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MOMENTS FOR STATIONARY MARKOV CHAINS WITH ASYMPTOTICALLY ZERO DRIFT

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Abstract: We consider a Markov chain on $\mathbb{R}^+$ with asymptotically zero drift and finite second moments of jumps. We assume that the chain has invariant distribution. The paper is devoted to the existence and nonexistence of moments of invariant distribution. Our analysis is based on the technique of test functions.

Keywords: stationary Markov chain, asymptotically zero drift, invariant distribution, heavy-tailed distribution, power moments, Weibull-type moments, test (Lyapunov) functions, equilibrium identity

§ 1. Introduction

Let $X = \{X_n, n \geq 0\}$ be a time homogeneous Markov chain taking values in $\mathbb{R}^+$. Denote by $\xi(x)$, $x \in \mathbb{R}^+$, a random variable corresponding to the jump of the chain at a point x, that is, a random variable with distribution

$$\mathbb{P}\{\xi(x) \in B\} = \mathbb{P}\{X_{n+1} - X_n \in B \mid X_n = x\}, \quad B \in \mathscr{B}(\mathbb{R}).$$

Denote the mean drift at x by $m(x) := \mathbb{E}\xi(x)$ and the second moment of the increment by $b(x) := \mathbb{E}\xi^2(x)$.

In this paper we study a stationary Markov chain X, which means that the distribution π_n of X_n does not change as n varies. Denote this invariant distribution by π. We study the sufficient conditions under which appropriate moments of invariant distribution are finite or infinite.

Mostly investigated is the case of random walk with delay at the origin which is described by the Lindley recursion $X_{n+1} = (X_n + \xi_{n+1})^+$ where ξ_n, $n \geq 1$, are independent identically distributed random variables; given a real x we put $x^+ := \max(x, 0)$ and $x^- := \max(-x, 0)$, so that $x = x^+ - x^-$. Here $\xi(x) = \max(\xi_1, -x)$ and $m(x) = \mathbb{E}\max(\xi, -x) = \mathbb{E}\xi_1 + o(1/x)$ as $x \to \infty$ provided that $b := \mathbb{E}\xi_1^2 < \infty$, and $b(x) \to b$. This Markov chain has invariant distribution if and only if $\mathbb{E}\xi_1 < 0$. In the context of queueing theory, it is well known that, for every $\gamma > 0$, the invariant distribution has finite γth moment, $\mathbb{E}X_0^\gamma < \infty$, if and only if $\mathbb{E}(\xi^+)^{\gamma+1} < \infty$ (see Kiefer and Wolfowitz [1]). Also, the invariant distribution is light-tailed if and only if so is the distribution of ξ; hereinafter we say that a random variable η has (right-)*light-tailed* distribution if $\mathbb{E}e^{\lambda\eta} < \infty$ for some $\lambda > 0$.

Clearly, the asymptotically zero drift, $m(x) \to 0$ as $x \to \infty$, makes the tail of the invariant distribution heavier and reduces the number of finite moments of the invariant distribution. In particular, in Section 2 we prove that almost every Markov chain with asymptotically zero drift has heavy-tailed invariant distribution; hereinafter we say that a random variable η has (right-)*heavy-tailed* distribution if $\mathbb{E}e^{\lambda\eta} = \infty$ for all $\lambda > 0$.

Consider, for example, the critical case where $m(x)$ behaves like $-c/x$ at large x. It is critical because typically the chain has no invariant distribution if $m(x) = o(1/x)$ as $x \to \infty$. The existence of an invariant distribution in the critical case was studied by Lamperti [2]; this study is based on considering the test function $V(x) = x^2$. Then the drift of V at x is equal to $\mathbb{E}\{V(X_{n+1}) - V(X_n) \mid X_n = x\} = 2xm(x) + b(x)$ and if $2xm(x) + b(x) < -\varepsilon$ for all sufficiently large x, then the chain is stable under some mild technical conditions (see [3, Chapter 11]).

[†)] Dedicated to my teacher on the occasion of his 80th birthday; the paper is related to my MSc and PhD Theses.

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In Section 4 we give rather general conditions for existence of moments, while in Section 5 we discuss conditions for nonexistence of moments. Section 6 is devoted to consequences for the critical case. We describe what power moments of the invariant distribution are finite.

In Section 7 we consider the corollaries for the Markov chains where $m(x)$ is slowly tending to zero; that is, where $m(x) \uparrow 0$ as $x \to \infty$ but $m(x)x \to \infty$. This case is intermediate between the critical case of Section 6 and the case of an asymptotically negative drift. For such a Markov chain, we show what Weibull-type moments of the typical form $\mathbb{E}e^{\gamma X_0^\beta}$ are finite and what are infinite. In Section 8 we discuss some ideas how to improve the results that are related to the Weibull type case.

