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Asymptotics for Sums of Random Variables with Local Subexponential Behaviour

Søren Asmussen,[1] Serguei Foss,[2] and Dmitry Korshunov[2]

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We study distributions F on $[0, \infty)$ such that for some $T \leqslant \infty$, $F^{*2}(x, x+T] \sim 2F(x, x+T]$. The case $T = \infty$ corresponds to F being subexponential, and our analysis shows that the properties for $T < \infty$ are, in fact, very similar to this classical case. A parallel theory is developed in the presence of densities. Applications are given to random walks, the key renewal theorem, compound Poisson process and Bellman–Harris branching processes.

KEY WORDS: Sums of independent random variables; subexponential distributions; distribution tails; local probabilities.

1. INTRODUCTION

For a probability distribution F on the real line, let $F(x) = F(-\infty, x]$ denote the distribution function and $\overline{F}(x) = F(x, \infty) = 1 - F(x)$ the tail. The class $\mathscr{S}$ of subexponential distributions is defined by the requirement $\overline{F^{*2}}(x) \sim 2\overline{F}(x)$ as $x \to \infty$ ($F^{*n} = n$th convolution power) and that the support is contained in $[0, \infty)$. This class plays an important role in many applications (see, e.g., Refs. 9, 14, 25, 22, and Ref. 2, Chap. IX). For example, one of the key results in the theory is:

Theorem 1. Let $S_n = \xi_1 + \cdots + \xi_n$ be a sequence of partial sums of i.i.d. random variables with common distribution F, and let τ be an

[1] Department of Mathematical Statistics, Lund University, Box 118, 221 00 Lund, Sweden. E-mail: asmus@maths.lth.se

[2] Sobolev Institute of Mathematics, 630090 Novosibirsk, Russia; and Heriot-Watt University, Edinburgh, EH14 4AS, United Kingdom. E-mail: S.Foss@ma.hw.ac.uk and D.Korshunov @ma.hw.ac.uk

independent integer-valued random variable. If $F \in \mathscr{S}$ and $\mathbf{E}(1+\delta)^{\tau} < \infty$ for some $\delta > 0$, then $\mathbf{P}(S_\tau > x) \sim \mathbf{E}\tau \cdot \overline{F}(x)$ as $x \to \infty$.

Special cases of this result provide asymptotics for tails of waiting times in the GI/G/1 queue, for ruin probabilities and Bellmann–Harris branching processes (see further the references in later parts of this paper).

Any subexponential distribution is long-tailed, i.e., for any fixed T, $\overline{F}(x+T) \sim \overline{F}(x)$ as $x \to \infty$. This easily yields $F^{*n}(x, x+T] = o(\overline{F}(x))$ for all $T < \infty$ and all n. Some applications, however, call for more detailed properties of $F^{*n}(x, x+T]$ when $T < \infty$, but the theory is more scattered so the references that we know of are few: Section 2 in Chover *et al.*[10] gave local theorems for some classes of lattice distributions; densities were considered in Section 2 in Ref. 10 (requiring continuity) and in Klüppelberg[20] who considered asymptotics of densities for a special case (see also Sgibnev[23] for some results on the densities on **R**); and finally Bertoin and Doney[6] and Asmussen *et al.*[4] dealt with the case where F is the ladder height distribution in a random walk in order to provide more detailed asymptotics of the random walk maximum than the standard consequences of Theorem 1.

The aim of the present paper is to develop a more systematic theory. Fix $0 < T \leqslant \infty$ and write $\Delta = (0, T]$,

$$x + \Delta \equiv \{x + y : y \in \Delta\} = (x, x+T], \qquad x \in \mathbf{R}.$$

Motivated from Ref. 4, we call F (concentrated on $[0, \infty)$) *Δ-subexponential* if the function $F(x+\Delta)$ is long-tailed (see Definition 1 later) and $F^{*2}(x+\Delta) \sim 2F(x+\Delta)$ (where $g(x) \sim h(x)$ means that $g(x)/h(x) \to 1$, $x \to \infty$). Here $T = \infty$ corresponds to ordinary subexponential distributions. We will see that all standard examples of subexponential distributions are also Δ-subexponential when $T < \infty$, and that the standard theory for $T = \infty$ carries over to $T < \infty$ practically without changes. We thereby provide a general theory covering both the classical subexponential case and some of the more refined questions encountered in Ref. 4, and we also give some further applications motivating this generalization, see for example the results from renewal theory in Section 6.

In Section 2, we derive the properties of Δ-subexponential distributions and prove a natural analogue of Theorem 1. In Section 3, we define distributions with subexponential densities and study their properties. In Section 4, sufficient conditions for Δ-subexponentiality are given. In Section 5, we apply results from Sections 2 and 3 to the asymptotic description of the distribution of the supremum of a random walk with negative drift. The rest of the paper contains further applications to Compound Poisson Processes, Infinitely Divisible Laws, Bellman–Harris Branching Processes, and the Key Renewal Theorem.

2. Δ-SUBEXPONENTIAL DISTRIBUTIONS

Definition 1. We say that a distribution F on $\mathbf{R}$ belongs to the class $\mathscr{L}_\Delta$ if $F(x+\Delta) > 0$ for all sufficiently large x and

$$\frac{F(x+t+\Delta)}{F(x+\Delta)} \to 1 \qquad \text{as} \quad x \to \infty, \tag{1}$$

uniformly in $t \in [0, 1]$.

Calling a function $g(x)$ *long-tailed* if $g(x+t)/g(x) \to 1$ uniformly in $t \in [0, 1]$, we see that the definition is equivalent to $F(x+\Delta)$ being long-tailed. If $T = \infty$, then we write $\mathscr{L}$ instead of $\mathscr{L}_\Delta$ and say that F is long-tailed. It follows from the definition that one can choose a function $h(x) \to \infty$ such that (1) holds uniformly in $|t| \leqslant h(x)$.

Proposition 1. Let the distributions F and G belong to the class $\mathscr{L}_\Delta$ for some Δ. Then $F * G \in \mathscr{L}_\Delta$ and

$$\liminf_{x \to \infty} \frac{(F * G)(x+\Delta)}{F(x+\Delta)+G(x+\Delta)} \geqslant 1. \tag{2}$$

Proof. Let ξ and η be two independent random variables with corresponding distributions F and G. Take an increasing function $h(x) \uparrow \infty$ such that $h(x) < x/2$, $F(x-y+\Delta) \sim F(x+\Delta)$, and $G(x-y+\Delta) \sim G(x+\Delta)$ as $x \to \infty$ uniformly in $|y| \leqslant h(x)$. Consider the event $B(x, t) = \{\xi+\eta \in x+t+\Delta\}$. The estimate (2) follows from the inequality

$$\mathbf{P}(B(x, 0)) \geqslant \mathbf{P}(B(x, 0), |\xi| \leqslant h(x)) + \mathbf{P}(B(x, 0), |\eta| \leqslant h(x))$$

combined with

$$\begin{aligned}
\mathbf{P}(B(x, 0), |\xi| \leqslant h(x)) &= \int_{-h(x)-0}^{h(x)} G(x-y+\Delta)\, F(dy) \\
&\sim G(x+\Delta) \int_{-h(x)-0}^{h(x)} F(dy) \\
&\sim G(x+\Delta), \\
\mathbf{P}(B(x, 0), |\eta| \leqslant h(x)) &\sim F(x+\Delta).
\end{aligned}$$

The probability of the event $B(x,t)$ is equal to the sum

$$\begin{aligned}\mathbf{P}(B(x,t),\ \xi\leqslant x-h(x))+\mathbf{P}(B(x,t),\ \eta\leqslant h(x))\\ +\mathbf{P}(B(x,t),\ \xi>x-h(x),\ \eta>h(x))\\ \equiv P_1(x,t)+P_2(x,t)+P_3(x,t).\end{aligned}$$

In order to prove that $F*G\in\mathcal{L}_\Delta$, we need to check that $\mathbf{P}(B(x,t))\sim\mathbf{P}(B(x,0))$ as $x\to\infty$ uniformly in $t\in[0,1]$. This follows from the relations

$$\begin{aligned}P_1(x,t)&=\int_{-\infty}^{x-h(x)}G(x+t-y+\Delta)\,F(dy)\\ &\sim\int_{-\infty}^{x-h(x)}G(x-y+\Delta)\,F(dy)=P_1(x,0),\end{aligned}$$

$P_2(x,t)\sim P_2(x,0)$, by the same reasons, and

$$\begin{aligned}P_3(x,t)&=\int_{x-h(x)}^{x-h(x)+t+T}\mathbf{P}(\eta\in x+t-y+\Delta,\ \eta>h(x))\,F(dy)\\ &\leqslant\mathbf{P}(\eta>h(x))\,F(x-h(x)+(0,t+T])=o(F(x+\Delta)).\end{aligned}$$

By induction, Proposition 1 yields

Corollary 1. Let $F\in\mathcal{L}_\Delta$ for some Δ. Then, for any $n\geqslant 2$, $F^{*n}\in\mathcal{L}_\Delta$ and

$$\liminf_{x\to\infty}\frac{F^{*n}(x+\Delta)}{F(x+\Delta)}\geqslant n.$$

Definition 2. Let F be a distribution on $\mathbf{R}^+$ with unbounded support. We say that F is *Δ-subexponential* and write $F\in\mathcal{S}_\Delta$ if $F\in\mathcal{L}_\Delta$ and

$$(F*F)(x+\Delta)\sim 2F(x+\Delta)\qquad\text{as}\quad x\to\infty.$$

Equivalently, a random variable ξ has a Δ-subexponential distribution if the function $\mathbf{P}(\xi\in x+\Delta)$ is long-tailed and, for two independent copies ξ_1 and ξ_2 of ξ,

$$\mathbf{P}(\xi_1+\xi_2\in x+\Delta)\sim 2\mathbf{P}(\xi\in x+\Delta)\qquad\text{as}\quad x\to\infty.$$

Note that $\mathcal{S}_{\Delta_1}\neq\mathcal{S}_{\Delta_2}$ if $\Delta_1\neq\Delta_2$; see the corresponding Examples 1 and 2 in Section 4.

