On lower limits and equivalences for distribution tails of randomly stopped sums [1]

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Abstract

For a distribution $F^{*\tau}$ of a random sum $S_\tau = \xi_1 + \ldots + \xi_\tau$ of i.i.d. random variables with a common distribution F on the half-line $[0, \infty)$, we study the limits of the ratios of tails $\overline{F^{*\tau}}(x)/\overline{F}(x)$ as $x \to \infty$ (here τ is an independent counting random variable). We also consider applications of obtained results to random walks, compound Poisson distributions, infinitely divisible laws, and sub-critical branching processes.

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1. Introduction. Let $\xi, \xi_1, \xi_2, \ldots,$ be independent identically distributed nonnegative random variables. We assume that their common distribution F on the half-line $[0, \infty)$ has an unbounded support, that is, $\overline{F}(x) \equiv F(x, \infty) > 0$ for all x. Put $S_0 = 0$ and $S_n = \xi_1 + \ldots + \xi_n$, $n = 1, 2, \ldots$.

Let τ be a counting random variable which does not depend on $\{\xi_n\}_{n \geq 1}$ and has finite mean. Denote by $F^{*\tau}$ the distribution of a randomly stopped sum $S_\tau = \xi_1 + \ldots + \xi_\tau$.

In this paper we discuss how does the tail behaviour of $F^{*\tau}$ relate to that of F and, in particular, under what conditions

$$\liminf_{x \to \infty} \frac{\overline{F^{*\tau}}(x)}{\overline{F}(x)} = \mathbf{E}\tau. \tag{1}$$

Relations on lower limits of ratios of tails have been first discussed by Rudin [21]. Theorem 2* of that paper states (for an integer p) the following

Theorem 1. *Let there exists a positive $p \in [1, \infty)$ such that $\mathbf{E}\xi^p = \infty$, but $\mathbf{E}\tau^p < \infty$. Then* (1) *holds.*

Rudin's studies were motivated by the paper [7] of Chover, Ney, and Wainger who considered, in particular, the problem of existence of a limit for the ratio

$$\frac{\overline{F^{*\tau}}(x)}{\overline{F}(x)} \qquad \text{as } x \to \infty. \tag{2}$$

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¿From Theorem 1, it follows that, if F and τ satisfy its conditions and if a limit of (2) exists, then that limit must be equal $\mathbf{E}\tau$.

Rudin proved Theorem 1 via probability generating functions techniques. Below we give an alternative and a more direct proof of Theorem 1 in the case of any positive p (i.e. not necessarily integer). Our method is based on truncation arguments; in this way, we propose a general scheme (see Theorem 4 below) which may be applied also to distributions with all finite moments.

The condition $\mathbf{E}\xi^p = \infty$ rules out a lot of distributions of interest, say, in the theory of subexponential distributions. For example, log-normal and Weibull-type distributions have all moments finite. Our first result presents a natural condition on a stopping time τ guaranteeing relation (1) for the whole class of heavy-tailed distributions.

Recall that a random variable ξ has *a light-tailed* distribution F on $[0, \infty)$ if $\mathbf{E}e^{\gamma\xi} < \infty$ with some $\gamma > 0$. Otherwise F is called *a heavy-tailed* distribution; this happens if and only if $\mathbf{E}e^{\gamma\xi} = \infty$ for all $\gamma > 0$.

Theorem 2. *Let F be a heavy-tailed distribution and τ have a light-tailed distribution. Then* (1) *holds.*

Proof of Theorem 2 is based on a new technical tool (see Lemma 2) and significantly differs from a proof of Theorem 1 in [15] where a particular case $\tau = 2$ was considered. Theorem 2 is restricted to the case of light-tailed τ, but here extends Rudin's result to the class of all heavy-tailed distributions. The reasons for the restriction to $\mathbf{E}e^{\gamma\tau} < \infty$ come from the proof of Theorem 2 but in fact are rather natural; the tail of τ should be lighter than the tail of any heavy-tailed distribution. Indeed, if $\xi_1 \geq 1$ then $\overline{F^{*\tau}}(x) \geq \mathbf{P}\{\tau > x\}$. This shows that the tail of $F^{*\tau}$ is at least as heavy as that of τ. Note that, in Theorem 1, in some sense, the tail of F is heavier than the tail of τ.