In [4], Menshikov and Popov investigated the behavior of the invariant distribution $\{\pi(x), x \in \mathbb{Z}^+\}$ for countable Markov chains with asymptotically zero drift and bounded jumps. Some rough theorems for the local probabilities $\pi(x)$ were proved; for instance, if $m(x) \sim -\mu/x$ and $b(x) \to \infty$ then, for every $\varepsilon > 0$, there exist constants $c_- > 0$ and $c_+ < \infty$ such that $c_- x^{-2\mu/b-\varepsilon} \le \pi(x) \le c_+ x^{-2\mu/b+\varepsilon}$. A similar result was proved for the case $m(x) \sim -\mu/x^\alpha$, $\alpha \in (0,1)$. To the best of our knowledge there are no results in the literature on the exact asymptotical behavior of $\pi(x)$.

§ 2. Heavy-Tailedness

We start with the following result which states that almost every stationary Markov chain with asymptotically zero drift generates heavy-tailed invariant distribution.

Theorem 1. *Let a Markov chain X have asymptotically zero drift, i.e. $m(x) \to 0$ as $x \to \infty$ and, in addition,*

$$\liminf_{x\to\infty} \ \mathbb{E}\{\xi^2(x); \xi(x) > 0\} > 0. \tag{1}$$

Then any right unbounded invariant distribution of X is heavy-tailed.

Proof. Assume on the contrary that X is stationary and right unbounded with finite exponential moment of order $\lambda > 0$. Then, for every x_0,

$$\mathbb{E}(V(X_1) - V(X_0)) = 0, \tag{2}$$

where $V(x) := \max(e^{\lambda x}, e^{\lambda x_0})$. Since

$$\mathbb{E}(V(X_1) - V(X_0)) \ge \mathbb{E}\{V(X_1) - V(X_0); X_0 > x_0\}$$

and X_0 has right unbounded support, it will be a contradiction with (2), if we prove that, for some x_0,

$$v(x) := \mathbb{E}\{V(X_1) - V(X_0) \mid X_0 = x\} > 0 \quad \text{for all } x > x_0. \tag{3}$$

For all $x > x_0$,

$$v(x) \ge \mathbb{E}e^{\lambda(x+\xi(x))} - e^{\lambda x} = e^{\lambda x}(\mathbb{E}e^{\lambda\xi(x)} - 1).$$

Since $e^y \ge 1 + y$ for all y and $e^y \ge 1 + y + y^2/2$ for all $y > 0$,

$$\mathbb{E}e^{\lambda\xi(x)} - 1 \ge \lambda m(x) + \frac{\lambda^2}{2}\mathbb{E}\{\xi^2(x); \xi(x) > 0\}.$$

Owing to $\lambda m(x) \to 0$ as $x \to \infty$ and the condition (1), there exists sufficiently large x_0 such that the sum on the right-hand side of the latter inequality is positive for all $x > x_0$ which proves (3) and hence the theorem.

Let us show by example that the condition (1) which is some kind of nondegeneracy of jumps, is essential for the theorem to hold. Consider the skip-free Markov chain X_n on $\mathbb{Z}^+$, that is, $\xi(x)$ takes only the values -1, 1, and 0, with probabilities $p_-(x)$, $p_+(x)$, and $1 - p_-(x) - p_+(x)$ respectively, $p_-(0) = 0$. Then the stationary probabilities $\pi(x)$, $x \in \mathbb{Z}^+$, satisfy the equations

$$\pi(x) = \pi(x-1)p_+(x-1) + \pi(x)(1 - p_+(x) - p_-(x)) + \pi(x+1)p_-(x+1),$$

which have the following solution:

$$\pi(x) = \pi(0) \prod_{k=1}^{x} \frac{p_+(k-1)}{p_-(k)}. \tag{4}$$

Consider the case now where $p_+(x) := 1/2(x+1)$ and $p_-(x) := 1/(x+1)$. In this case we have asymptotically zero drift but the stationary probabilities are equivalent to $c/2^x$ so that the invariant distribution is light-tailed. Clearly, here (1) is violated.

§ 3. Equilibrium Identities

As particularly seen from the last section, our approach to proving the existence or nonexistence of moments is based on calculating the mean drift of the appropriate test function of a Markov chain. It is a specific property of the exponential function that its derivative is proportional to the function itself, which makes calculations much easier. For various test functions, say for the power ones, the derivatives are increasing slower than the original function, which makes it necessary to develop some technique for equilibrium identities.

If, for some test function V, the mean of $V(X_0)$ is finite in the stationary regime, then $\mathbb{E}V(X_1) - \mathbb{E}V(X_0) = 0$. Then conditioning on X_0 leads to the equilibrium identity $\mathbb{E}v(X_0) = 0$ where $v(x) := \mathbb{E}\{V(X_1) - V(x) \mid X_0 = x\}$. But if the mean of $V(X_0)$ does not exist, the equilibrium identity may fail.