Remark 1. The class of $\mathbf{R}^+$-subexponential distributions coincides with the standard class $\mathcal{S}$ of subexponential distributions. Typical examples of $\mathscr{S}_\Delta$ distributions (for all $T>0$) are the same, in particular the Pareto, lognormal, and Weibull (with parameter between 0 and 1) distributions, as will be shown in Section 4. Also, many properties of $\mathscr{S}_\Delta$-distributions with finite Δ are very close to those of subexponential distributions, as will be shown below. However, a main difference is that for $T<\infty$, the function $F(x+\Delta)$ may be non-monotone in x, whereas it is non-increasing for $T=\infty$.

Remark 2. It follows from the definition that, if $F \in \mathscr{S}_\Delta$ for some finite interval $\Delta=(0,T]$, then $F \in \mathscr{S}_{n\Delta}$ for any $n=2,3,\ldots$ and $F \in \mathscr{S}$. Indeed, for any $n \in \{2,3,\ldots,\infty\}$,

$$\mathbf{P}(\xi_1+\xi_2 \in x+n\Delta) = \sum_{k=0}^{n-1} \mathbf{P}(\xi_1+\xi_2 \in x+kT+\Delta)$$

$$\sim 2\sum_{k=0}^{n-1} \mathbf{P}(\xi \in x+kT+\Delta) = 2\mathbf{P}(\xi \in x+n\Delta).$$

Remark 3. In Ref. 10, the authors consider the class of distributions concentrated on the integers and such that $F(\{n+1\}) \sim F(\{n\})$ and $F^{*2}(\{n\}) \sim 2F(\{n\})$ as $n\to\infty$. These distributions are Δ-subexponential with $\Delta=(0,1]$.

Proposition 2. Assume $F[0,\infty)=1$ and $F \in \mathscr{L}_\Delta$ for some Δ. Let ξ_1 and ξ_2 be two i.i.d. random variables with distribution F. The following assertions are equivalent:

(i) $F \in \mathscr{S}_\Delta$;

(ii) there exists a function h such that $h(x)\to\infty$, $h(x)<x/2$, and $F(x-y+\Delta) \sim F(x+\Delta)$ as $x\to\infty$ uniformly in $|y| \leqslant h(x)$,

$$\mathbf{P}(\xi_1+\xi_2 \in x+\Delta,\ \xi_1>h(x),\ \xi_2>h(x)) = o(F(x+\Delta)) \qquad \text{as} \quad x\to\infty; \tag{3}$$

(iii) the relation (3) holds for every function h such that $h(x)\to\infty$.

Proof. Note that if (3) is valid for some $h(x)$, then it follows for any $h_1 \geqslant h$. For $h(x)<x/2$, the probability of the event $B=\{\xi_1+\xi_2 \in x+\Delta\}$ is equal to

$$\mathbf{P}(B,\ \xi_1 \leqslant h(x)) + \mathbf{P}(B,\ \xi_2 \leqslant h(x)) + \mathbf{P}(B,\ \xi_1>h(x),\ \xi_2>h(x))$$

and the conclusions of the proposition follow from

$$\begin{aligned}\mathbf{P}(B, \xi_1 \leqslant h(x)) &= \mathbf{P}(B, \xi_2 \leqslant h(x)) \\ &= \int_0^{h(x)} F(x-y+\Delta)\, F(dy) \\ &\sim F(x+\Delta) \int_0^{h(x)} F(dy) \sim F(x+\Delta).\end{aligned}$$

Now we prove that the class $\mathcal{S}_\Delta$ is closed under a certain local tail equivalence relation.

Lemma 1. Assume that $F \in \mathcal{S}_\Delta$ for some Δ. If the distribution G on $\mathbf{R}^+$ belongs to $\mathcal{L}_\Delta$ and

$$0 < \liminf_{x\to\infty} \frac{G(x+\Delta)}{F(x+\Delta)} \leqslant \limsup_{x\to\infty} \frac{G(x+\Delta)}{F(x+\Delta)} < \infty, \tag{4}$$

then $G \in \mathcal{S}_\Delta$. In particular, $G \in \mathcal{S}_\Delta$, provided $G(x+\Delta) \sim cF(x+\Delta)$ as $x \to \infty$ for some $c \in (0, \infty)$.

Proof. Take a function $h(x) \to \infty$ such that $h(x) < x/2$ and $G(x-y+\Delta) \sim G(x+\Delta)$ as $x \to \infty$ uniformly in $|y| \leqslant h(x)$. Let ζ_1 and ζ_2 be independent random variables with common distribution G. By Proposition 2(ii), it is sufficient to prove that

$$I \equiv \mathbf{P}(\zeta_1 + \zeta_2 \in x+\Delta, \zeta_1 > h(x), \zeta_2 > h(x)) = o(G(x+\Delta)).$$

We have

$$\begin{aligned}I &= \int_{h(x)}^{x-h(x)} G(x-y+\Delta)\, G(dy) \\ &\quad + \int_{x-h(x)}^{x-h(x)+T} \mathbf{P}(\zeta_1 \in x-y+\Delta, \zeta_1 > h(x))\, G(dy) \\ &\equiv I_1 + I_2,\end{aligned}$$

where

$$I_2 \leqslant \mathbf{P}(\zeta_1 > h(x))\, G(x-h(x)+\Delta) = o(G(x+\Delta))$$

and, by condition (4), for some $c_1 < \infty$ and for all sufficiently large x,

$$I_1 \leqslant c_1 \int_{h(x)}^{x-h(x)} F(x-y+\Delta)\, G(dy)$$

$$\leqslant c_1 \mathbf{P}(\zeta_1 + \xi_2 \in x+\Delta,\, \zeta_1 > h(x),\, \xi_2 > h(x))$$

$$= c_1 \int_{h(x)}^{x-h(x)} G(x-y+\Delta)\, F(dy)$$

$$+ c_1 \int_{x-h(x)}^{x-h(x)+T} \mathbf{P}(\zeta_1 \in x-y+\Delta,\, \zeta_1 > h(x))\, F(dy).$$

Here ξ_1 and ξ_2 are independent random variables with common distribution F. Hence, by using the same arguments as before and Proposition 2(iii),

$$I_1 \leqslant c_1^2 \int_{h(x)}^{x-h(x)} F(x-y+\Delta)\, F(dy) + c_1 \mathbf{P}(\zeta_1 > h(x))\, F(x-h(x)+\Delta)$$

$$\leqslant c_1^2 \mathbf{P}(\xi_1 + \xi_2 \in x+\Delta,\, \xi_1 \geqslant h(x),\, \xi_2 \geqslant h(x)) + o(F(x+\Delta))$$

$$= o(F(x+\Delta)) = o(G(x+\Delta)).$$

Proposition 3. Assume that $F \in \mathscr{S}_\Delta$ for some Δ. Let G_1, G_2 be two distributions on $\mathbf{R}_+$ such that $G_1(x+\Delta)/F(x+\Delta) \to c_1$ and $G_2(x+\Delta)/F(x+\Delta) \to c_2$ as $x \to \infty$, for some constants $c_1, c_2 \geqslant 0$. Then

$$\frac{(G_1 * G_2)(x+\Delta)}{F(x+\Delta)} \to c_1 + c_2 \qquad \text{as} \quad x \to \infty.$$

If $c_1 + c_2 > 0$ then, by Lemma 1, $G_1 * G_2 \in \mathscr{S}_\Delta$.

Proof. Take two independent random variables ζ_1 and ζ_2 with distributions G_1 and G_2. Take a function h as before. The probability of the event $B = \{\zeta_1 + \zeta_2 \in x+\Delta\}$ is equal to the sum

$$\mathbf{P}(B, \zeta_1 \leqslant h(x)) + \mathbf{P}(B, \zeta_2 \leqslant h(x)) + \mathbf{P}(B, \zeta_1 > h(x), \zeta_2 > h(x)).$$

We have that (see the proof of Proposition 1), as $x \to \infty$,

$$\frac{\mathbf{P}(B, \zeta_1 \leqslant h(x))}{F(x+\Delta)} \to c_2, \qquad \frac{\mathbf{P}(B, \zeta_2 \leqslant h(x))}{F(x+\Delta)} \to c_1.$$

Following the arguments of Lemma 1, we obtain that

$$\mathbf{P}(B, \zeta_1 > h(x), \zeta_2 > h(x)) = o(F(x+\Delta)).$$

The proposition is proved.

By induction, Proposition 3 implies the following

Corollary 2. Assume that $F \in \mathscr{S}_\Delta$ for some $T \in (0, \infty]$ and $G(x+\Delta)/F(x+\Delta) \to c \geqslant 0$ as $x \to \infty$. Then for any $n \geqslant 2$, $G^{*n}(x+\Delta)/F(x+\Delta) \to nc$ as $x \to \infty$. If $c > 0$, then $G^{*n} \in \mathscr{S}_\Delta$.

