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TRANSIENT PHENOMENA FOR REAL-VALUED MARKOV CHAINS*

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(*Translated by N. A. Berestova*)

This paper considers transient phenomena arising in investigations of stationary real-valued ergodic Markov chains. They are similar in a sense to nonergodic chains having paths tending to infinity. This approach enables one to construct approximations for the stationary distributions of the chains.

Let $\{X_n^{(\varepsilon)}\}_{n=0}^{\infty}$ be a sequence (in ε) of homogeneous real-valued Markov chains (in n) with transition function $P^{(\varepsilon)}(x, B)$, $x \in R$, $B \in \mathfrak{B}(R)$, where $\mathfrak{B}(R)$ is the σ-algebra of the Borel sets in R. An invariant measure $\pi^{(\varepsilon)}$ corresponding to the chain $\{X_n^{(\varepsilon)}\}$, i.e., a measure satisfying the equation

$$\pi^{(\varepsilon)}(B) = \int_R P^{(\varepsilon)}(x, B)\, \pi^{(\varepsilon)}(dx), \qquad \pi^{(\varepsilon)}(R) = 1, \tag{1}$$

is our main subject of study. If the chains $\{X_n^{(\varepsilon)}\}$ are ergodic for $\varepsilon > 0$, then the asymptotic behavior of their stationary distribution (as $\varepsilon \to 0$) will be discussed. In the sequel, it is supposed that equation (1) has a unique solution when $\varepsilon > 0$. This is the case if conditions hold for the chains $\{X_n^{(\varepsilon)}\}$ to be ergodic involving a "mean drift" of the chain towards some compact set (see Theorem A) and a "mixing" condition of Doob-Doeblin type (see [2]). In this situation the distribution $P^{(\varepsilon)}(x, n, \cdot)$ converges in variation to $\pi^{(\varepsilon)}(\cdot)$ with the measure $\pi^{(\varepsilon)}(\cdot)$ unique.

Consider a family of random variables (r.v.'s) $\xi^{(\varepsilon)}(x)$ whose distribution coincides with the distribution of the step of the chain $\{X_n^{(\varepsilon)}\}$ from the state x: $P\{x + \xi^{(\varepsilon)}(x) \in B\} = P^{(\varepsilon)}(x, B)$. Below we shall use some regularity conditions. The first one concerns the assumption of "loadability" of the Markov chains $\{X_n^{(\varepsilon)}\}$ meaning that the "average drift" tends to zero: $E\xi^{(\varepsilon)}(x) \to 0$ as $x \to \infty$ and $\varepsilon \downarrow 0$. We then assume that the transient kernel is "weakly continuous" (we shall omit the index (0) for the parameters of the limiting chain $X_n \equiv X_n^{(0)}$): $P^{(\varepsilon)}(x, \cdot) \Longrightarrow P(y, \cdot)$ as $x \to y$, $\varepsilon \downarrow 0$ for any $y \in R$, and that the limiting kernel

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$P(x, \cdot)$ is such that for any compact $K \subset R$ and any $x \in R$ there exists a natural number n_0 such that $P\{X_{n_0} \notin K \mid X_0 = x\} > 0$.

We wish to study the asymptotic behavior of the distribution of $\pi^{(\varepsilon)}$ (as $\varepsilon \downarrow 0$) under moment assumptions on the r.v. $\xi^{(\varepsilon)}(x)$ (listed below). If the limit chain $\{X_n\}$ has a unique invariant distribution, then this is a problem on stability or continuous dependence of $\pi^{(\varepsilon)} \Longrightarrow \pi$ on the parameter $\varepsilon \downarrow 0$. If the chain $\{X_n\}$ does not have a proper distribution ($|X_n| \to \infty$ in probability as $n \to \infty$) then the problem cited above concerns transient phenomena that describe the asymptotic behavior of the invariant distribution of the chain $\{X_n^{(\varepsilon)}\}$ as $\varepsilon \downarrow 0$.