We start with the following lemma which was proved in [5].

Lemma 1. *Let X be a Markov chain on a measurable space $(S, \mathscr{S})$, and let $V(x)$ be a nonnegative measurable function, $V : S \to \mathbb{R}^+$. If X has invariant distribution π and*

$$\int_S v^+(x)\pi(dx) < \infty, \tag{5}$$

then

$$\int_S v(x)\pi(dx) \geq 0.$$

Without strengthening (5), it is impossible to prove that the integral is equal to zero. Indeed, consider the Markov chain on the set of points $\{2^k,\ k = 0, 1, \dots\}$ with the following transition probabilities: $\mathbb{P}\{X_1 = 1 \mid X_0 = 2^k\} = 1/2$ and $\mathbb{P}\{X_1 = 2^{k+1} \mid X_0 = 2^k\} = 1/2$, so that $m(2^k) = 1/2$ for all k. This Markov chain is stable with invariant probabilities $\pi(2^k) = 2^{-k-1}$. Here (5) is satisfied for $V(x) = x$ but clearly $\int_S v(x)\pi(dx) = 1/2 \neq 0$.

In the next lemma we prove some sufficient condition for the equilibrium identity to hold for the general random variables; it improves Lemma 2.3.1 from [6, Chapter 2] (which assumes that η_1, $\eta_2 \geq 0$ and $\mathbb{E}|\eta_1 - \eta_2| < \infty$) and Lemma 10 from [7] (where η_1 and η_2 are two first members of a stationary sequence and the proof is based on this). A result for Markov chains will be stated at the end of this section.

Lemma 2. *Let η_1 and η_2 be identically distributed random variables. If*

$$\mathbb{E}(\eta_2 - \eta_1)^+ < \infty, \tag{6}$$

then $\mathbb{E}|\eta_2 - \eta_1| < \infty$ and $\mathbb{E}(\eta_2 - \eta_1) = 0$.

Proof. Given $A > 0$, consider random variables

$$\eta_k^{[A]} := \begin{cases} -A, & \text{if } \eta_k < -A, \\ \eta_k, & \text{if } \eta_k \in [-A, A], \\ A, & \text{if } \eta_k > A. \end{cases}$$

Then $\left|\eta_2^{[A]} - \eta_1^{[A]}\right| \le 2A$ and, by the identical distribution of $\eta_1^{[A]}$ and $\eta_2^{[A]}$, $\mathbb{E}\big(\eta_2^{[A]} - \eta_1^{[A]}\big) = 0$. Furthermore, by definition

$$\eta_2^{[A]} - \eta_1^{[A]} \le (\eta_2 - \eta_1)^+, \tag{7}$$

$$\left|\eta_2^{[A]} - \eta_1^{[A]}\right| \le |\eta_2 - \eta_1|. \tag{8}$$

Also, we have the everywhere convergence $\eta_2^{[A]} - \eta_1^{[A]} \to \eta_2 - \eta_1$ as $A \to \infty$. Therefore, by Fatou's lemma applicable due to (7) and (6), we have

$$0 = \limsup_{A\to\infty} \mathbb{E}\big(\eta_2^{[A]} - \eta_1^{[A]}\big) \le \mathbb{E} \limsup_{A\to\infty} \big(\eta_2^{[A]} - \eta_1^{[A]}\big) = \mathbb{E}(\eta_2 - \eta_1). \tag{9}$$

Hence, $\mathbb{E}(\eta_2 - \eta_1)^- < \infty$, because otherwise (9) fails, in view of (6). Together with (6) it yields the first assertion, $\mathbb{E}|\eta_2 - \eta_1| < \infty$. In turn, owing to (8) it allows us to apply the dominated convergence theorem to conclude that $\mathbb{E}\big(\eta_2^{[A]} - \eta_1^{[A]}\big) = 0$ gives in limit $\mathbb{E}(\eta_2 - \eta_1) = 0$. The proof is complete.

As a corollary we obtain the following result for a stationary Markov chain X, with $\eta_k = V(X_k)$; it generalizes the second assertion of Lemma 1 in [5].

Lemma 3. *Let X be a Markov chain on a measurable space $(S, \mathscr{S})$ and let $V : S \to \mathbb{R}$ be a measurable function. If X has invariant distribution π and*

$$\int_S \mathbb{E}\{(V(X_1) - V(x))^+ \mid X_0 = x\}\pi(dx) < \infty, \tag{10}$$

then

$$\int_S v(x)\pi(dx) = 0.$$

Note that (10) does not hold for the Markov chain described after Lemma 1 with $V(x) = x$, because here $\mathbb{E}\{(X_1 - 2^k)^+ \mid X_0 = 2^k\} = 2^k/2$ and so the series

$$\sum_k \mathbb{E}\{(X_1 - 2^k)^+ \mid X_0 = 2^k\}\pi(2^k) = \sum_k 2^{k-1}2^{-k-1}$$

diverges.