Let $\{\xi_n\}$ and $\{\zeta_n\}$ be two sequences of i.i.d. non-negative random variables with common distributions $F(B) = \mathbf{P}(\xi_1 \in B)$ and $G(B) = \mathbf{P}(\zeta_1 \in B)$ respectively. Put $S_n = \zeta_1 + \cdots + \zeta_n$.

Proposition 4. Assume that $F \in \mathscr{S}_\Delta$ for some Δ and $G(x+\Delta) = O(F(x+\Delta))$ as $x \to \infty$. Then, for any $\varepsilon > 0$, there exist $x_0 = x_0(\varepsilon) > 0$ and $V(\varepsilon) > 0$ such that, for any $x > x_0$ and for any $n \geqslant 1$,

$$G^{*n}(x+\Delta) \leqslant V(\varepsilon)(1+\varepsilon)^n F(x+\Delta).$$

Proof. For $x_0 \geqslant 0$ and $k \geqslant 1$, put

$$A_k \equiv A_k(x_0) = \sup_{x > x_0} \frac{G^{*k}(x+\Delta)}{F(x+\Delta)}.$$

Take any $\varepsilon > 0$. Following the arguments of Lemma 1, we conclude the relation, as $x \to \infty$,

$$\mathbf{P}(\xi_1 + \zeta_2 \in x+\Delta, \xi_1 > h(x), \zeta_2 > h(x)) = o(F(x+\Delta)).$$

Hence, there exists x_0 such that, for any $x > x_0$,

$$\mathbf{P}(\xi_1 + \zeta_2 \in x+\Delta, \zeta_2 \leqslant x - x_0) \leqslant (1+\varepsilon/2)\, F(x+\Delta).$$

For any $n > 1$ and $x > x_0$,

$$\mathbf{P}(S_n \in x+\Delta) = \mathbf{P}(S_n \in x+\Delta, \zeta_n \leqslant x - x_0) + \mathbf{P}(S_n \in x+\Delta, \zeta_n > x - x_0)$$

$$\equiv P_1(x) + P_2(x),$$

where, by the definition of A_{n-1} and x_0,

$$P_1(x) = \int_0^{x-x_0} \mathbf{P}(S_{n-1} \in x-y+\Delta)\, \mathbf{P}(\zeta_n \in dy)$$

$$\leqslant A_{n-1} \int_0^{x-x_0} F(x-y+\Delta)\, \mathbf{P}(\zeta_n \in dy)$$

$$= A_{n-1}\mathbf{P}(\xi_1+\zeta_n \in x+\Delta, \zeta_n \leqslant x-x_0)$$

$$\leqslant A_{n-1}(1+\varepsilon/2)\, F(x+\Delta). \tag{5}$$

Further,

$$P_2(x) = \int_0^{x_0+T} \mathbf{P}(\zeta_n \in x-y+\Delta, \zeta_n > x-x_0)\, \mathbf{P}(S_{n-1} \in dy)$$

$$\leqslant \sup_{0<t\leqslant x_0} \mathbf{P}(\zeta_n \in x-t+\Delta) \int_0^{x_0+T} \mathbf{P}(S_{n-1} \in dy)$$

$$\leqslant \sup_{0<t\leqslant x_0} \mathbf{P}(\zeta_n \in x-t+\Delta).$$

Thus, if $x > 2x_0$, then

$$P_2(x) \leqslant A_1 \sup_{0<t\leqslant x_0} F(x-t+\Delta) \leqslant A_1 L_1 F(x+\Delta),$$

where

$$L_1 = \sup_{0<t\leqslant x_0,\, y>2x_0} \frac{F(y-t+\Delta)}{F(y+\Delta)}.$$

If $x_0 < x \leqslant 2x_0$, then $P_2(x) \leqslant 1$ implies

$$\frac{P_2(x)}{F(x+\Delta)} \leqslant \frac{1}{\inf_{x_0<x\leqslant 2x_0} F(x+\Delta)} \equiv L_2.$$

Since $F \in \mathscr{L}_\Delta$, both L_1 and L_2 are finite for x_0 sufficiently large. Put $R = A_1 L_1 + L_2$. Then, for any $x > x_0$,

$$P_2(x) \leqslant RF(x+\Delta). \tag{6}$$

It follows from (5) and (6) that $A_n \leqslant A_{n-1}(1+\varepsilon/2)+R$ for $n>1$. Therefore, an induction argument yields:

$$A_n \leqslant A_1(1+\varepsilon/2)^{n-1}+R\sum_{l=0}^{n-2}(1+\varepsilon/2)^l \leqslant Rn(1+\varepsilon/2)^{n-1}.$$

This implies the conclusion of the proposition.

Let us consider now some random time τ with distribution $p_n = \mathbf{P}(\tau=n)$, $n\geqslant 0$ which is independent of $\{\zeta_n\}$. Then the distribution of the randomly stopped sum S_τ is equal to

$$\mathbf{P}(S_\tau \in B) = \sum_{n\geqslant 0} p_n G^{*n}(B).$$

Theorem 2. Let $0<T\leqslant\infty$. Assume $F[0,\infty)=1$, $G(x+\Delta)/F(x+\Delta) \to c\geqslant 0$ as $x\to\infty$, and $\mathbf{E}\tau<\infty$.

(i) If $F\in\mathscr{S}_\Delta$ and $\mathbf{E}(1+\delta)^\tau<\infty$ for some $\delta>0$, then

$$\frac{\mathbf{P}(S_\tau\in x+\Delta)}{F(x+\Delta)} \to c\cdot\mathbf{E}\tau \qquad \text{as} \quad x\to\infty. \tag{7}$$

(ii) If (7) holds, $c>0$, $p_n>0$ for some $n\geqslant 2$, and, in the case of a finite Δ, $F\in\mathscr{L}_\Delta$, then $F\in\mathscr{S}_\Delta$.

Proof. (i) follows from Corollary 2, Proposition 4, and the dominated convergence theorem.

We prove the second assertion. First, for any $n\geqslant 2$,

$$\liminf_{x\to\infty}\frac{G^{*n}(x+\Delta)}{G(x+\Delta)} \geqslant n. \tag{8}$$

Indeed, if $\Delta=(0,\infty)$, then (8) follows from Lemma 1 in Ref. 9. If the interval Δ is finite, $F\in\mathscr{L}_\Delta$, and $c>0$, then $G\in\mathscr{L}_\Delta$ and (8) follows from Corollary 1.

If $p_n>0$ for some $n\geqslant 2$, then it follows from (8) and (7) that

$$G^{*n}(x+\Delta)\sim nG(x+\Delta) \qquad \text{as} \quad x\to\infty \tag{9}$$

(the proof is a straightforward argument by contradiction).

If $\Delta=(0,\infty)$, then (9) implies the subexponentiality of G, by Lemma 7 in Ref. 15. If Δ is a finite interval and $F\in\mathscr{L}_\Delta$, then $G\in\mathscr{L}_\Delta$ and, by

Corollary 1, the convolution $G^{*(n-1)}$ belongs to the class $\mathscr{L}_\Delta$ too. Thus, by Proposition 1,

$$n = \limsup_{x\to\infty} \frac{G^{*n}(x+\Delta)}{G(x+\Delta)} = \limsup_{x\to\infty} \frac{(G * G^{*(n-1)})(x+\Delta)}{G(x+\Delta)}$$
$$\geqslant 1 + \limsup_{x\to\infty} \frac{G^{*(n-1)}(x+\Delta)}{G(x+\Delta)}.$$

By induction we deduce from this estimate that

$$\limsup_{x\to\infty} \frac{G^{*2}(x+\Delta)}{G(x+\Delta)} \leqslant 2,$$

which implies the Δ-subexponentiality of G. Now $F \in \mathscr{S}_\Delta$ by Lemma 1.

In Theorem 2, assertion (i) is valid for any Δ-subexponential distribution. For a fixed distribution F, the condition $\mathbf{E}(1+\delta)^\tau < \infty$ may be substantially weakened. We can illustrate that by the following example. Consider the case of the infinite interval $\Delta = (0, \infty)$. Assume that $G = F$ and there exist finite positive constants c and α such that $\overline{F}(x/n) \leqslant cn^\alpha \overline{F}(x)$ for any $x > 0$ and $n \geqslant 1$ (for instance, the Pareto distribution with parameter α satisfies this condition). Then $\mathbf{P}(S_\tau > x) \sim \mathbf{E}\tau \cdot \overline{F}(x)$ as $x \to \infty$ provided $\mathbf{E}\tau^{1+\alpha}$ is finite, as follows by combining dominated convergence with

$$\mathbf{P}(S_n > x) \leqslant \mathbf{P}(n \cdot \max_{k \leqslant n} \zeta_k > x) \leqslant n\mathbf{P}(\zeta_1 > x/n) \leqslant n^{1+\alpha}\overline{F}(x).$$

Proposition 4 implies also the following corollary. For $x \geqslant 0$, put $\eta(x) = \min\{n \geqslant 1 : S_n > x\}$ and $\chi(x) = S_{\eta(x)} - x$. As earlier, let τ be a non-negative integer-valued random variable which does not depend on ζ's.

Corollary 3. Assume that $G \in \mathscr{S}_\Delta$ and $\mathbf{E}(1+\delta)^\tau < \infty$ for some $\delta > 0$. Then $\mathbf{P}(\chi(x) \in y+\Delta, \eta(x) \leqslant \tau) \sim \mathbf{E}\tau \cdot G(x+y+\Delta)$ as $\min(x, y) \to \infty$.

