Local asymptotics for the time of first return to the origin of transient random walk

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Abstract

We consider a transient random walk on $\mathbb{Z}^d$ which is asymptotically stable, without centering, in a sense which allows different norming for each component. The paper is devoted to the asymptotics of the probability of the first return to the origin of such a random walk at time n.

Keywords: multidimensional random walk, transience, first return to the origin, local limit theorem, defective renewal function, locally subexponential distributions, Banach algebra of distributions
2000 MSC: 60G50

1. Introduction

Let S_n, $n \geq 0$, be a random walk in $\mathbb{Z}^d$ generated by independent identically distributed steps $\xi_n = (\xi_{n1}, \ldots, \xi_{nd})$, $n \geq 1$, that is, $S_0 = 0$, $S_n = \xi_1 + \ldots + \xi_n$. Define $\tau_0 = 0$ and recursively $\tau_{n+1} = \min\{k > \tau_n : S_k = 0\}$; by standard convention $\min \emptyset = \infty$. Then $\tau = \tau_1$ is the first return to the origin of the random walk S_n.

In this paper we study the asymptotic behavior of

$$p_n := \mathbb{P}\{\tau = n\} = \mathbb{P}\{\text{the first return to zero occurs at time } n\}$$

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as $n \to \infty$. Put $G(B) = \mathbb{P}\{\tau \in B\}$ and

$$p := \sum_{n=1}^{\infty} p_n = \mathbb{P}\{\tau < \infty\} = G[1, \infty) \leq 1.$$

The measure $u_n := \mathbb{P}\{S_n = 0\}$ on $\mathbb{Z}^+$ is actually the renewal measure generated by the τ's, that is,

$$u_n = \sum_{k=0}^{\infty} \mathbb{P}\{\tau_k = n\} = \sum_{k=0}^{\infty} G^{*k}(n). \tag{1}$$

Then

$$\sum_{n=0}^{\infty} \mathbb{P}\{S_n = 0\} = \sum_{k=0}^{\infty} G^{*k}[0, \infty) = \sum_{k=0}^{\infty} p^k = \frac{1}{1-p}, \tag{2}$$

which implies

$$p = \frac{\sum_{n=1}^{\infty} \mathbb{P}\{S_n = 0\}}{1 + \sum_{n=1}^{\infty} \mathbb{P}\{S_n = 0\}}. \tag{3}$$

The random walk S_n is called aperiodic if Z^d is a minimal lattice for S_n in the sense that, for every $\varepsilon > 0$,

$$\sup_{\lambda \in [-\pi,\pi]^d \setminus [-\varepsilon,\varepsilon]^d} |\mathbb{E} e^{i(\lambda, \xi_1)}| < 1.$$

Aperiodicity is clearly no essential restriction, as the state space can always be redefined, if necessary, so as to make a random walk aperiodic.

In what follows we will be studying aperiodic random walks on $\mathbb{Z}^d$ which are *asymptotically stable* in the following sense: there is sequence $c_n = (c_{n1}, \ldots, c_{nd})$ such that

$$X_n := (S_{n1}/c_{n1}, \ldots, S_{nd}/c_{nd}) \xrightarrow{D} Y = (Y_1, \ldots, Y_d),$$

where Y is a strictly d-dimensional stable random variable. Since this implies that each component of X_n is asymptotically stable, we know that each c_{nr} is in the class $RV(1/\alpha_r)$ of regularly varying at infinity with index $1/\alpha_r$ sequences (see, e.g. Bingham et al. 1987, Section 1.9), where $\alpha_r \in (0, 2]$ is the index of the univariate stable random variable Y_r. Thus $C_n := \prod_1^d c_{nr}$ is in $RV(\eta)$, where $\eta = \sum_1^d 1/\alpha_r \geq d/2$. We need the following local limit theorem, in which g denotes the density function of Y.