§ 4. Existence of Moments

Let $h(x)$ be a decreasing function tending to zero and such that $\liminf_{x\to\infty} h(x)x > 0$. Then the function

$$H(x) := \int_0^x h(y)dy$$

goes to infinity as $x \to \infty$ and is concave.

In order to understand the properties of the invariant distribution in the case $m(x) = -h(x)$, consider the special skip-free Markov chain introduced in Section 2. If we take $p_+(x) = (1 - h(x))/2$ and $p_-(x) = (1 + h(x))/2$, then $m(x) = -h(x)$ and $b(x) = 1$. By (4)

$$\begin{aligned} \log \pi(x) &= \log \pi(0) + \sum_{k=1}^{x} (\log(1 - h(k)) - \log(1 + h(k))) \\ &= \log \pi(0) - 2\sum_{k=1}^{x} h(k) - \frac{2}{3}\sum_{k=1}^{x} h^3(k) - \frac{2}{5}\sum_{k=1}^{x} h^5(k) - \cdots . \end{aligned} \tag{11}$$

In sufficiently regular cases the leading second term behaves like $-2H(x)$, so that $\pi(x) \approx e^{-2H(x)}$. This provides the intuition behind the following result.

Theorem 2. *Let $h(x)$ be a differentiable function and*

$$\frac{d}{dx}\frac{1}{h(x)} \to c \in [0,\infty) \quad \text{as } x \to \infty. \tag{12}$$

Assume that

$$\sup_x b(x) < \infty \tag{13}$$

and that there exists $\gamma > 0$ such that the family of random variables

$$(\xi^+(x))^2 e^{\gamma H(\xi^+(x))}, \quad x \geq 0, \tag{14}$$

is uniformly integrable and

$$\limsup_{x\to\infty}\Bigl(2\frac{m(x)}{h(x)} + (c+\gamma)b(x)\Bigr) < 0. \tag{15}$$

Then $\mathbb{E}e^{\gamma H(X_0)} < \infty$ in a stationary regime.

Proof. The first two derivatives of the test function

$$V(x) := \int_0^x U(y)\,dy, \quad \text{where } U(x) := \int_0^x e^{\gamma H(y)}\,dy,$$

are equal to $U(x)$ and $e^{\gamma H(x)}$. The ratio of the derivatives of $U(x)$ and $\frac{1}{h(x)}e^{\gamma H(x)}$ is equal to

$$\frac{e^{\gamma H(x)}}{\bigl(\frac{d}{dx}\frac{1}{h(x)} + \gamma\bigr)e^{\gamma H(x)}} \to \frac{1}{c+\gamma} \quad \text{as } x \to \infty,$$

since $\frac{d}{dx}\frac{1}{h(x)} \to c$. Also considering that $H(x) \to \infty$, we obtain by l'Hôpital's rule that

$$U(x) \sim \frac{1}{(c+\gamma)h(x)}e^{\gamma H(x)} \quad \text{as } x \to \infty. \tag{16}$$

The increment of the test function $V(x)$ possesses the following Taylor's expansion:

$$V(x+y) = V(x) + U(x)y + e^{\gamma H(x+\theta y)}y^2/2, \tag{17}$$

where $0 \leq \theta = \theta(x,y) \leq 1$. Since H is increasing, we obtain the following upper bound:

$$\begin{aligned} V(x+y) &\leq V(x) + U(x)y + e^{\gamma H(x+y^+)}y^2/2 \\ &= V(x) + U(x)y + e^{\gamma H(x)}e^{\gamma(H(x+y^+)-H(x))}y^2/2. \end{aligned} \tag{18}$$

Since $H'(x) = h(x) \downarrow 0$, $H(x+y^+) - H(x) \to 0$ as $x \to \infty$ for every y. In addition, concavity of H implies that $H(x+y^+) - H(x) \leq H(y^+)$. Then, by (14) the family of the random variables $(\xi^+(x))^2 e^{\gamma(H(x+\xi^+(x))-H(x))}$, $x \geq 0$, is uniformly integrable. Together with the convergence $H(x+\xi^+(x)) - H(x) \to 0$ as $x \to \infty$, it implies that

$$\mathbb{E}e^{\gamma(H(x+\xi^+(x))-H(x))}\xi^2(x) = \mathbb{E}\xi^2(x) + o(1) \quad \text{as } x \to \infty.$$