Proof. Let h be such that $h(y) \leqslant y/2$, $h(y) \uparrow \infty$ as $y \to \infty$, and $G(y+t+\Delta) \sim G(y+\Delta)$ uniformly in $|t| \leqslant h(y)$. Put $z = \min(h(y), x)$. For any $n \geqslant 2$,

$$\mathbf{P}(\chi(x) \in y+\Delta, \eta(x) = n) = \mathbf{P}(S_{n-1} \leqslant x, S_n \in x+y+\Delta)$$
$$\geqslant \int_0^z \mathbf{P}(S_{n-1} \in dt)\, G(x+y-t+\Delta)$$
$$\sim G(x+y+\Delta).$$

On the other hand,

$$\begin{aligned}\mathbf{P}(S_{n-1} \leqslant x, S_n \in x+y+\Delta) &= \mathbf{P}(S_n \in x+y+\Delta) - \mathbf{P}(S_{n-1} > x, S_n \in x+y+\Delta) \\ &\leqslant \mathbf{P}(S_n \in x+y+\Delta) - \mathbf{P}(\zeta_n \leqslant z, S_n \in x+y+\Delta) \\ &= (n+o(1))\, G(x+y+\Delta) - (n-1+o(1))\, G(x+y+\Delta).\end{aligned}$$

Thus, for any fixed $n \geqslant 1$,

$$\mathbf{P}(\chi(x) \in y+\Delta, \eta(x) = n) \sim G(x+y+\Delta).$$

Now Proposition 4 and the dominated convergence theorem complete the proof, since $\mathbf{P}(\chi(x) \in y+\Delta, \eta(x) = n) \leqslant \mathbf{P}(S_n \in x+y+\Delta)$ and

$$\begin{aligned}\mathbf{P}(\chi(x) \in y+\Delta, \eta(x) \leqslant \tau) &= \sum_{k=1}^{\infty} \mathbf{P}(\tau = k) \sum_{n=1}^{k} \mathbf{P}(\chi(x) \in y+\Delta, \eta(x) = n) \\ &\sim G(x+y+\Delta) \sum_{k=1}^{\infty} k \mathbf{P}(\tau = k).\end{aligned}$$

3. DISTRIBUTIONS WITH SUBEXPONENTIAL DENSITIES

Similar results (with similar proofs!) hold for densities of absolutely continuous distributions. More precisely, in this section we consider a class of distributions $\{F\}$ with the following property: each distribution F has a density $f(x)$ for all sufficiently large values of x, i.e., for a certain $\hat{x} = \hat{x}(F)$ and for any Borel set $B \subseteq [\hat{x}, \infty)$,

$$F(B) = \int_B f(y)\, dy.$$

We say that a density f on $[\hat{x}(F), \infty)$ is *long-tailed* (and write $f \in \mathscr{L}$) if the function $f(x)$ is bounded on $[\hat{x}, \infty)$, $f(x) > 0$ for all sufficiently large x, and $f(x+t) \sim f(x)$ as $x \to \infty$ uniformly in $t \in [0, 1]$. In particular, if $f \in \mathscr{L}$, then $f(x) \to 0$ as $x \to \infty$.

A distribution F on $\mathbf{R}_+$ with a density $f(x)$ on $[\hat{x}, \infty)$ is said to belong to the class $\mathscr{S}_{ac}$ (the density f is *subexponential*) if $f \in \mathscr{L}$ and, as $x \to \infty$,

$$f^{*2}(x) \equiv 2 \int_0^{\hat{x}} f(x-y)\, F(dy) + \int_{\hat{x}}^{x-\hat{x}} f(x-y)\, f(y)\, dy \sim 2f(x).$$

Typical examples of $\mathscr{S}_{ac}$ are given by the Pareto, lognormal, and Weibull (with parameter between 0 and 1) distributions (for the proof, see Section 4). Note that distribution with subexponential density is Δ-subexponential for any $0 < T \leqslant \infty$.

Proposition 5. Let F and G have densities f and g on $[\hat{x}, \infty)$ belonging to the class $\mathscr{L}$. Then the density $f * g$ of the convolution $F * G$ is long-tailed and

$$\liminf_{x \to \infty} \frac{(f * g)(x)}{f(x) + g(x)} \geqslant 1. \tag{10}$$

In particular, if $f \in \mathscr{L}$, then $f^{*n} \in \mathscr{L}$ and $\liminf_{x \to \infty} f^{*n}(x)/f(x) \geqslant n$.

Proof. Take a function $h(x) \uparrow \infty$ such that $\hat{x} \leqslant h(x) < x/2$, $f(x-y) \sim f(x)$ and $g(x-y) \sim g(x)$ as $x \to \infty$ uniformly in $|y| \leqslant h(x)$. Then

$$(f * g)(x+t) = \int_{-\infty}^{x-h(x)} f(x+t-y)\, G(dy) + \int_{-\infty}^{h(x)} g(x+t-y)\, F(dy)$$

$$+ \int_{x-h(x)}^{x+t-h(x)} f(x+t-y)\, g(y)\, dy$$

$$\equiv I_1(x, t) + I_2(x, t) + I_3(x, t).$$

Now the conclusion of the proposition follows from $I_1(x, t) \sim I_1(x, 0)$ and $I_2(x, t) \sim I_2(x, 0)$ as $x \to \infty$ uniformly in $t \in (0, 1]$ and the estimate

$$I_3(x, t) \leqslant \sup_{y \in [h(x), h(x)+t)} f(y) \int_{x-h(x)}^{x+t-h(x)} g(y)\, dy \sim t f(h(x))\, g(x) = o(g(x)).$$

Proposition 6. Assume that the distribution F on $\mathbf{R}^+$ has a density $f \in \mathscr{L}$ on $[\hat{x}, \infty)$. Then the following assertions are equivalent:

(i) the density f is subexponential;

(ii) for some function h such that $h(x) \to \infty$, $h(x) < x/2$, and $f(x-y) \sim f(x)$ as $x \to \infty$ uniformly in $|y| \leqslant h(x)$,

$$\int_{h(x)}^{x-h(x)} f(x-y)\, f(y)\, dy = o(f(x)) \qquad \text{as} \quad x \to \infty; \tag{11}$$

(iii) the relation (11) holds for every function h such that $h(x) \to \infty$.

Proof. For $\hat{x} \leqslant h(x) < x/2$,

$$f^{*2}(x) = 2\int_0^{h(x)} f(x-y)\,F(dy) + \int_{h(x)}^{x-h(x)} f(x-y)\,f(y)\,dy.$$

Here the first integral is equivalent to $f(x)$ as $x \to \infty$. This completes the proof.

Lemma 2. Let f be a subexponential density on $[\hat{x}, \infty)$. Assume that the density g on $[\hat{x}, \infty)$ is long-tailed and

$$0 < \liminf_{x\to\infty} g(x)/f(x) \leqslant \limsup_{x\to\infty} g(x)/f(x) < \infty.$$

Then g is subexponential too. In particular, $g \in \mathscr{S}_{ac}$, given $g(x) \sim cf(x)$ as $x \to \infty$ for some $c \in (0, \infty)$.

Proof. The result follows by Proposition 6(iii). Indeed, one can choose $c_1 < \infty$ such that $g(x) \leqslant c_1 f(x)$ for all sufficiently large x and

$$\int_{h(x)}^{x-h(x)} g(x-y)\,g(y)\,dy \leqslant c_1^2 \int_{h(x)}^{x-h(x)} f(x-y)\,f(y)\,dy.$$

Proposition 7. Let f be a subexponential density on $[\hat{x}, \infty)$. Let f_1, f_2 be two densities on $[\hat{x}, \infty)$ such that $f_1(x)/f(x) \to c_1$ and $f_2(x)/f(x) \to c_2$ as $x \to \infty$, for some constants $c_1, c_2 \geqslant 0$. Then

$$\frac{(f_1 * f_2)(x)}{f(x)} \to c_1 + c_2 \qquad \text{as} \quad x \to \infty.$$

If $c_1 + c_2 > 0$ then, by Lemma 2, the convolution $f_1 * f_2$ is a subexponential density.

Proof. Take a function h as before. Then

$$\begin{aligned} f_1 * f_2(x) &= \int_0^{h(x)} f_1(x-y)\,F_2(dy) + \int_0^{h(x)} f_2(x-y)\,F_1(dy) \\ &\quad + \int_{h(x)}^{x-h(x)} f_1(x-y)\,f_2(y)\,dy \\ &\equiv I_1(x) + I_2(x) + I_3(x). \end{aligned}$$

We have $I_1(x)/f(x) \to c_1$ and $I_2(x)/f(x) \to c_2$ as $x \to \infty$. Finally,

$$I_3(x) \leqslant (c_1 c_2 + o(1)) \int_{h(x)}^{x-h(x)} f(x-y)\, f(y)\, dy = o(f(x)),$$

which completes the proof.

Corollary 4. Assume that $F \in \mathscr{S}_{ac}$. Then for any $n \geqslant 2$, $f^{*n}(x) \sim nf(x)$ as $x \to \infty$ and $F^{*n} \in \mathscr{S}_{ac}$.

Let $\{\xi_n\}$ be a sequence of i.i.d. non-negative random variables with a common distribution $F(B) = \mathbf{P}(\xi_1 \in B)$. Put $S_n = \xi_1 + \cdots + \xi_n$.

