly from Remark 3 at the end of Lemma 2 and from Lemmas 5 and 8. This completes the proof of Theorem 4.

3. Application to the ALOHA network

We illustrate the preceding sections with the Markov chain arising from the model of the original ALOHA packet switching network, originally proposed by Abramson [1], and which was indeed the motivation of our study. Let us first briefly recall the salient features of the system.

(a) A single error-free channel is shared among an infinite population of users (or stations), which retransmit messages of constant length (packets). Time is slotted and may be considered discrete. Users are synchronized with respect to the slots, so that packets are transmitted at the beginning of slots only. Each slot is equal to the time required to transmit a packet.

(b) Each transmission is within reception range of every user. When more than one user transmits simultaneously, packets collide (interfere) and none is received correctly. These collisions are treated as transmission errors and each user must strive to retransmit its colliding packet until it is correctly received. The users all employ the same algorithm for this purpose and have to resolve the contention without the benefit of any other source of information on other user's activity save the common channel.

(c) Each user with a colliding packet will repeatedly transmit each time with a certain probability, until it hits a free slot and thus succeeds.

The main drawback of the ALOHA protocol described above is that, left to their own devices, the nodes congest the channel which, in the absence of additional control, is non-ergodic. The approach suggested in [11] was to let retransmission probabilities be a function of the number of blocked stations at time t. Such a retransmission control policy (RCP) can stabilize the channel. This has been proved in [5], [6], under 'Markov' assumptions, but remains true when the external input process is only stationary [4], [7]. We shall deal here with the Markov situation.

Let A_k be the number of new packets generated by the stations during the kth slot. We shall assume the $\{A_k, k \geqq 0\}$ form a sequence of i.i.d. random variables, with $\mathbb{P}(A_k = i) = c_i$, $i \geqq 0$, and $\mathbb{E}(A_k) = \lambda$, $k \geqq 0$. Let X_k, $k \geqq 0$, be the number of blocked stations at time k (i.e observed at the beginning of the kth slot) and $f(X_k)$ the probability that a blocked station retransmits during this kth slot. Given $\{X_k = n\}$, the random number of messages in the kth slot has a binomial distribution. Hence, $X = \{X_k, k \geqq 0\}$ form a Markov chain. Define the quantity

$$d_n = c_0 n f(n)(1 - f(n))^{n-1} + c_1(1 - f(n))^n. \tag{46}$$

Thus d_n represents, in the wide sense, the mean downward drift of X in the kth slot, given the event $\{X_k = n\}$. We recall, without proof, the main result (see [4], [5], [6], [7]).

Theorem 5

(i) If $\lambda < \liminf_{i\to\infty} d_i$, X is ergodic;

(ii) If $\lambda > \limsup_{i\to\infty} d_i$, X is transient.

Our goal is to analyse the stability and ergodicity of a class of RCPs, in the limit 'zero drift' case. These RCPs are chosen (see [5], [6]) such that there exists

$$r = \lim_{i\to\infty} if(i). \tag{47}$$

Then, (46) gives

$$d = \lim_{i\to\infty} d_i = e^{-r}(rc_0 + c_1). \tag{48}$$

For the problem to be meaningful, we have to choose $c_0 > c_1$. Otherwise d, given by (48), would be a decreasing function of r; then $\lambda > c_1 > d$ and the system could never be ergodic. In fact, we do not restrict the generality by taking $f(i) = r/i$, $i > 0$. Now, with the notation of Section 1, combining (46), (47), (48), we get by a direct computation

$$\begin{aligned} m_i &= \lambda - d - \frac{\mu}{i} + O\left(\frac{1}{i^2}\right), \\ b_i &= \mathbb{E}[(X_{k+1} - X_k)^2 \mid X_k = i] = \varphi + c_0 i f(i)(1 - f(i))^{i-1} - c_1(1 - f(i))^i, \end{aligned} \tag{49}$$

where

$$\mu = \frac{r^2 e^{-r}}{2}[c_0(2-r) - c_1], \qquad \varphi = \mathbb{E}[A_k^2].$$

Thus $b = \lim_{i\to\infty} b_i = \varphi + (c_0 r - c_1)e^{-r}$.

The rest of this section is devoted to the limiting case $\lambda = d$.

Theorem 6. If $\lambda = d$, the Markov chain X of the ALOHA protocol is non-ergodic and the two following main situations can take place:

(i) If $\mu \geqq 0$, which is equivalent to $0 < r \leqq 2 - c_1/c_0$, then X is null recurrent;

(ii) If $b < -2\mu$, then X is transient.

Proof. Putting $s = b - 2\mu$, we first show that s is strictly positive. One can easily write $e^r s = e^r\varphi + (r-1)[c_1(1+r) + c_0 r(r-1)]$. Thus, for $r \geqq 1$, the result is obvious. When $0 \leqq r \leqq 1$, it follows, since $\varphi > \lambda = d = e^{-r}(rc_0 + c_1)$, that

$$\frac{e^r\varphi}{1-r} - [c_1(1+r) + c_0 r(r-1)] > \frac{rc_0 + c_1}{1-r} - [c_1(1+r) + c_0 r(r-1)],$$

which finally yields

$$\frac{rc_0[1 + (1-r)^2] + c_1 r^2}{1-r} > 0.$$

Hence, $s > 0$. Now it is not difficult to see that the proof of Theorem 4 remains valid for the ALOHA process without assuming (4), since the random variables $\{A_k, k \geqq 0\}$ are i.i.d (more precisely, the quantity denoted by V_i in this later theorem is still $o(1)$, as $i \to \infty$). This completes the proof of Theorem 6.

Remark 4. An interesting situation is the *optimal policy* [5], [6], which aims at maximizing d_i (the throughput of the system), with respect to the retransmission probability f, for any $0 < i \leqq \infty$. In this case,

$$f(i) = \frac{c_0 - c_1}{ic_0 - c_1}$$

and, from (45) and (46),

$$r = \frac{c_0 - c_1}{c_0}, \quad d = c_0 e^{-r}. \tag{50}$$

Moreover, when $\lambda = d$, one can check that the expansion in (49) is still valid, provided that r is replaced by its value specified in (50). In this case, Theorem 6 shows that X is *null recurrent* since, now, $\mu = c_0 r^2 e^{-r}/2 > 0$.

Coming back to the general RCPs introduced in this section, let us suppose the input sequence $\{A_k^{(\varepsilon)}, k \geqq 0\}$ is perturbed and depends on some parameter ε, so that $\lambda^{(\varepsilon)} = \mathbb{E}[A_k^{(\varepsilon)}] = d - \varepsilon$, $\varepsilon \geqq 0$. Assume also that the basic conditions (2) and (3) hold, together with

$$m_i^{(\varepsilon)} = -\varepsilon - \frac{\mu}{i} + o\left(\varepsilon + \frac{1}{i}\right),$$

which, from a practical point of view, does not seem to be a drastic restriction. On account of Theorem 6, the Markov chain $X^{(0)}$ is non-ergodic and the sequence $X^{(\varepsilon)}$ is amenable to Theorem 2, depending on the parameter values coming in Theorem 6.

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