Theorem 1. *If S_n is an aperiodic random walk on $\mathbb{Z}^d$ which is asymptotically stable in the above sense, it holds that uniformly for $x \in \mathbb{Z}^d$*

$$C_n \mathbb{P}\{S_n = x\} = g(x_1/c_{n1}, \dots, x_d/c_{nd}) + o(1) \text{ as } n \to \infty.$$

In particular, $u_n = \mathbb{P}\{S_n = 0\} \sim g(0, \dots, 0)/C_n$ as $n \to \infty$.

For $d = 1$ this is the classical local limit theorem of Gnedenko (see Kolmogorov and Gnedenko (1954, § 50); for the case $d = 2$ it is proved in Doney (1991), and as remarked there, the proof extends in a straightforward way to the case $d > 2$. It can also be viewed as a special case of Theorem 6.4 in Griffin (1986), where the more general case of matrix norming is treated.

Since it is known that g and its derivatives are bounded, we deduce the following

Corollary 2. *If S_n is an aperiodic random walk on $\mathbb{Z}^d$ which is asymptotically stable, there exists a constant A such that*

$$\mathbb{P}\{S_n = x\} \quad \leq \quad A/C_n \qquad \text{for all } x \in \mathbb{Z}^d,$$

and, for every fixed k,

$$\mathbb{P}\{S_{n-k} = x\} = \mathbb{P}\{S_n = x\} + o(1/C_n)$$

as $n \to \infty$ uniformly for $x \in \mathbb{Z}^d$.

Remark 3. Since every random walk is transient when $d \geq 3$, a result which for the simplest symmetric random walk goes back to Polya (1921), the requirement of transience only features for $d = 1$ and $d = 2$, when under our assumptions it is equivalent to $\sum_{n=1}^{\infty} 1/C_n < \infty$.

Suppose we know that $\mathbb{P}\{\tau = n\} \in RV(-\gamma)$ for some $\gamma \geq 1$. Then G/p is the so-called *locally subexponential* distribution. In this case, the local asymptotics of the defective renewal function is described in Asmussen et al. (2003, Proposition 12): the following limit exists:

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{\infty} G^{*k}(n)}{G(n)} = \sum_{k=0}^{\infty} kp^{k-1} = \frac{1}{(1-p)^2},$$

so that $\mathbb{P}\{S_n = 0\} \sim \mathbb{P}\{\tau = n\}/(1-p)^2$. This is the intuition behind the following, which is our main result.

Theorem 4. *Let S_n be an aperiodic, transient random walk on $\mathbb{Z}^d$, $d \geq 1$, which is asymptotically stable in the above sense. Then as $n \to \infty$,*

$$\mathbb{P}\{\tau = n\} \sim (1-p)^2 \mathbb{P}\{S_n = 0\} \sim \frac{(1-p)^2 g(0, \ldots, 0)}{C_n}.$$

Remark 5. The special case where $d \geq 3, \mathbb{E}\xi_n = 0$ and $\mathbb{E}\xi_{nj}\xi_{nj} = B_{ij} < \infty$ with $\det B \neq 0$ reads

$$\mathbb{P}\{\tau = n\} \sim (1-p)^2 \mathbb{P}\{S_n = 0\} \sim \frac{(1-p)^2}{(2\pi)^{d/2}\sqrt{\det B}} n^{-d/2}.$$

This was proved in an unpublished communication by one of us, and is quoted in Chapter A.6 of Giacomin (2007); this illustrates the increasing importance of local results such as this in Mathematical Physics. To the best of our knowledge, Theorem 4 was not proved in the literature even in the case of the simplest symmetric random walk on $\mathbb{Z}^d$ with $d \geq 3$.

In the next section we prove Theorem 4 by analytic means via a Banach algebra technique; this is the method that was used in proving the above special case. Then in Section 3 we give a probabilistic proof capturing the most probable way that large values of τ occur.

Note also that the recurrent one-dimensional case $d = 1$ was first studied by Kesten (1963) where it was proved in Theorems 7 and 8 that when $\mathbb{E}\xi = 0$ and $\mathbb{E}\xi^2 < \infty$, we have $\mathbb{P}\{\tau = n\} \sim \sqrt{\frac{\mathrm{Var}\xi}{2\pi}} n^{-3/2}$ as $n \to \infty$; the recurrent case of convergence to a stable law with index $\alpha \in [1, 2]$ was also considered. Two different approaches for proving this equivalence may be found in Bender et al. (2004, Theorem 1.2).