Proposition 8. Assume that $F \in \mathscr{S}_{ac}$. Then, for any $\varepsilon > 0$, there exist $x_0 = x_0(\varepsilon) \geqslant \hat{x}$ and $V(\varepsilon) > 0$ such that, for any $x > x_0$ and for any integer $n \geqslant 1$,

$$f^{*n}(x) \leqslant V(\varepsilon)(1+\varepsilon)^n f(x).$$

Proof. For $x_0 \in \mathbf{R}_+$ and $k \geqslant 1$, put $A_k \equiv A_k(x_0) = \sup_{x > x_0} f^{*k}(x)/f(x)$. Take any $\varepsilon > 0$. Fix an integer j such that $(j+1+\varepsilon)^{1/j} < 1+\varepsilon/2$. By Corollary 4, $A_{j+1}(x_0) \to j+1$ as $x_0 \to \infty$. Choose $x_0 \geqslant \hat{x}$ such that

$$A_{j+1}(x_0) \leqslant (j+1+\varepsilon) < (1+\varepsilon/2)^j.$$

For any $n > j$ and $x > 2x_0$,

$$f^{*n}(x) = \int_0^{x-x_0} f^{*(n-j)}(x-y)\, F^{*j}(dy) + \int_0^{x_0} f^{*j}(x-y)\, F^{*(n-j)}(dy)$$

$$\equiv I_1(x) + I_2(x),$$

where, by the definition of A_{n-j} and A_{j+1},

$$I_1(x) \leqslant A_{n-j} \int_0^{x-x_0} f(x-y)\, F^{*j}(dy) \leqslant A_{n-j} f^{*(j+1)}(x) \leqslant A_{n-j} A_{j+1} f(x) \tag{12}$$

and

$$I_2(x) \leqslant \max_{0 < t \leqslant x_0} f^{*j}(x-t) \leqslant A_j \max_{0 < t \leqslant x_0} f(x-t) \leqslant A_j L_1 f(x), \tag{13}$$

where $L_1 = \sup_{0<t\leqslant x_0,\, y>2x_0} f(y-t)/f(y)$. If $x_0 < x \leqslant 2x_0$, then

$$\frac{f^{*n}(x)}{f(x)} \leqslant \frac{\sup_{x\in(x_0,2x_0]} f^{*n}(x)}{\inf_{x_0<x\leqslant 2x_0} f(x)} \equiv L_2. \tag{14}$$

Since $f\in\mathcal{L}$, we may choose x_0 such that L_1 and $L_2<\infty$. Put $R = A_j L_1 + L_2$. It follows from (12)–(14) that, for any $x > x_0$,

$$f^{*n}(x) \leqslant (A_{n-j}A_{j+1} + R)\, f(x).$$

Hence, for $n > j$, $A_n \leqslant A_{n-j}A_{j+1} + R$. The remaining part of the proof is the same as that of Proposition 4.

Theorem 3. Let $\{p_n\}_{n\geqslant 1}$ be a non-negative sequence such that $m_p \equiv \sum_{n\geqslant 1} np_n$ is finite. Denote

$$g(x) = \sum_{n\geqslant 1} p_n f^{*n}(x).$$

(i) If a distribution F has a subexponential density f on $[\hat{x},\infty)$ and $\sum_{n\geqslant 1}(1+\delta)^n p_n < \infty$ for some $\delta > 0$, then

$$g(x) \sim m_p f(x) \qquad \text{as} \quad x\to\infty. \tag{15}$$

(ii) If equivalence (15) holds, $p_1 < 1$, $F[0,\infty) = 1$, and $f\in\mathcal{L}$, then $F\in\mathcal{S}_{ac}$.

Proof. Assertion (i) follows from Corollary 4, Proposition 8, and the dominated convergence theorem. We prove the second assertion. By Lemma 5, for any $n\geqslant 2$,

$$\liminf_{x\to\infty} f^{*n}(x)/f(x) \geqslant n.$$

If $p_n > 0$ for some $n \geqslant 2$, then this estimate and (15) imply that

$$f^{*n}(x) \sim nf(x) \qquad \text{as} \quad x\to\infty. \tag{16}$$

By Proposition 5, $f^{*(n-1)}\in\mathcal{L}$ and

$$n = \limsup_{x\to\infty}\frac{f^{*n}(x)}{f(x)} = \limsup_{x\to\infty}\frac{(f * f^{*(n-1)})(x)}{f(x)} \geqslant 1 + \limsup_{x\to\infty}\frac{f^{*(n-1)}(x)}{f(x)}.$$

By induction we deduce from this estimate that $\limsup_{x\to\infty} f^{*2}(x)/f(x) \leqslant 2$, which implies the subexponentiality of f.

4. SUFFICIENT CONDITIONS FOR Δ-SUBEXPONENTIALITY AND SUBEXPONENTIALITY OF DENSITIES

The sufficient conditions for distributions to be subexponential are well-known (see, e.g., Ref. 24 and Section 2.5.3 in Ref. 22). In this section, we propose similar conditions for distributions to belong either to $\mathcal{S}_\Delta$ for a finite T, or to $\mathcal{S}_{ac}$.

Proposition 9. Let a distribution F on $\mathbf{R}^+$ belong to the class $\mathcal{L}_\Delta$ for some finite $T > 0$. Assume that there exist $c > 0$ and $x_0 < \infty$ such that $F(x+t+\Delta) \geqslant cF(x+\Delta)$ for any $t \in (0, x]$ and $x > x_0$. Then $F \in \mathcal{S}_\Delta$.

Proof. Let a function $h(x) \to \infty$ be such that $h(x) < x/2$. Then

$$\begin{aligned}
&\mathbf{P}(\xi_1 + \xi_2 \in x + \Delta, \xi_1 > h(x), \xi_2 > h(x))\\
&\quad\leqslant 2 \int_{h(x)}^{x/2+T} F(x-y+\Delta)\, F(dy)\\
&\quad\leqslant 2(c+o(1)) \int_{h(x)}^{x/2+T} F(x+\Delta)\, F(dy) = o(F(x+\Delta))
\end{aligned}$$

as $x \to \infty$. Applying now Lemma 2(ii) we conclude that $F \in \mathcal{S}_\Delta$.

The Pareto distribution (with the tail $\overline{F}(x) = x^{-\alpha}$, $\alpha > 0$, $x \geqslant 1$) satisfies conditions of Proposition 9. The same is true for any distribution F such that $\mathbf{P}(\xi \in x + \Delta)$ is regularly varying at infinity, i.e., for $F(x+\Delta) \sim x^{-\alpha} l(x)$, where $l(x)$ is slowly varying at infinity.

Proposition 10. Let a distribution F on $\mathbf{R}^+$ belong to the class $\mathcal{L}_\Delta$ for some finite Δ. Let there exist x_0 such that the function $g(x) \equiv -\ln F(x+\Delta)$ is concave for $x \geqslant x_0$. Let there exist a function $h(x) \uparrow \infty$ as $x \to \infty$ such that $F(x+t+\Delta) \sim F(x+\Delta)$ uniformly in $|t| \leqslant h(x)$ and $xF(h(x)+\Delta) \to 0$. Then $F \in \mathcal{S}_\Delta$.

Proof. Due to Lemma 1, without loss of generality assume $x_0 = 0$. Since $g(x)$ is concave, the minimum of the sum $g(x-y) + g(y)$ on the interval $y \in [h(x), x-h(x)]$ is equal to $g(x-h(x)) + g(h(x))$. Therefore,

$$\begin{aligned}
\int_{h(x)}^{x-h(x)} F(x-y+\Delta)\, F(dy) &\leqslant c_1 \int_{h(x)}^{x-h(x)} F(x-y+\Delta)\, F(y+\Delta)\, dy\\
&= c_1 \int_{h(x)}^{x-h(x)} e^{-(g(x-y)+g(y))}\, dy\\
&\leqslant c_1 x e^{-(g(x-h(x))+g(h(x)))}.
\end{aligned}$$

Since $e^{-g(x-h(x))} \sim e^{-g(x)}$,

$$\int_{h(x)}^{x-h(x)} F(x-y+\Delta)\, F(dy) = O(e^{-g(x)} x e^{-g(h(x))}) = o(F(x+\Delta)),$$

which completes the proof.

Consider the Weibull distribution, $\overline{F}(x) = e^{-x^\beta}$, $x \geqslant 0$, $\beta \in (0, 1)$. Then

$$F(x+\Delta) \sim \beta T x^{\beta-1} \exp(-x^\beta) \qquad \text{as} \quad x \to \infty.$$

Consider the distribution $\hat{F}$ with the tail $\overline{\hat{F}}(x) = \min(1, x^{\beta-1} e^{-x^\beta})$. Let x_0 be the unique positive solution to the equation $x^{1-\beta} = e^{-x^\beta}$. Then the function $\hat{g}(x) = -\ln \hat{F}(x+\Delta)$ is concave for $x \geqslant x_0$, and conditions of Proposition 10 are satisfied with $h(x) = x^\gamma$, $\gamma \in (0, 1-\beta)$. Therefore, $\hat{F} \in \mathscr{S}_\Delta$ and, due to Lemma 1, $F \in \mathscr{S}_\Delta$.

Similarly, one can check that, for the lognormal distribution with the density $f(x) = e^{-(\ln x - \ln a)^2/2\sigma^2} / x\sqrt{2\pi\sigma^2}$,

$$F(x+\Delta) \sim T f(x),$$

the function $g(x) = -\ln(x^{-1} e^{-(\ln x - \ln a)^2/2\sigma^2}) = \ln x + (\ln x - \ln a)^2/2\sigma^2$ is eventually concave, and conditions of Proposition 10 are satisfied with any $h(x) = o(x)$. Thus, $F \in \mathscr{S}_\Delta$.