Inserting this convergence into (18), we obtain the following upper bound for the mean drift of the test function V at x as $x \to \infty$:

$$\begin{aligned} \mathbb{E}V(x+\xi(x)) - V(x) &\leq U(x)m(x) + e^{\gamma H(x)}(b(x)/2 + o(1)) \\ &= \frac{m(x)}{h(x)}\frac{1+o(1)}{c+\gamma}e^{\gamma H(x)} + e^{\gamma H(x)}(b(x)/2 + o(1)) \\ &= \frac{1+o(1)}{2(c+\gamma)}\Bigl(2\frac{m(x)}{h(x)} + (c+\gamma)b(x) + o(b(x)+1)\Bigr)e^{\gamma H(x)}. \end{aligned}$$

Owing to the conditions (15) and (13), there exist x_0 and $\varepsilon > 0$ such that $\mathbb{E}V(x+\xi(x))-V(x) \le -\varepsilon e^{\gamma H(x)}$ for all $x > x_0$. By Lemma 1, this implies that

$$\int_{x_0}^{\infty} e^{\gamma H(x)}\pi(dx) \le \frac{1}{\varepsilon} \sup_{x\in[0,x_0]} (\mathbb{E}V(x+\xi(x)) - V(x)),$$

so that $\mathbb{E}e^{\gamma H(X_0)}$ is finite. The proof is complete.

Note that a theorem of [8] can be used instead of Lemma 1. The first result on the existence of moments for Harris-recurrent stationary Markov chains was proved by Tweedie in [9, Theorem 1] (also see [3, Chapter 14]).

§ 5. Nonexistence of Moments

Theorem 3. *Suppose that* (12) *holds and*

$$\mathbb{E}\xi(x) = O(h(x)) \quad \text{as } x \to \infty. \tag{19}$$

Suppose also that there exists $\gamma > 0$ *such that*

$$\sup_{x\ge 0} \mathbb{E}(\xi^+(x))^2 e^{\gamma H(\xi^+(x))} < \infty \tag{20}$$

and that the family $\{(\xi^-(x))^2, x \ge 0\}$ *is uniformly integrable, i.e.,*

$$\sup_{x\ge 0} \mathbb{E}\{(\xi^-(x))^2; \xi^-(x) > A\} \to 0 \quad \text{as } A \to \infty, \tag{21}$$

so that (13) *follows. If*

$$\liminf_{x\to\infty} \left(2\frac{m(x)}{h(x)} + (c+\gamma)b(x) \right) > 0, \tag{22}$$

then $\mathbb{E}e^{\gamma H(X_0)} = \infty$ *in a stationary regime provided that the corresponding invariant distribution has right-unbounded support.*

Proof. Since $H(x)$ increases, we deduce from (17) the following lower bound:

$$\begin{aligned} V(x+y) &\ge V(x) + U(x)y + e^{\gamma H(x-y^-)} y^2/2 \\ &= V(x) + U(x)y + e^{\gamma H(x)} e^{\gamma(H(x-y^-)-H(x))} y^2/2. \end{aligned} \tag{23}$$

The function $H(x)$ is concave, and so $H(x-y^-)+H(y^-) \to 0$ as $x \to \infty$ for every y. Together with (21), this implies that

$$\mathbb{E}e^{\gamma(H(x-\xi^-(x))-H(x))}\xi^2(x) = b(x) + o(1) \quad \text{as } x \to \infty.$$

Inserting this and (16) into (23), we conclude the following lower bound for the mean drift of V at x as $x \to \infty$:

$$\begin{aligned} \mathbb{E}V(x+\xi(x)) - V(x) &\ge (1+o(1))\frac{m(x)}{(c+\gamma)h(x)} e^{\gamma H(x)} + e^{\gamma H(x)}(b(x)/2 + o(1)) \\ &= \frac{1+o(1)}{2(c+\gamma)} \left(2\frac{m(x)}{h(x)} + (c+\gamma)b(x) + o(b(x)+1) \right) e^{\gamma H(x)}. \end{aligned}$$

By (13) and (22), there exist x_* and $\varepsilon > 0$ such that $v(x) := \mathbb{E}V(x+\xi(x))-V(x) \ge \varepsilon e^{\gamma H(x)}$ for all $x > x_*$. Consider the test function $V_*(x) := \max(V(x_*), V(x))$. Then by definition $\mathbb{E}V_*(x+\xi(x)) - V_*(x) \ge 0$ for all $x \in [0, x_*]$ and

$$v_*(x) := \mathbb{E}V_*(x+\xi(x)) - V_*(x) \ge \mathbb{E}V(x+\xi(x)) - V(x) > 0$$

for all $x \geq x_*$. Owing to the unboundedness of the support of invariant distribution, this yields that

$$\mathbb{E}v_*(X_0) > 0. \tag{24}$$

To prove the theorem, assume on the contrary that $\mathbb{E}e^{\gamma H(X_0)}$ is finite in a stationary regime. Then we are going to prove that

$$\mathbb{E}v_*(X_0) = 0. \tag{25}$$

This contradiction with (24) shows that $\mathbb{E}e^{\gamma H(X_0)}$ could not be finite.