In dimension 2 Jain and Pruitt (1972, Theorem 4.1) proved, assuming zero mean and finite covariance B, that $\mathbb{P}\{\tau = n\} \sim 2\pi\sqrt{\det B} n^{-1} \log^{-2} n$ as $n \to \infty$.

The only local result for dimensions 3 and higher we found is one by Kesten and Spitzer (1963, Theorem 1b) where they proved that $\mathbb{P}\{\tau = n + 1\} \sim \mathbb{P}\{\tau = n\}$ as $n \to \infty$ given $\mathbb{E}\xi = 0$.

In conclusion note that different mechanisms are involved in formation of large deviations of τ in dimensions $d = 1$, $d = 2$ and $d \geq 3$. In the one-dimensional recurrent case, the equation (1) says that we deal with the renewal process generated by τ's where τ has an infinite mean. In principle the same is true for the case $d = 2$ but here the tail of τ is very heavy, it is slowly varying at infinity. In the case $d \geq 3$ transience holds, so that we have the renewal process generated by a defective distribution which yields that the renewal atoms u_n are proportional to p_n; this is the main topic addressed to in the present article.

2. Banach algebra approach

The proof of Theorem 4 is straightforward, if we assume additionally that $\sum_{n=1}^{\infty} u_n < 1$ which is equivalent to $p < 1/2$. The discrete renewal relation (1) between the u's and p's implies that

$$1 + u(s) = \frac{1}{1 - p(s)},$$

where we put $u(s) = \sum_{n=1}^{\infty} u_n s^n$ and $p(s) = \sum_{n=0}^{\infty} p_n s^n$, so that

$$p(s) \quad = \quad \frac{u(s)}{1 + u(s)} = \sum_{n=1}^{\infty} (-1)^{n+1} (u(s))^n, \quad |s| \leq 1.$$

This is equivalent to

$$p_n = \sum_{k=1}^{n} (-1)^{k+1} u_n^{*(k)}$$

where $u_n^{*(k)}$ is the k-fold convolution of $\{u_n\}$. Note that u_n is regularly varying at infinity so that the defective distribution $\{u_n\}$ is locally subexponential, see Asmussen et al. (2003) or Foss et al. (in press, Chapter 4). Then, as follows from Foss et al. (2011, Theorem 4.30),

$$\frac{p_n}{u_n} = \sum_{k=1}^{n} (-1)^{k+1} \frac{u_n^{*(k)}}{u_n} \rightarrow \sum_{k=1}^{\infty} (-1)^{k+1} k (u(1))^{k-1} = \frac{1}{(1 + u(1))^2}.$$

Note $1 + u(1) = 1/(1 - p)$, so $p_n \sim (1 - p)^2 u_n$ as $n \to \infty$.

What happens if $\sum_{n=1}^{\infty} u_n \geq 1$? In this case we cannot expand $u(s)/(1 + u(s))$ as a power series in $u(s)$. Nevertheless this is an analytic function in $u(s)$, for all (complex) s in $|s| \leq 1$, because $u(1) < \infty$ by the assumed transience. We can then apply Theorem 1 from the paper Chover et al. (1973) and get the same result; this reference is based on Banach algebra techniques.

3. Probabilistic approach

The starting point of the probabilistic proof of Theorem 4 is the following result, which holds in any dimension.