Similarly to Propositions 9 and 10 we obtain the following

Proposition 11. Let a distribution F on $\mathbf{R}^+$ have a long-tailed density $f(x)$. Let one of the following conditions hold:

(i) there exists $c > 0$ such that $f(y) \geqslant c f(x)$ for any $y \in (x, 2x]$;

(ii) the function $g(x) \equiv -\ln f(x)$ is concave for $x \geqslant x_0$ and, for some $h(x) \to \infty$, $f(x+t) \sim f(x)$ uniformly in $|t| \leqslant h(x)$ and $x e^{-g(h(x))} \to 0$.

Then f is subexponential.

The density of the Pareto distribution satisfies condition (i) of Proposition 11. The density of the Weibull distribution with parameter $\beta \in (0, 1)$ satisfies condition (ii) of Proposition 11 with $h(x) = \ln^{2/\beta} x$.

Example 1. Assume that ξ takes positive integer values only, $\mathbf{P}(\xi = 2k) = \gamma/k^2$ and $\mathbf{P}(\xi = 2k+1) = \gamma/2^k$, where γ is a normalizing constant. Then ξ has a lattice distribution F with span 1. By Proposition 9, $F \in \mathscr{S}_{(0,2]}$, but F cannot belong to any $\mathscr{S}_{(0,a]}$ if a is not infinity or an even integer.

Example 2. Assume that ξ is a sum of two independent random variables: $\xi = \eta + \zeta$ where η is distributed uniformly on $(-1/8, 1/8)$ and $\mathbf{P}(\zeta = k) = \gamma/k^2$. Then the distribution F of ξ is absolutely continuous. It may be checked that $F \in \mathcal{S}_{(0,1]}$, but F cannot belong to any $\mathcal{S}_{(0,a]}$ if a is not infinity or an integer.

Example 3. Consider a long-tailed function $f(x)$ in the range $f(x) \in [1/x^2, 2/x^2]$ for any $x > 0$. Let us choose the function f in such a way that f is not asymptotically equivalent to a non-increasing function.

For instance, one can define f as follows. Consider the increasing sequence $x_n = 2^{n/4}$. Put $f(x_{2n}) = 1/x_{2n}^2$ and $f(x_{2n+1}) = 2/x_{2n+1}^2$. Then assume that f is linear between any two consecutive points.

Consider the lattice distribution F on the set of natural numbers with $F(\{n\}) = f(n)$ for all sufficiently large integer n. Then by Lemma 1, $F \in \mathcal{S}_{(0,1]}$, but $f(n) = F((n-1, n])$ is not asymptotically equivalent to a non-increasing function.

Example 4. Let G_+ be the ascending ladder height distribution of a random walk with increment distribution F. It is shown in Ref. 4 that $G_+ \in \mathcal{S}_\Delta$ for all $T < \infty$ when F is non-lattice. However, G_+ cannot have a subexponential density when F is singular (say concentrated on $\{-1, \sqrt{2}\}$) since then also G_+ is singular.

5. SUPREMUM OF A RANDOM WALK

Theorems 2 and 3 give us a unified approach for obtaining the local and integral asymptotic theorems for the supremum of a random walk.

Let $\{\xi_n\}$ be a sequence of independent random variables with a common distribution $F(B) = \mathbf{P}(\xi_n \in B)$ and $\mathbf{E}\xi_1 = -m < 0$. Let

$$F^I(x) \equiv 1 - \overline{F^I}(x) = 1 - \min\left(1, \int_x^\infty \overline{F}(y)\, dy\right)$$

denote the integrated-tail distribution function. It is easy to see that

(a) if F is long-tailed, then F^I is long-tailed, too;

(b) F^I is long-tailed if and only if $\overline{F}(x) = o(\overline{F^I}(x))$ as $x \to \infty$.

Put $S_0 = 0$, $S_n = \xi_1 + \cdots + \xi_n$. By the SLLN, $M = \sup_{n \geqslant 0} S_n$ is finite with probability 1. Write $\pi(B) = \mathbf{P}(M \in B)$, $\pi(x) \equiv \pi(-\infty, x] = 1 - \bar{\pi}(x)$.

It is well-known (see, e.g., Refs. 2, 13, 16 and references therein) that if $F^I \in \mathscr{S}$, then, as $x \to \infty$,

$$\bar{\pi}(x) \sim \frac{1}{m} \overline{F^I}(x). \tag{17}$$

In particular, $\pi \in \mathscr{S}$. Korshunov[21] proved the converse: (17) implies $F^I \in \mathscr{S}$.

Recently, Asmussen *et al.*[4] proved that if $F \in \mathscr{S}^*$, i.e., if

$$\int_0^x \bar{F}(x-y)\, \bar{F}(y)\, dy \sim 2\mathbf{E} \max(\xi_1, 0)\, \bar{F}(x), \qquad x \to \infty,$$

then, for any $T \in (0, \infty)$,

$$\pi(x+\Delta) \sim \frac{T}{m} \bar{F}(x) \tag{18}$$

(if the distribution F is lattice then x and T should be restricted to values of the lattice span). In particular, $\pi \in \mathscr{S}_\Delta$ for any $0 < T < \infty$.

In the lattice case, (18) was proved earlier by Bertoin and Doney.[6] They also sketched a proof of (18) for non-lattice distributions.

It follows from Theorem 2(b) in Ref. 18 that the converse is also true: if (18) holds for any $T \in (0, \infty)$ and F is long tailed, then $F \in \mathscr{S}^*$.

Remark 4. Since (18) holds for any $T > 0$, it implies that, for any $T_0 > 0$,

$$\pi(x+\Delta) \sim \frac{1}{m} \int_x^{x+T} \bar{F}(y)\, dy \tag{19}$$

as $x \to \infty$ uniformly in $T \in [T_0, \infty]$.

One can see that Theorem 2 gives a unified approach for obtaining (17) and (18). We start with the following

Lemma 3. Let $v(x)$ be a long-tailed function and let

$$V(x) \equiv \int_x^\infty v(y)\, dy.$$

Assume that $V(0) < \infty$. For any n, define the event $A_n = \{S_j \leqslant 0$ for all $j \leqslant n\}$ and put $p = \mathbf{P}(M > 0)$. Then, as $x \to \infty$,

$$\sum_{n=0}^{\infty} \mathbf{E}(v(x-S_n); A_n) \sim \frac{1-p}{m} V(x).$$

Proof. Since v is long-tailed, V is long-tailed, too, and $v(x) = o(V(x))$.

Assume that the distribution F is non-lattice (the proof in the lattice case is similar). For $n \geqslant 0$, consider the measures

$$H_n(B) = \mathbf{P}\{S_j \leqslant 0 \text{ for any } j \leqslant n, S_n \in B\}, \qquad B \subseteq (-\infty, 0]$$

and the corresponding taboo renewal function

$$H(B) = \sum_{n=0}^{\infty} H_n(B).$$

It is well-known that, for a non-lattice distribution,

$$H(y+(0,1]) \sim (1-p)/m \qquad \text{as} \quad y \to -\infty. \tag{20}$$

Since

$$\mathbf{E}(v(x-S_n); A_n) = \int_{-\infty}^{0} v(x-y)\, H_n(dy)$$

and the function $v(x)$ is long-tailed, we obtain

$$\sum_{n=0}^{\infty} \mathbf{E}(v(x-S_n); A_n) = \int_0^{\infty} v(x+y)\, H(-dy) \sim \sum_{j=0}^{\infty} v(x+j)\, H((-j-1, -j]).$$

Take an integer-valued function $h(x) \to \infty$ such that $v(x+t) \sim v(x)$ uniformly in $|t| \leqslant h(x)$ and $v(x)\, h(x) = o(V(x))$. Then, by (20),

$$\begin{aligned}\sum_{n=0}^{\infty} \mathbf{E}(v(x-S_n); A_n) &\sim \sum_{j=h(x)}^{\infty} v(x+j)\, H((-j-1, -j]) \\ &\sim \frac{1-p}{m} \sum_{j=h(x)}^{\infty} v(x+j) \\ &\sim \frac{1-p}{m} \int_x^{\infty} v(y)\, dy.\end{aligned}$$

The proof is complete.

Consider the defective stopping time

$$\eta = \inf\{n \geqslant 1 : S_n > 0\} \leqslant \infty$$

and let $\{\psi_n\}$ be i.i.d. random variables with common distribution function

$$G(x) \equiv \mathbf{P}(\psi_n \leqslant x) = \mathbf{P}(S_\eta \leqslant x \mid \eta < \infty).$$

It is well-known (see, e.g., Chap. 12 in Feller[17]) that the distribution of the maximum M coincides with the distribution of the randomly stopped sum $\psi_1 + \cdots + \psi_\nu$, where the stopping time ν is independent of the sequence $\{\psi_n\}$ and is geometrically distributed with parameter $p = \mathbf{P}(M > 0) < 1$, i.e., $\mathbf{P}(\nu = k) = (1-p)\, p^k$ for $k = 0, 1, \ldots$. Equivalently,

$$\mathbf{P}(M \in B) = (1-p) \sum_{k=0}^{\infty} p^k G^{*k}(B).$$

From Chap. 4, Theorem 10 in Borovkov,[7] if F^I is long-tailed, then

$$\overline{G}(x) \sim \frac{1-p}{pm} \overline{F^I}(x). \tag{21}$$

For any $T \in (0, \infty)$ and $\Delta = (0, T]$, if the function $v(x) = F(x+\Delta)$ is long-tailed, then by Lemma 3,

$$G(x+\Delta) = \mathbf{P}(S_\eta \in x+\Delta)/\mathbf{P}(\eta < \infty) = \frac{1}{p} \sum_{n=1}^{\infty} \mathbf{P}(S_n \in x+\Delta, \eta = n)$$

$$\sim \frac{1-p}{pm} \int_x^{\infty} F(y+\Delta) \sim \frac{(1-p)\, T}{pm} \overline{F}(x). \tag{22}$$

Now (17) and (18) follow from (21) and (22) by Theorem 2.