Consider the test functions $V_n(x)$, $n \geq 1$, defined as a linear continuation of V_* at $x_* + n$, specifically

$$V_n(x) := \begin{cases} V_*(x), & \text{if } x \leq x_* + n, \\ V_*(x_* + n) + V_*'(x_* + n)(x - x_* - n), & \text{if } x > x_* + n. \end{cases}$$

The function V is convex, since its second derivative equals $e^{\gamma H(x)} > 0$. Hence, by construction, V_n is convex too, $V_n'(x) \leq V_*'(x_* + n)$ for all x. Thus,

$$u_n(x) := \mathbb{E}|V_n(x + \xi(x)) - V_n(x)| \leq V_*'(x_* + n)\mathbb{E}|\xi(x)|.$$

Owing to the condition (13), we have $\sup_x \mathbb{E}|\xi(x)| < \infty$ which yields $\mathbb{E}u_n(X_0) < \infty$. Then, by Lemma 3,

$$\mathbb{E}v_n(X_0) = 0, \tag{26}$$

where $v_n(x) := \mathbb{E}V_n(x + \xi(x)) - V_n(x)$. The monotone sequence of functions V_n converges to V_*. Now our goal is to show why it is possible to pass to the limit in (26) as $n \to \infty$, in order to obtain (25).

First, by the monotone convergence $V_n \to V_*$, we see for every x that

$$v_n(x) \to v_*(x) \quad \text{as } n \to \infty. \tag{27}$$

Second, by Taylor's expansion,

$$v_n(x) = V_n'(x)m(x) + \mathbb{E}V_n''(x + \theta\xi(x))\xi^2(x)/2. \tag{28}$$

Since $V_n'(x) \leq V_*'(x)$ and, by (16), $V_*'(x) = O(e^{\gamma H(x)}/h(x))$ as $x \to \infty$,

$$V_n'(x)|m(x)| \leq V_*'(x)|m(x)| \leq c_1 e^{\gamma H(x)}, \tag{29}$$

where $c_1 < \infty$ due to (19). We have $V_n''(x) \leq V_*''(x) = e^{\gamma H(x)}$ for $x > x_*$, and so

$$\mathbb{E}V_n''(x + \theta\xi(x))\xi^2(x) \leq \mathbb{E}e^{\gamma H(x+\xi^+(x))}\xi^2(x) \leq e^{\gamma H(x)}\mathbb{E}e^{\gamma H(\xi^+(x))}\xi^2(x)$$

by concavity of H. Applying (20) and (21), we find that

$$\mathbb{E}V_n''(x + \theta\xi(x))\xi^2(x) \leq c_2 e^{\gamma H(x)}. \tag{30}$$

Inserting (29) and (30) into (21) we deduce that v_n possesses the following upper bound:

$$|v_n(x)| \leq (c_1 + c_2/2)e^{\gamma H(x)}.$$

Then $(c_1 + c_2/2)e^{\gamma H(X_0)}$ is an integrable majorant for the family of random variables $|v_n(X_0)|$, $n \geq 1$. By the dominated convergence theorem, we can now integrate (27) against the invariant distribution, so that (26) indeed yields (25). Since (25) contradicts (24), the proof is complete.

§ 6. Power Moments

Taking $h(x) = \min\,(1/x, 1)$, so that $c = 1$ in (12), we obtain from Theorems 2 and 3 the following conditions for existence of power moments.

Corollary 1. *Let the second moments $b(x)$ be bounded and assume that there exists $\gamma > 0$ such that*

$$\sup_{x\geq 0} \mathbb{E}\{(\xi^+(x))^{2+\gamma}; \xi(x) > A\} \to 0 \quad \text{as } A \to \infty. \tag{31}$$

If $2xm(x) + (1+\gamma)b(x) \leq -\varepsilon < 0$ for all sufficiently large x, then the moment of order γ of the invariant distribution of X is finite.

Corollary 2. *Let $m(x) = O(1/x)$ as $x \to \infty$, the family $\{(\xi^-(x))^2, x \geq 0\}$ be uniformly integrable, and there exists $\gamma > 0$ such that*

$$\sup_{x\geq 0} \mathbb{E}(\xi^+(x))^{2+\gamma} < \infty.$$

If $2xm(x) + (1+\gamma)b(x) \geq \varepsilon > 0$ for all sufficiently large x, then the moment of order γ of the invariant distribution of X with right-unbounded support is infinite.

In the case of regular behavior of the first and second moments, we deduce the following criterion.