Lemma 6. *Let a sequence Q_n which is regularly varying at infinity be such that*

$$\begin{aligned} \mathbb{P}\{S_n = x\} &\leq Q_n \quad \text{for all } x \in \mathbb{Z}^d, & (4) \\ \sum_{n=1}^{\infty} Q_n &< \infty, & (5) \end{aligned}$$

and, for every fixed k,

$$\mathbb{P}\{S_{n-k} = x\} = \mathbb{P}\{S_n = x\} + o(Q_n) \tag{6}$$

as $n \to \infty$ uniformly in all $x \in \mathbb{Z}^d$. Let r_n be any fixed unboundedly increasing sequence. Then

$$\mathbb{P}\{S_n = x, \tau > n\} = (1-p)\mathbb{P}\{S_n = x\} + o(Q_n)$$

as $n \to \infty$ uniformly in $x \in \mathbb{Z}^d$ such that $\|x\| > r_n$.

Proof. It is equivalent to prove the relation

$$\mathbb{P}\{S_n = x, \tau < n\} = p\mathbb{P}\{S_n = x\} + o(Q_n). \tag{7}$$

We start with the following decomposition:

$$\begin{aligned} \mathbb{P}\{S_n = x, \tau < n\} &= \sum_{k=1}^{n-1} \mathbb{P}\{S_n = x, \tau = k\} \\ &= \sum_{k=1}^{n-1} \mathbb{P}\{S_{n-k} = x\}\mathbb{P}\{\tau = k\}. \end{aligned}$$

For every fixed N, we have

$$\begin{aligned} \sum_{k=N}^{n-N} \mathbb{P}\{S_{n-k} = x\}\mathbb{P}\{\tau = k\} &\leq \sum_{k=N}^{n-N} \mathbb{P}\{S_{n-k} = x\}\mathbb{P}\{S_k = 0\} \\ &\leq \sum_{k=N}^{n-N} Q_k Q_{n-k}, & (8) \end{aligned}$$

by the condition (4). For every fixed k, $\mathbb{P}\{S_k = x\} \to 0$ as $\|x\| \to \infty$. Together with (4) it implies that, for every fixed N,

$$\begin{aligned} \sum_{k=n-N}^{n-1} \mathbb{P}\{S_{n-k} = x\}\mathbb{P}\{\tau = k\} &\leq \sum_{k=n-N}^{n-1} \mathbb{P}\{S_{n-k} = x\}Q_k \\ &\leq \sum_{k=n-N}^{n-1} Q_k o(1) = o(Q_n) \end{aligned} \tag{9}$$

as $n \to \infty$ uniformly in $\|x\| \geq r_n$. Finally, by (6), for every fixed N,

$$\sum_{k=1}^{N} \mathbb{P}\{S_{n-k} = x\}\mathbb{P}\{\tau = k\} = \mathbb{P}\{S_n = x\} \sum_{k=1}^{N} \mathbb{P}\{\tau = k\} + o(Q_n) \tag{10}$$

as $n \to \infty$ uniformly in all x.

Combining (8)–(10) we obtain that

$$\begin{aligned} &\limsup_{n\to\infty} \frac{1}{Q_n} \Big| \sum_{k=1}^{n-1} \mathbb{P}\{S_n = x, \tau = k\} - \mathbb{P}\{S_n = x\}\mathbb{P}\{\tau \leq N\} \Big| \\ &\qquad\qquad \leq \limsup_{n\to\infty} \frac{1}{Q_n} \sum_{k=N}^{n-N} Q_k Q_{n-k}. \end{aligned}$$

By regular variation of Q_n at infinity, taking into account that the series (5) converges, we conclude that the right hand side can be made as small as we please by choosing N sufficiently large; this is also a well-known property in the theory of locally subexponential distributions, see e.g Foss et al. (in press, Chapter 4). Now the proof of (7) follows by letting $N \to \infty$.

Lemma 7. *Under the conditions of Lemma 6, as $n \to \infty$,*

$$\mathbb{P}\{\tau = n\} = (1-p)^2 \mathbb{P}\{S_n = 0\} + o(Q_n).$$

Proof. Let m be such that $n = 2m$ in the case of even n and $n = 2m+1$ otherwise. We have

$$\mathbb{P}\{\tau = n\} = \sum_{x \in \mathbb{Z}^d \setminus 0} \mathbb{P}\{S_m = x, \tau = n\}.$$