In the present paper we limit ourselves to theorems on transient phenomena for Markov chains assuming values on the positive half-line. The study of the behavior of invariant distributions for chains given on the whole real line in many respects reduces to studying chains given on half-line.

Thus, let $X_n^{(\varepsilon)} \geqq 0$. We use the following notation: $m^{(\varepsilon)}(x) = E\,\xi^{(\varepsilon)}(x)$, and $b^{(\varepsilon)}(x) = E\,(\xi^{(\varepsilon)}(x))^2$. The main regularity condition relates to the behavior of $m^{(\varepsilon)}(x)$ and $b^{(\varepsilon)}(x)$ as $x \to \infty$, $\varepsilon \downarrow 0$ and is stated completely in (3). In particular, this condition implies that $\lim_{x\to\infty} xm(x) = \mu$, $-\infty \leqq \mu \leqq \infty$, and $\lim_{x\to\infty} b(x) = b$, $0 < b < \infty$, $\sup_x b(x) < \infty$ exist for the "limiting chain". The parameters μ and b characterizing the asymptotic behavior of the first two moments of jumps in the process play the decisive role in classifying the asymptotic behavior of the distribution $\pi^{(\varepsilon)}$. They are also essential to the fulfillment of the ergodicity conditions for the chain $\{X_n\}$ (see, e.g., [4]). The following statement holds true. Put $\tau(x) = \min\{n \geqq 1\colon X_n \leqq A \mid X_0 = x\}$.

THEOREM A [4]. *If* $2xm(x) + b(x) \leqq -\delta < 0$ *for* $x \geqq A$ *and* $2xm(x) + b(x) \leqq C < \infty$ *for* $x < A$, *then* $\sup_{x\leqq A} E\,\tau(x) < \infty$ (*uniform positive recurrence of the set* [0, A]).

As is generally known, the uniform positive recurrence of $[0, A]$ implies ergodicity of $\{X_n\}$ under wide assumptions. For example, the existence of a probability measure φ on R, a number $p > 0$, and a natural number $n_0 \geqq 1$ such that

$$P\{X_{n_0} \in B \mid X_0 = x\} \geqq p\varphi(B) \tag{2}$$

for every $x \in [0, A]$, $B \in \mathfrak{B}$, and the aperiodicity of the chain $\{X_n\}$ are sufficient (see, for instance, [5] and [8]). In our case fulfillment or nonfulfillment of the hypotheses of Theorem A is determined by the value of $2\mu/b$. From the statement cited above it follows that $\{X_n\}$ is ergodic if $2\mu/b \leqq -1$ and (2) holds.

We now formulate the regularity conditions for $m^{(\varepsilon)}(x)$ and $b^{(\varepsilon)}(x)$. We consider the dependence of $P^{(\varepsilon)}(x, \cdot)$ on the parameter ε such that $\lim_{x\to\infty} m^{(\varepsilon)}(x) = -\varepsilon$ and the chains $\{X_n^{(\varepsilon)}\}$ are ergodic when $\varepsilon > 0$ and (2) is satisfied. Moreover, we suppose that

$$\begin{gathered} m^{(\varepsilon)}(x) = -\varepsilon + \mu/x + o(\varepsilon + 1/x), \qquad x \to \infty, \quad \varepsilon \downarrow 0, \quad -\infty < \mu < \infty, \\ \sup_{x,\varepsilon} b^{(\varepsilon)}(x) < \infty, \qquad \lim_{x\to\infty,\varepsilon\downarrow 0} b^{(\varepsilon)}(x) = b, \quad 0 < b < \infty. \end{gathered} \tag{3}$$

The sequence $\{X_n^{(\varepsilon)}\}$ determined by $X_{n+1}^{(\varepsilon)} = (X_n^{(\varepsilon)} + \xi_n^{(\varepsilon)})^+$, where $x^+ = \max(0, x)$, and $\xi_n^{(\varepsilon)}$ are independent uniformly distributed r.v.'s with $E\,\xi_n^\varepsilon = -\varepsilon$, $E\,(\xi_n^{(\varepsilon)})^2 \to b$, is a special case of such a double array. Limit theorems for a random walk of this type are considered in [1]–[3] (in this case $\mu = 0$).