Theorem 2 may be applied in various areas where randomly stopped sums do appear – see Sections 8–11 (random walks, compound Poisson distributions, infinitely divisible laws, and branching processes) and, e.g. [17] for further examples.

For any distribution on $[0, \infty)$, let

$$\varphi(\gamma) = \int_0^\infty e^{\gamma x} F(dx) \in (0, \infty], \quad \gamma \in \mathbf{R},$$

and

$$\widehat{\gamma} = \sup\{\gamma : \varphi(\gamma) < \infty\} \in [0, \infty].$$

Note that the moment-generating function $\varphi(\gamma)$ is monotone continuous in the interval $(-\infty, \widehat{\gamma})$, and $\varphi(\widehat{\gamma}) = \lim_{\gamma \uparrow \widehat{\gamma}} \varphi(\gamma) \in [1, \infty]$.

Theorem 3. *Let $\varphi(\widehat{\gamma}) < \infty$ and $\mathbf{E}(\varphi(\widehat{\gamma}) + \varepsilon)^\tau < \infty$ for some $\varepsilon > 0$. Assume that*

$$\frac{\overline{F^{*\tau}}(x)}{\overline{F}(x)} \rightarrow c \quad as\ x \to \infty,$$

where $c \in (0, \infty]$. Then $c = \mathbf{E}(\tau\varphi^{\tau-1}(\widehat{\gamma}))$.

For (comments on) earlier partial results in the case $\tau = 2$, see, e.g., papers [6–8, 10, 15, 19, 20, 23] and further references therein. The proof of Theorem 3 follows from Lemmas 3 and 4 in Section 7.

2. Preliminary result. We start with the following

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Theorem 4. *Let there exist a non-decreasing concave function $h : \mathbf{R}^+ \to \mathbf{R}^+$ such that*

$$\mathbf{E}e^{h(\xi)} < \infty \quad \text{and} \quad \mathbf{E}\xi e^{h(\xi)} = \infty. \tag{3}$$

For any $n \geq 1$, put $A_n = \mathbf{E}e^{h(\xi_1+\dots+\xi_n)}$. Assume that F is heavy-tailed and

$$\mathbf{E}\tau A_{\tau-1} \quad < \quad \infty. \tag{4}$$

Then, for any light-tailed distribution G on $[0, \infty)$,

$$\liminf_{x \to \infty} \frac{\overline{G * F^{*\tau}}(x)}{\overline{F}(x)} \quad = \quad \mathbf{E}\tau. \tag{5}$$

By considering G concentrated at 0, we get the following

Corollary 1. *In conditions of Theorem 4, (1) holds.*

Proof of Theorem 4. First we restate Theorem 1^* of Rudin [21] in Lemma 1 below in terms of probability distributions and stopping times.

Lemma 1. *For any distribution F on $[0, \infty)$ with unbounded support and any independent counting random variable τ,*

$$\liminf_{x \to \infty} \frac{\overline{F^{*\tau}}(x)}{\overline{F}(x)} \quad \geq \quad \mathbf{E}\tau.$$

Proof. For any two distributions F_1 and F_2 on $[0, \infty)$ with unbounded supports,

$$\begin{aligned} \overline{F_1 * F_2}(x) &\geq (F_1 \times F_2)((x, \infty) \times [0, x]) + (F_1 \times F_2)([0, x] \times (x, \infty)) \\ &\sim \overline{F}_1(x) + \overline{F}_2(x) \quad \text{as } x \to \infty. \end{aligned}$$

By induction arguments, it implies that, for any $n \geq 1$,

$$\liminf_{x \to \infty} \frac{\overline{F^{*n}}(x)}{\overline{F}(x)} \quad \geq \quad n.$$

Applying Fatou's lemma to the representation

$$\frac{\overline{F^{*\tau}}(x)}{\overline{F}(x)} \quad = \quad \sum_{n=1}^{\infty} \mathbf{P}\{\tau = n\} \frac{\overline{F^{*n}}(x)}{\overline{F}(x)},$$

we arrive at the lemma conclusion.