Similarly, Theorem 3 allows us to get the asymptotics for the density of π.

Theorem 4. Assume that $F \in \mathcal{S}^*$ and that the density f on $[x(F), \infty)$ of the distribution F is long-tailed. Then, as $x \to \infty$, the density of π is equivalent to $\overline{F}(x)/m$.

Indeed, if the density f of the distribution F is long-tailed, then by Lemma 3 (with $v(x) = f(x)$), G has a density g on the interval $[\hat{x}(F), \infty)$ which is long-tailed and

$$g(x) \sim \frac{1-p}{pm} \overline{F}(x).$$

Further, if $F \in \mathscr{S}^*$, then G has a subexponential density. The density of the distribution π may be represented as

$$(1-p) \sum_{k=1}^{\infty} p^k g^{*k}(x),$$

and, by Theorem 3(i), is equivalent to

$$g(x)(1-p) \sum_{k=1}^{\infty} kp^k \sim \bar{F}(x)/m \qquad \text{as} \quad x \to \infty. \tag{23}$$

Remark 5. The result of Theorem 4 is new. In Proposition 1 in Ref. 4, it was claimed that the same asymptotics may be obtained under different conditions.

6. THE RENEWAL FUNCTION AND THE KEY RENEWAL THEOREM

Let G be a non-negative measure on $(0, \infty)$. We will assume throughout that $\theta \leqslant 1$ where $\theta = G(0, \infty)$. Then the renewal measure

$$U = \sum_{n=0}^{\infty} G^{*n}$$

is well-defined and finite on compact sets. In addition, if $\theta < 1$ then U is a finite measure (in fact, $U[0, \infty) = (1-\theta)^{-1}$). See, e.g., Ref. 17 or Ref. 1 for this and further basic facts from renewal theory.

Blackwell's renewal theorem states that when $\theta = 1$ and G is non-lattice, then $U(x+\Delta) \sim T/\mu_G$ where μ_G is the mean of G. When $\theta < 1$ and G is light-tailed, it is easy to see by standard techniques (Ref. 1, VI.5) that $U(x+\Delta)$ decreases exponentially fast. Callaert and Cohen[8] gave an asymptotic expression for a special heavy-tailed case with $\theta < 1$, $T = \infty$. Here is a more complete and local version. We will say that $G \in \mathscr{S}_\Delta$ if $F \in \mathscr{S}_\Delta$ where F is the probability measure G/θ.

Proposition 12. Assume $0 < \theta < 1$. If $T < \infty$, assume also that $G \in \mathscr{L}_\Delta$. Then $U(x+\Delta) \sim (1-\theta)^{-2} G(x+\Delta)$ as $x \to \infty$ if and only if $G \in \mathscr{S}_\Delta$.

Proof. By Theorem 2,

$$U(x+\Delta) = \sum_{n=1}^{\infty} G^{*n}(x+\Delta) = \sum_{n=1}^{\infty} \theta^n F^{*n}(x+\Delta)$$

is asymptotically eqiuvalent to

$$\sum_{n=1}^{\infty} n\theta^n F(x+\Delta) = \frac{\theta}{(1-\theta)^2} F(x+\Delta) = \frac{1}{(1-\theta)^2} G(x+\Delta)$$

if and only if $G \in \mathscr{S}_\Delta$.

Alternatively, one may use the representation $U = H/(1-\theta)$ where H is the distribution of $X_1 + \cdots + X_\tau$ where $\mathbf{P}(\tau = n) = (1-\theta)\,\theta^n$, $n = 0, 1, 2,...$ and the X_k are i.i.d. with distribution $F = G/\theta$ (see Ref. 1, Proposition 2.6, p. 114). Hence by Theorem 2

$$U(x+\Delta) = \frac{H(x+\Delta)}{1-\theta} \sim \frac{\mathbf{E}\tau F(x+\Delta)}{1-\theta}$$

$$= \frac{\theta}{(1-\theta)^2} F(x+\Delta) = \frac{1}{(1-\theta)^2} G(x+\Delta).$$

We now turn to the renewal equation

$$Z(x) = z(x) + \int_0^x Z(x-y)\, G(dy), \qquad x \geqslant 0, \tag{24}$$

where $z \geqslant 0$ and z is locally bounded. This together with $\theta \leqslant 1$ is more than sufficient to ensure that

$$Z(x) = \int_0^x z(x-y)\, U(dy)$$

is the unique locally bounded solution. The key renewal theorem states that $Z(x)$ has limit $\mu_G^{-1} \int_0^\infty z(y)\, dy$ when $\theta = 1$. Light-tailed asymptotics of $Z(x)$ when $\theta < 1$ is also available (see Ref. 17 or Ref. 1, VI.5) and has found numerous applications. Therefore, it is surprising that heavy-tailed asymptotics when $\theta < 1$ appears not to have been discussed before a specific application came up in Asmussen.[3] A result was stated there which contains the basic intuition, but the proof is heuristic as well as the conditions are not formulated in a precise form. The analysis of the preceding parts of this paper allows for a more rigorous treatment, and we shall show (see Refs. 1 and 17 for the definition of z to be d.R.i. = directly Riemann integrable):

Theorem 5. Assume $\theta < 1$ and define $g(x) = G(x, x+1]$, $I = \int_0^\infty z(y)\, dy$. Then:

(i) if $G \in \mathscr{S}_\Delta$ for all $T < \infty$, z is d.R.i., and $z(x)/g(x) \to 0$, then

$$Z(x) \sim \frac{I}{(1-\theta)^2}\, g(x);$$

(ii) if $G \in \mathscr{S}_\Delta$ for all $T < \infty$, z is d.R.i., and $z(x)/g(x) \to c \in (0, \infty)$, then

$$Z(x) \sim \left(\frac{I}{(1-\theta)^2} + \frac{c}{1-\theta}\right) g(x);$$

(iii) if the probability density $z(y)/I$ is subexponential and $z(x)/g(x) \to \infty$, then

$$Z(x) \sim \frac{1}{1-\theta}\, z(x).$$

Proof. In (i) and (ii), the assumptions imply $G(x, x+1/n] \sim g(x)/n$ for all n and $g(x+y)/g(x) \to 1$ uniformly in $|y| < y_0 < \infty$. Therefore applying Proposition 2 to the probability measure $(1-\theta)\, U$ and appealing to Proposition 12 with $T = 1/n$ shows that for each n we can find $h_n(x) \to \infty$ such that $h_n(x) < x/2$ and

$$U(x-(k+1)/n, x-k/n] \sim \frac{g(x)}{n(1-\theta)^2} \qquad \text{uniformly in} \quad k \leqslant nh_n(x), \tag{25}$$

$$\int_0^{x-h_n(x)} g(x-y)\, U(dy) \sim (1-\theta)^{-1}\, g(x), \tag{26}$$

$$\int_{h_n(x)}^{x-h_n(x)} g(x-y)\, U(dy) = o(g(x)) \tag{27}$$

(without loss of generality, we may assume that $nh_n(x)$ is an integer). We will use the decomposition $Z(x) = J_{1,n} + J_{2,n} + J_{3,n}$ where $J_{1,n} = \int_0^{h_n(x)} z(x-y)\, U(dy)$ and similarly $J_{2,n}$, $J_{3,n}$ are the integrals over $(h_n(x), x-h_n(x)]$, resp. $(x-h_n(x), x]$.

In (i), we replace h_n by a smaller h_n if necessary to ensure $z(x-y)/g(x) \to 0$ uniformly in $|y| \leqslant h_n(x)$ (this is possible since $g \in \mathscr{L}$), implying $J_{1,n} = o(g(x))$. Next,

$$J_{2,n} = o(1) \int_{h_n(x)}^{x-h_n(x)} g(x-y)\, U(dy) = o(g(x))$$

by (27). Finally, writing $\bar{z}_n(x) = \sup_{|y-x|\leqslant 1/n} z(y)$, (25) yields

$$J_{3,n} \leqslant \sum_{k=0}^{nh_n(x)} \bar{z}_n(k/n)\, U(x-(k+1)/n, x-k/n]$$

$$\sim \frac{g(x)/n}{(1-\theta)^2} \sum_{k=0}^{nh_n(x)} \bar{z}_n(k/n)$$

$$\sim \frac{g(x)/n}{(1-\theta)^2} \sum_{k=0}^{\infty} \bar{z}_n(k/n).$$

Since z is d.R.i., $n^{-1}\sum \cdots \to I$ as $n\to\infty$, yielding $\limsup Z(x)/g(x) \leqslant (1-\theta)^{-2} I$ in (i). The proof of $\liminf Z(x)/g(x) \geqslant (1-\theta)^{-2} I$ is similar.

In (ii), we may assume $z(x-y)/g(x) \to c$ uniformly in $|y| \leqslant h_n(x)$ and then get

$$J_{1,n} \sim cg(x)\, U(h_n(x)) \sim cg(x)\, U(\infty) = cg(x)(1-\theta)^{-1}.$$

For $J_{2,n}$, we have to replace $o(1)$ by $O(1)$, but the result remains $o(g(x))$. Finally, $J_{3,n}$ can be treated just as in (i), and (ii) is proved.