We assume that the following conditions hold in the theorems:

the chains $\{X_n^{(\varepsilon)}\}$ are homogeneous,

the chains $\{X_n^{(\varepsilon)}\}$ have a unique invariant distribution for $\varepsilon > 0$,

the transient kernel satisfies the continuity condition.

THEOREM 1 (stability, $2\mu < -b$). *Let the asymptotic representations* (3) *hold and let*

$$\sup_{x,\varepsilon} E\left\{\left(\xi^{(\varepsilon)}(x)\right)^2;\ \xi^{(\varepsilon)}(x) > N\right\} \longrightarrow 0, \qquad N \to \infty.$$

If $2\mu < -b$ *and the chain* $\{X_n\}$ *has a unique invariant distribution* π, *then* $\pi^{(\varepsilon)} \Longrightarrow \pi$ *weakly as* $\varepsilon \downarrow 0$.

In the sequel it will be more convenient for us to study the asymptotic behavior of $\pi^{(\varepsilon)}$ not in terms of this distribution itself but in terms of $X^{(\varepsilon)}$ having the distribution $\pi^{(\varepsilon)}$. We denote the distribution of X by $\mathfrak{L}(X)$.

THEOREM 2 (convergence to Γ distribution, $2\mu > -b$). *Let the asymptotic representations* (3) *hold and let*

(4) $$\sup_{x,\varepsilon} E\left\{\left(\xi^{(\varepsilon)}(x)\right)^2;\ \left|\xi^{(\varepsilon)}(x)\right| > N\right\} \longrightarrow 0, \qquad N \to \infty.$$

If $\infty > 2\mu > -b$, *then*

$$\mathfrak{L}(2\varepsilon X^{(\varepsilon)}) \Longrightarrow \Gamma_{1/b,1+2\mu/b}$$

weakly, where $\Gamma_{\alpha,\lambda}$ *is the Gamma distribution with parameters* α *and* λ.

It turns out that without improvement of the remainders in the asymptotic representation (3) in the case $2\mu = -b$, there does not exist, generally speaking, a collective limit theorem for the r.v. $X^{(\varepsilon)}$. Introduce notation for the iterated logarithms and their products:

$$l_0(x) \equiv x, \qquad l_{k+1}(x) = \log(l_k(x)), \qquad L_k(x) = \prod_{m=1}^{k} l_m(x).$$

THEOREM 3 (the critical case, $2\mu = -b$). *Suppose that* $1 \leqq k < \infty$,

$$m^{(\varepsilon)}(x) = -\varepsilon + \mu/x + \sum_{s=1}^{k} \alpha_s/x\, L_s(x) + o\left(\varepsilon + 1/x\, L_k(x)\right),$$

$$b^{(\varepsilon)}(x) = b + \sum_{s=1}^{k} \beta_s/L_s(x) + o\left(\varepsilon + 1/L_k(x)\right),$$

as $x \to \infty, \varepsilon \downarrow 0$ *and* $\sup_{x,\varepsilon} E\,|\xi^{(\varepsilon)}(x)|^{2+\delta} < \infty$ *for some* $\delta > 0$. *Let* $2\mu = -b, 2\alpha_1 + \beta_1 = -b, \ldots,$ *and* $2\alpha_{k-1} + \beta_{k-1} = -b$.

(a) *If* $2\alpha_k + \beta_k < -b$ *and the chain* $\{X_n\}$ *has a unique invariant distribution, then* $\pi^{(\varepsilon)} \Longrightarrow \pi$ *weakly as* $\varepsilon \downarrow 0$.