It follows from Lemma 1 that it is sufficient to prove the upper bound in Theorem 4. Assume the contrary, i.e. there exist $\delta > 0$ and x_0 such that

$$\overline{G * F^{*\tau}}(x) \quad \geq \quad (\mathbf{E}\tau + \delta)\overline{F}(x) \quad \text{for all } x > x_0. \tag{6}$$

For any positive $b > 0$, consider a concave function

$$h_b(x) \quad \equiv \quad \min\{h(x), bx\}. \tag{7}$$

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Since F is heavy-tailed, $h(x) = o(x)$ as $x \to \infty$. Therefore, for any fixed b, there exists x_1 such that $h_b(x) = h(x)$ for all $x > x_1$. Hence, by the condition (3),

$$\mathbf{E}e^{h_b(\xi)} < \infty \quad \text{and} \quad \mathbf{E}\xi e^{h_b(\xi)} = \infty. \tag{8}$$

For any x, we have the convergence $h_b(x) \downarrow 0$ as $b \downarrow 0$. Then, for any fixed n,

$$A_{n,b} \equiv \mathbf{E}e^{h_b(\xi_1 + \ldots + \xi_n)} \quad \downarrow \quad 1 \quad \text{as } b \downarrow 0.$$

This and the condition (4) imply that there exists b such that

$$\mathbf{E}\tau A_{\tau-1,b} \quad \leq \quad \mathbf{E}\tau + \delta/8. \tag{9}$$

Let η be a random variable with distribution G which does not depend on $\{\xi_n\}_{n \geq 1}$ and τ. Since G is light-tailed,

$$\mathbf{E}\eta e^{h_b(\eta)} < \infty. \tag{10}$$

In addition, we may choose $b > 0$ so small that

$$\mathbf{E}e^{h_b(\eta)}(\mathbf{E}\tau + \delta/8) \quad \leq \quad \mathbf{E}\tau + \delta/4. \tag{11}$$

For any real a and t, put $a^{[t]} = \min\{a, t\}$. Then

$$\frac{\mathbf{E}(\eta + \xi_1^{[t]} + \ldots + \xi_\tau^{[t]})e^{h_b(\eta + \xi_1 + \ldots + \xi_\tau)}}{\mathbf{E}\xi_1^{[t]}e^{h_b(\xi_1)}} = \sum_{n=1}^{\infty} \frac{\mathbf{E}\eta e^{h_b(\eta + \xi_1 + \ldots + \xi_n)}}{\mathbf{E}\xi_1^{[t]}e^{h_b(\xi_1)}}\mathbf{P}\{\tau = n\}$$
$$+ \sum_{n=1}^{\infty} n\frac{\mathbf{E}\xi_1^{[t]}e^{h_b(\eta + \xi_1 + \ldots + \xi_n)}}{\mathbf{E}\xi_1^{[t]}e^{h_b(\xi_1)}}\mathbf{P}\{\tau = n\}.$$

By concavity of the function h_b,

$$\sum_{n=1}^{\infty} \frac{\mathbf{E}\eta e^{h_b(\eta + \xi_1 + \ldots + \xi_n)}}{\mathbf{E}\xi_1^{[t]}e^{h_b(\xi_1)}}\mathbf{P}\{\tau = n\} \quad \leq \quad \sum_{n=1}^{\infty} \frac{\mathbf{E}\eta e^{h_b(\eta) + h_b(\xi_1 + \ldots + \xi_n)}}{\mathbf{E}\xi_1^{[t]}e^{h_b(\xi_1)}}\mathbf{P}\{\tau = n\}$$
$$= \quad \frac{\mathbf{E}\eta e^{h_b(\eta)}}{\mathbf{E}\xi_1^{[t]}e^{h_b(\xi_1)}}\mathbf{E}A_{\tau,b}$$
$$\to \quad 0 \quad \text{as } t \to \infty,$$

due to (10), (9), and (8). Again by concavity of the function h_b,

$$\sum_{n=1}^{\infty} n\frac{\mathbf{E}\xi_1^{[t]}e^{h_b(\eta + \xi_1 + \ldots + \xi_n)}}{\mathbf{E}\xi_1^{[t]}e^{h_b(\xi_1)}}\mathbf{P}\{\tau = n\} \quad \leq \quad \sum_{n=1}^{\infty} n\frac{\mathbf{E}\xi_1^{[t]}e^{h_b(\eta) + h_b(\xi_1) + h_b(\xi_2 + \ldots + \xi_n)}}{\mathbf{E}\xi_1^{[t]}e^{h_b(\xi_1)}}\mathbf{P}\{\tau = n\}$$
$$= \quad \mathbf{E}e^{h_b(\eta)}\sum_{n=1}^{\infty} nA_{n-1,b}\mathbf{P}\{\tau = n\}$$
$$\leq \quad \mathbf{E}\tau + \delta/4,$$