Corollary 3. *Let, for some $b > 0$ and $\mu > b/2$,*

$$m(x) \sim -\mu/x \quad \text{and} \quad b(x) \to b \quad \text{as } x \to \infty.$$

Let the condition (31) hold for some $\gamma > 0$ and the family of random variables $\{(\xi^-(x))^2, x \geq 0\}$ be uniformly integrable. Then the moment of order γ of the invariant distribution of X is finite if $\gamma < 2\mu/b - 1$, and infinite if the invariant distribution is right-unbounded and $\gamma > 2\mu/b - 1$.

We see that in contrast to the random walks with asymptotically negative drift, in this corollary the finiteness of the γth moment of the invariant distribution requires two (not one) additional moments on jumps and heavily depends on the asymptotic behavior of $m(x)$ and $b(x)$.

We leave out of consideration the critical case $\mu = b/2$ where it is still possible to have invariant distribution (see [10, 11]). Stability of a Markov chain X takes place if, for instance,

$$m(x) = -\frac{b}{2x} - \frac{c_1 + o(1)}{x \log x}, \quad b(x) = b - \frac{c_2}{\log x} \quad \text{as } x \to \infty$$

and $2c_1 + c_2 > b$. Taking $p_+(x) = b/2 - b/4x + c_1/2x\log x + c_2/2\log x$, $p_-(x) = b/2 + b/4x - c_1/2x\log x + c_2/2\log x$ and $p_0(x) = 1 - b - c_2/\log x$ in the example of a skip-free Markov chain from Section 2, we conclude that only the logarithmical moments $\mathbb{E}\log^\gamma X_0$ of order $\gamma < 2c_1 + c_2 - 1$ may be finite in this critical case.

§ 7. Weibull Type Moments

In this section we consider the case where $h(x)x \to \infty$ as $x \to \infty$. The particular cases are $m(x) \sim -\mu x^{-1}\log x$ and $m(x) \sim -\mu/x^\alpha$, $\alpha \in (0, 1)$.

As follows from Corollary 1, in the case $xm(x) \to -\infty$ the invariant distribution has all moments finite provided that so are $\xi^+(x)$, $x \geq 0$. Here we specify Theorem 2 as follows.

Corollary 4. *Suppose that $h(x)$ is differentiable and $\frac{d}{dx}\frac{1}{h(x)} \to 0$ as $x \to \infty$. Suppose also that the condition (13) holds as well as (14) for some $\gamma > 0$. If*

$$\limsup_{x\to\infty} \left(2\frac{m(x)}{h(x)} + \gamma b(x)\right) < 0,$$

then $\mathbb{E}e^{\gamma H(X_0)} < \infty$ for X in a stationary regime.

Theorem 3 implies the following conditions for the nonexistence of Weibull type moments.

Corollary 5. *Suppose that $h(x)$ is differentiable and $\frac{d}{dx}\frac{1}{h(x)} \to 0$ as $x \to \infty$. Suppose also that* (19)–(21) *hold. If*

$$\liminf_{x\to\infty}\left(2\frac{m(x)}{h(x)} + \gamma b(x)\right) > 0,$$

then $\mathbb{E}e^{\gamma H(X_0)} = \infty$ for X in a stationary regime with the right-unbounded support of the invariant distribution.

In the case of regular behavior of the first and second moments we deduce the following criterions. The first is about partial exponential moments.

Corollary 6. *Let, for some $\mu > 0$, $\alpha \in (0,1)$, and $b > 0$, $m(x) \sim -\mu/x^\alpha$ and $b(x) \to b$ as $x \to \infty$. Let the family of random variables $\{(\xi^-(x))^2, x \ge 0\}$ be uniformly integrable and let, for some $\gamma > 0$, the family of random variables $\{e^{\gamma(\xi^+(x))^{1-\alpha}}(\xi^+(x))^2, x \ge 0\}$ be uniformly integrable too. Then $\mathbb{E}e^{\gamma X_0^{1-\alpha}} < \infty$ if $\gamma < 2\mu/(1-\alpha)b$, and $\mathbb{E}e^{\gamma X_0^{1-\alpha}} = \infty$ if $\gamma > 2\mu/(1-\alpha)b$ and the invariant distribution has right-unbounded support.*

The second criterion deals with lognormal moments.

Corollary 7. *Let, for some $\mu > 0$ and $b > 0$, $m(x) \sim -\mu\frac{\log x}{x}$ and $b(x) \to b$ as $x \to \infty$. Let the family of random variables $\{(\xi^-(x))^2, x \ge 0\}$ be uniformly integrable and let, for some $\gamma > 0$, the family of random variables $\{e^{\frac{\gamma}{2}\log^2(1+\xi^+(x))}(\xi^+(x))^2, x \ge 0\}$ be uniformly integrable too. Then $\mathbb{E}e^{\frac{\gamma}{2}\log^2(1+X_0)} < \infty$ if $\gamma < 2\mu/b$, and $\mathbb{E}e^{\frac{\gamma}{2}\log^2(1+X_0)} = \infty$ if $\gamma > 2\mu/b$ and the invariant distribution has right-unbounded support.*