Since S_m and the sequence $\{S_{m+k} - S_m, k \geq 1\}$ are independent, the xth summand in the latter sum is equal to the product

$$\mathbb{P}\{S_m = x, \tau > m\} \\ \times \mathbb{P}\{S_{m+k} - S_m \neq -x \text{ for all } k = 1, \dots, n-m-1, S_n - S_m = -x\}.$$

The second probability here is equal to

$$\begin{aligned} &\mathbb{P}\{S_n - S_{m+k} \neq 0 \text{ for all } k = 1, \dots, n-m-1, S_n - S_m = -x\} \\ &= \mathbb{P}\{\widetilde{S}_k \neq 0 \text{ for all } k = 1, \dots, n-m-1, \widetilde{S}_{n-m} = -x\} \\ &= \mathbb{P}\{\widetilde{\tau} > n-m, \widetilde{S}_{n-m} = -x\}, \end{aligned}$$

where $\widetilde{S}_k := \widetilde{\xi}_1 + \ldots + \widetilde{\xi}_k$, $\widetilde{\xi}_k := \xi_{n-k+1}$, and $\widetilde{\tau}$ is the first return time to zero of the random walk $\widetilde{S}_k$. The random walk $\widetilde{S}_k$ has the same distribution as S_k, so

$$\begin{aligned} \mathbb{P}\{\tau = n\} &= \sum_{x \in \mathbb{Z}^d \setminus 0} \mathbb{P}\{S_m = x, \tau > m\} \mathbb{P}\{\widetilde{S}_{n-m} = -x, \widetilde{\tau} > n-m\} \\ &= \sum_{x \in \mathbb{Z}^d \setminus 0} \mathbb{P}\{S_m = x, \tau > m\} \mathbb{P}\{S_{n-m} = -x, \tau > n-m\} \\ &= \Sigma_1 + \Sigma_2, \end{aligned} \tag{11}$$

where Σ_1 is the sum over $\|x\| \leq \log n =: r_n$ and Σ_2 is the sum over $\|x\| > \log n$. By the condition (4) and regular variation of Q_n,

$$\begin{aligned} \Sigma_1 &\leq \sum_{\|x\| \leq \log n} \mathbb{P}\{S_m = x\} \mathbb{P}\{S_{n-m} = -x\} \\ &\leq Q_m Q_{n-m} (2 \log n)^d = o(Q_n) \quad \text{as } n \to \infty. \end{aligned} \tag{12}$$

By Lemma 6, as $n \to \infty$,

$$\begin{aligned} \Sigma_2 &= (1-p) \sum_{\|x\| > \log n} \mathbb{P}\{S_m = x, \tau > m\} [\mathbb{P}\{S_{n-m} = -x\} + o(Q_n)] \\ &= (1-p) \sum_{\|x\| > \log n} \mathbb{P}\{S_m = x, \tau > m\} \mathbb{P}\{S_{n-m} = -x\} + o(Q_n). \end{aligned}$$

Repeating these arguments to the first multiple we obtain that

$$\Sigma_2 = (1-p)^2 \sum_{\|x\| > \log n} \mathbb{P}\{S_m = x\} \mathbb{P}\{S_{n-m} = -x\} + o(Q_n).$$

Taking also into account (12) we finally obtain that

$$\Sigma_2 = (1-p)^2 \mathbb{P}\{S_n = 0\} + o(Q_n). \tag{13}$$

Substituting (12) and (13) into (11), we arrive at the desired conclusion.

The proof of Theorem 4 is now immediate, since by Corollary 2 the sequence $Q_n = A/C_n$ satisfies all conditions of Lemma 6, for suitable A. The proof is complete.

Remark 8. The same question may be addressed in the more general setting of matrix norming with the help of results by Griffin (1986). The key point is that the norming sequence $|B_n|$ in his Theorem 6.4 is automatically regularly varying: we owe this comment to Phil Griffin, in a private communication. It is then easy to see that our result extends to this situation whenever $d \geq 3$ or $d = 2$ and transience is assumed.

Acknowledgment

The authors are grateful to the referees for their careful reading and for helpful comments and remarks.

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