In (iii), consider the probability measure K with density $z(x)/I$. The measure K is Δ-subexponential for any Δ. Put $\Delta = (0, 1]$ and write

$$\int_0^x z(x-y)\, U(dy) = \int_0^{x-h} z(x-y)\, U(dy) + \int_{x-h}^x z(x-y)\, U(dy)$$

$$= I_1(x,h) + I_2(x,h),$$

$$(K*U)(x+\Delta) = \int_0^{x-h} K(x-y+\Delta)\, U(dy) + \int_{x-h}^{x+1} K(x-y+\Delta)\, U(dy)$$

$$= I_1'(x,h) + I_2'(x,h).$$

For any fixed h we have, as $x\to\infty$,

$$I_2(x,h) \leqslant h \cdot \sup_{y\leqslant h} |z(y)| \cdot U(x-h, x] = o(z(x))$$

and, by the same reasons, $I_2'(x,h) = o(z(x))$. Then it is possible to choose $h(x)\uparrow\infty$ such that still $I_2(x,h(x)) = o(z(x))$ and $I_2'(x,h(x)) = o(z(x))$. Since $z \in \mathscr{L}$, $z(x) \sim I\cdot K(x+\Delta)$ and $I_1(x,h(x)) \sim I\cdot I_1'(x,h(x))$. Combining these estimates we deduce

$$\int_0^x z(x-y)\, U(dy) \sim I\cdot (K*U)(x+\Delta).$$

Applying Proposition 3 with $G_1 = K$, $G_2 = U(1-\theta)$, $c_1 = 1$, and $c_2 = 0$ finally yields

$$I \cdot (K * U)(x+\Delta) \sim \frac{I}{1-\theta} K(x+\Delta) \sim \frac{z(x)}{1-\theta}.$$

7. THE COMPOUND POISSON PROCESS

Let F be a distribution on $\mathbf{R}_+$ and μ a positive constant. Let G be the compound Poisson distribution

$$G(B) = e^{-\mu} \sum_{n \geqslant 0} \frac{\mu^n}{n!} F^{*n}(B).$$

Theorem 6. Let $0 < T \leqslant \infty$. If $T < \infty$, then assume $F \in \mathscr{L}_\Delta$. Then the following assertions are equivalent:

(i) $F \in \mathscr{S}_\Delta$;

(ii) $G(x+\Delta) \sim \mu F(x+\Delta)$ as $x \to \infty$.

The proof follows from Theorem 2, with $p_n = \mu^n e^{-\mu}/n!$.

The case $T = \infty$ was considered, for regularly varying tails, in Refs. 8 and 12, and for subexponential tails, in Ref. 15.

8. INFINITELY DIVISIBLE LAWS

Let F be an infinitely divisible law on $[0, \infty)$. The Laplace transform of an infinitely divisible law F can be expressed as

$$\int_0^\infty e^{-\lambda x} F(dx) = e^{-a\lambda - \int_0^\infty (1-e^{-\lambda x})\, \nu(dx)}$$

(see, for example, Ref. 17, p. 450). Here $a \geqslant 0$ is a constant and the Lévy measure ν is a Borel measure on $(0, \infty)$ with the properties $\mu = \nu((1, \infty)) < \infty$ and $\int_0^1 x\nu(dx) < \infty$. Put $G(B) = \nu(B \cap (1, \infty))/\mu$.

The relations between the tail behaviour of measure F and the corresponding Lévy measure ν were considered in Theorem 1 in Ref. 15. We prove the following local analogue of that result.

Theorem 7. Let $0 < T \leqslant \infty$. If $T < \infty$, then assume $G \in \mathscr{L}_\Delta$. Then the following assertions are equivalent:

(i) $G \in \mathscr{S}_\Delta$;

(ii) $\nu(x+\Delta) \sim F(x+\Delta)$ as $x \to \infty$.

Proof. It is pointed out in Ref. 15 that the distribution F admits the representation $F = F_1 * F_2$, where $F_1(x, \infty) = O(e^{-\varepsilon x})$ for some $\varepsilon > 0$ and

$$F_2(B) = e^{-\mu} \sum_{n=0}^{\infty} \frac{\mu^n}{n!} G^{*n}(B).$$

Now, by Theorem 2, with $p_n = \mu^n e^{-\mu}/n!$ we have

$$F_2(x+\Delta) \sim \mu G(x+\Delta) = \nu(x+\Delta) \qquad \text{as} \quad x \to \infty.$$

Since $F_1(x+\Delta) = o(G(x+\Delta))$ as $x \to \infty$, by Proposition 3

$$F(x+\Delta) = (F_1 * F_2)(x+\Delta) \sim F_2(x+\Delta) \sim \nu(x+\Delta).$$

9. BRANCHING PROCESSES

In this section we consider the limit behaviour of sub-critical, age-dependent branching processes for which the Malthusian parameter does not exist.

Let $h(z)$ be the particle production generating function of an age-dependent branching process with particle lifetime distribution F (see Chap. IV in Ref. 5 and Chap. VI in Ref. 19 for background). We take the process to be sub-critical, i.e., $A \equiv h'(1) < 1$. Let $Z(t)$ denote the number of particles at time t. It is known (see, for example, Ref. 5, Chap. IV, Section 5 or Ref. 9) that $A(t) = \mathbf{E}Z(t)$ admits the representation

$$A(t) = (1-A) \sum_{n=1}^{\infty} A^{n-1}(1-F^{*n}(t)). \tag{28}$$

It was proved in Ref. 9 for sufficiently small values of A and then in Refs. 10 and 11 for any $A < 1$ that $A(t) \sim \overline{F}(t)/(1-A)$ as $t \to \infty$, provided $F \in \mathscr{S}$.

Applying Theorem 2 with $p_n = (1-A)\, A^{n-1}$ (see also Proposition 12), we deduce

Theorem 8. If $F \in \mathscr{L}_\Delta$, then the following are equivalent:

(i) $F \in \mathscr{S}_\Delta$;

(ii) $A(t) - A(t+T) \sim F(t+\Delta)/(1-A)$ as $t \to \infty$.

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REFERENCES

1. Asmussen, S. (1987). *Applied Probability and Queues*, Wiley, Chichester (2nd ed. (2003) Springer, New York).
2. Asmussen, S. (2000). *Ruin Probabilities*, World Scientific, Singapore.
3. Asmussen, S. (1998). A probabilistic look at the Wiener–Hopf equation. *SIAM Rev.* **40**, 189–201.
4. Asmussen, S., Kalashnikov, V., Konstantinides, D., Klüppelberg, C., and Tsitsiashvili, G. (2002). A local limit theorem for random walk maxima with heavy tails. *Statist. Probab. Lett.* **56**, 399–404.
5. Athreya, K., and Ney, P. (1972). *Branching Processes*, Springer, Berlin.
6. Bertoin, J., and Doney, R. A. (1994). On the local behaviour of ladder height distributions. *J. Appl. Probab.* **31**, 816–821.
7. Borovkov, A. A. (1976). *Stochastic Processes in Queueing Theory*, Springer, Berlin.
8. Callaert, H., and Cohen, J. W. (1972). A lemma on regular variation of a transient renewal function. *Z. Wahr. Verw. Gebiete* **24**, 275–278.
9. Chistyakov, V. P. (1964). A theorem on sums of independent positive random variables and its application to branching random processes. *Theory Probab. Appl.* **9**, 640–648.
10. Chover, J., Ney, P., and Wainger, S. (1973). Functions of probability measures. *J. Anal. Math.* **26**, 255–302.
11. Chover J., Ney P., and Wainger S. (1973). Degeneracy properties of subcritical branching processes. *Ann. Probab.* **1**, 663–673.
12. Cohen, J. W. (1973). Some results on regular variation for distributions in queueing and fluctuation theory. *J. Appl. Probab.* **10**, 343–353.
13. Embrechts, P., Klüppelberg, C., and Mikosch, T. (1997). *Modelling Extremal Events for Insurance and Finance*, Springer, Berlin.
14. Embrechts, P., and Goldie, C. M. (1982). On convolution tails. *Stochastic Process. Appl.* **13**, 263–278.
15. Embrechts, P., Goldie, C. M., and Veraverbeke, N. (1979). Subexponentiality and infinite divisibility. *Z. Wahr. Verw. Gebiete* **49**, 335–347.
16. Embrechts, P., and Veraverbeke, N. (1982). Estimates for the probability of ruin with special emphasis on the possibility of large claims. *Insurance Math. Econom.* **1**, 55–72.
17. Feller, W. (1971). *An Introduction to Probability Theory and Its Applications*, Vol. 2, Wiley, New York.
18. Foss, S., and Zachary, S. (2003). The maximum on a random time interval of a random walk with long-tailed increments and negative drift. *Ann. Appl. Probab.* **13**, 37–53.
19. Harris, T. (1963). *The Theory of Branching Processes*, Springer, Berlin.
20. Klüppelberg, C. (1989). Subexponential distributions and characterization of related classes. *Probab. Theory Related Fields* **82**, 259–269.
21. Korshunov, D. (1997). On distribution tail of the maximum of a random walk. *Stochastic Process. Appl.* **72**, 97–103.
22. Rolski, T., Schmidli, H., Schmidt, V., and Teugels, J. (1998). *Stochastic Processes for Insurance and Finance*, Wiley, Chichester.

23. Sgibnev, M. S. (1981). Banach algebras of functions that have identical asymptotic behaviour at infinity. *Siberian Math. J.* **22**, 179–187.
24. Teugels, J. L. (1975). The class of subexponential distributions. *Ann. Probab.* **3**, 1000–1011.
25. Veraverbeke, N. (1977). Asymptotic behavior of Wiener–Hopf factors of a random walk. *Stochastic Process. Appl.* **5**, 27–37.

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