§ 8. Further Results on the Weibull Case

In the present section, without loss of generality we assume that $X_n \ge 1$. Corollary 4 states a rough result in a sense that the constant multiple γ in $e^{\gamma H(x)}$ has a strong impact on the function behavior. The question arises, provided that $m(x) = -h(x)$ and $b(x)$ converges to some $b > 0$, is it possible to take $\gamma = 2/b$, say, is it possible to prove the existence of the moment $\mathbb{E}\frac{1}{X_0^2}e^{\frac{2}{b}H(X_0)}$ or something like this? Our main contribution in this section is that it is possible only in the case $h(x) = o(1/\sqrt{x})$. In the case where $h(x) \ge c/\sqrt{x}$ the answer will be different. Here the situation is similar to that in [12, § 7] where the phenomenon of ranges $[0,1/2)$, $[1/2,2/3)$, $[2/3,3/4),\ldots$, for the parameter of Weibull distribution appeared too.

In order to understand what we can expect, turn again to the special skip-free Markov chain of Section 2. As follows from (11), if $h(x) = 1/x^\alpha$, $\alpha > 1/3$, then

$$\pi(x) = \pi(0)e^{-2\sum_{k=1}^{x}h(k)+O(1)} = e^{-2x^{1-\alpha}/(1-\alpha)+O(1)},$$

so that the asymptotics of $\pi(x)$ is determined only by $H(x)$. If $\alpha \in [1/3, 1/5)$ then it follows again from (11) that

$$\pi(x) = \pi(0)e^{-2\sum_{k=1}^{x}h(k)-\frac{2}{3}\sum_{k=1}^{x}h^3(k)+O(1)} = e^{-2x^{1-\alpha}/(1-\alpha)-2x^{1-3\alpha}/2(1-3\alpha)+O(1)}$$

and the asymptotics of $\pi(x)$ is determined by $H(x) + \int_0^x h^3(y)dy$. And so on.

Note that in the skip-free case all odd moments of $\xi(x)$'s are asymptotically zero. This leads to decomposition only with even powers of $h(x)$. In general it is not so hence the terms $h^{2j+1}(x)$ with odd powers are not excluded.

Since here we are interested in the impact of the rate of decay of $h(x)$ to the existence of moments, we restrict considerations for the case of bounded jumps for simplicity of calculations. We consider only the case $h(x) = o(1/\sqrt{x})$.

Theorem 4. *Suppose that $h(x) = o(1/\sqrt{x})$ as $x \to \infty$, h is differentiable, $h'(x) = o(1/x)$, and*

$$m(x) \le -h(x) + o(1/x) \quad \text{and} \quad b(x) \le b + o(1/xh(x)). \tag{32}$$

Assume that, for some $A < \infty$, $|\xi(x)| \le A$, and $\liminf_{x\to\infty} b(x) > 0$. Then, for every $\delta > 0$,

$$\mathbb{E}\frac{1}{X_0^{1+\delta}} e^{\frac{2}{b}H(X_0)} < \infty,$$

if X is in a stationary regime.

Proof. Consider the test function

$$V(x) := \int_1^x y^{-\delta} e^{\frac{2}{b}H(y)}\, dy,$$

so that $V''(x) = (-\delta x^{-1-\delta} + 2x^{-\delta}h(x)/b)e^{\frac{2}{b}H(x)}$ and

$$V'''(x) = o((x^{-1-\delta} + x^{-\delta}h^2(x))e^{\frac{2}{b}H(x)}) = o(x^{-1-\delta}e^{\frac{2}{b}H(x)}).$$

Since the jumps of the chain are bounded by A, the mean drift of $V(x)$ possesses the following Taylor's expansion:

$$\mathbb{E}V(x+\xi(x)) - V(x) = V'(x)m(x) + V''(x)b(x)/2 + O(V'''(x)) \quad \text{as } x \to \infty.$$

By the conditions on the first two moments,

$$\mathbb{E}V(x+\xi(x)) - V(x) = (x^{-\delta}m(x)$$
$$+(-\delta x^{-1-\delta} + 2x^{-\delta}h(x)/b)b(x)/2 + o(x^{-1-\delta}))e^{\frac{2}{b}H(x)}$$
$$\le (-\delta x^{-1-\delta}b(x)/2 + o(x^{-1-\delta}))e^{\frac{2}{b}H(x)}.$$

The second moments are bounded away from zero and, therefore, there exists $\varepsilon > 0$ such that

$$\mathbb{E}V(x+\xi(x)) - V(x) \le -\varepsilon x^{-1-\delta}e^{\frac{2}{b}H(x)}$$

for all sufficiently large x. As earlier in the proof of Theorem 2, this completes the proof.

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