(b) *If* $2\alpha_k + \beta_k > -b$, *then*

$$\mathfrak{L}\left(\left(l_k(X^{(\varepsilon)})/l_k(1/\varepsilon)\right)^{1+(2\alpha_k+\beta_k)/b}\right) \Longrightarrow U[0,\ 1]$$

weakly, where $U[0, 1]$ *is the uniform distribution on* $[0, 1]$.

THEOREM 4 (convergence to a normal distribution, $\mu = \infty$). *Suppose that*

$$m^{(\varepsilon)}(x) = -\varepsilon + \alpha/x^{\lambda} + o\left(\varepsilon^{(1+\lambda)/2\lambda} + 1/x^{(1+\lambda)/2}\right), \qquad b^{\varepsilon}(x) = b + o(1)$$

as $x \to \infty$, $\varepsilon \downarrow 0$ *and condition* (4) *is satisfied. If* $\alpha > 0$ *and* $0 < \lambda < 1$, *then* $E X_n^{(\varepsilon)} \sim (\alpha/\varepsilon)^{1/\lambda}$ *as* $\varepsilon \downarrow 0$ *and*

$$\mathfrak{L}\left(\left(X^{(\varepsilon)} - E\,X^{(\varepsilon)}\right)\varepsilon^{(1+\lambda)/2\lambda}\right) \Longrightarrow N(0,\, b\,\alpha^{1/\lambda}/2\lambda)$$

weakly, where $N(\beta, \sigma^2)$ *is the normal distribution with parameters* β *and* σ^2.

If $m^{(\varepsilon)}(x) = -\varepsilon + l(x)/x^{\lambda} + o(\cdot)$, where $0 < \lambda < 1$ and $l(x) > 0$ is a slowly varying function, then there is convergence to the normal law under very broad conditions on the function $l(x)$ as before.

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ESTIMATION OF THE MAXIMUM OF A NONPARAMETRIC SIGNAL TO WITHIN A CONSTANT*

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(*Translated by A. E. Shemyakin*)

1. Introduction. Let a stochastic process $X_\varepsilon(t)$ be observed having on the interval [0, 1] the stochastic differential

$$dX_\varepsilon(t) = S(t)\,dt + \varepsilon\,db(t), \tag{1}$$

where $\varepsilon > 0$ is a small parameter and $b(\cdot)$ is a standard Wiener process. It is required to estimate the functional

$$F(\cdot) = F(S(\cdot)) = \sup_{t\in[0,1]} S(t)$$

from the observations over a trajectory of the process $X_\varepsilon(t)$, $0 \leqq t \leqq 1$, under the following a priori assumptions on the signal $S(\cdot)$. Denote by $\Sigma(\beta, L)$, $0 < \beta \leqq 1$, $L > 0$, the class of functions $g(\cdot)$ satisfying a Hölder condition on [0,1] with exponent β and constant L:

$$|g(t_1) - g(t_2)| \leqq L|t_1 - t_2|^\beta, \qquad t_1,\, t_2 \in [0,1].$$

We will assume the signal $S(\cdot)$ to belong to $\Sigma(\beta, L)$ for some known β and L.

Let us consider for an arbitrary measurable function of $X_\varepsilon(\cdot)$ (an estimator $\widetilde{\theta}_\varepsilon = \widetilde{\theta}_\varepsilon(X_\varepsilon(\cdot))$), a risk of the form

$$R_\varepsilon(\widetilde{\theta}_\varepsilon, \beta) = \sup_{S\in\Sigma(\beta,L)} E_{S(\cdot)} w\left(\varphi_\varepsilon^{-1}\left(\widetilde{\theta}_\varepsilon - F(S(\cdot))\right)\right). \tag{2}$$

Here $\varphi_\varepsilon = \{\varepsilon^2 \log(1/\varepsilon)\}^{\beta/(2\beta+1)}$ is a normalizing factor, $E_{S(\cdot)}$ is the expectation with respect to the measure generated by the process $X_\varepsilon(\cdot)$ providing that the true value of the signal in (1) is $S(\cdot)$ and $w(\cdot)$ is the loss function (l.f.) with the customary (see [1]) properties: It is

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