by (9) and (11). Hence, for sufficiently large t,

$$\frac{\mathbf{E}(\eta + \xi_1^{[t]} + \ldots + \xi_\tau^{[t]})e^{h_b(\eta + \xi_1 + \ldots + \xi_\tau)}}{\mathbf{E}\xi_1^{[t]}e^{h_b(\xi_1)}} \quad \leq \quad \mathbf{E}\tau + \delta/2. \tag{12}$$

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On the other hand, since $(\eta + \xi_1 + \ldots + \xi_\tau)^{[t]} \leq \eta + \xi_1^{[t]} + \ldots + \xi_\tau^{[t]}$,

$$\frac{\mathbf{E}(\eta + \xi_1^{[t]} + \ldots + \xi_\tau^{[t]})e^{h_b(\eta + \xi_1 + \ldots + \xi_\tau)}}{\mathbf{E}\xi_1^{[t]}e^{h_b(\xi_1)}} \geq \frac{\mathbf{E}(\eta + \xi_1 + \ldots + \xi_\tau)^{[t]}e^{h_b(\eta + \xi_1 + \ldots + \xi_\tau)}}{\mathbf{E}\xi_1^{[t]}e^{h_b(\xi_1)}}$$

$$= \frac{\int_0^\infty x^{[t]}e^{h_b(x)}(G * F^{*\tau})(dx)}{\int_0^\infty x^{[t]}e^{h_b(x)}F(dx)}. \qquad (13)$$

The right side, after integration by parts, is equal to

$$\frac{\int_0^\infty \overline{G * F^{*\tau}}(x)d(x^{[t]}e^{h_b(x)})}{\int_0^\infty \overline{F}(x)d(x^{[t]}e^{h_b(x)})}.$$

Since $\mathbf{E}\xi_1 e^{h_b(\xi_1)} = \infty$, both integrals (the divident and the divisor) in this fraction tend to infinity as $t \to \infty$. For the *non-decreasing* function $h_b(x)$, the latter and the assumption (6) imply that

$$\liminf_{t \to \infty} \frac{\int_0^\infty \overline{G * F^{*\tau}}(x)d(x^{[t]}e^{h_b(x)})}{\int_0^\infty \overline{F}(x)d(x^{[t]}e^{h_b(x)})} = \liminf_{t \to \infty} \frac{\int_{x_0}^\infty \overline{G * F^{*\tau}}(x)d(x^{[t]}e^{h_b(x)})}{\int_{x_0}^\infty \overline{F}(x)d(x^{[t]}e^{h_b(x)})} \geq \mathbf{E}\tau + \delta.$$

Substituting this into (13) we get a contradiction to (12) for sufficiently large t. The proof is complete.

3. Proof of Theorem 1. Take an integer k such that $p - 1 \leq k < p$. Without loss of generality, we may assume that $\mathbf{E}\xi^k < \infty$ (otherwise we may consider a smaller p).

Consider a concave non-decreasing function $h(x) = (p - 1)\ln x$. Then $\mathbf{E}e^{h(\xi_1)} < \infty$ and $\mathbf{E}\xi_1 e^{h(\xi_1)} = \infty$. Thus,

$$A_n \equiv \mathbf{E}e^{h(\xi_1 + \ldots + \xi_n)} = \mathbf{E}(\xi_1 + \ldots + \xi_n)^{p-1}$$
$$\leq (\mathbf{E}(\xi_1 + \ldots + \xi_n)^k)^{(p-1)/k},$$

since $(p - 1)/k \leq 1$. Further,

$$\mathbf{E}(\xi_1 + \ldots + \xi_n)^k = \sum_{i_1, \ldots, i_k = 1}^{n} \mathbf{E}(\xi_{i_1} \cdot \ldots \cdot \xi_{i_k}) \leq cn^k,$$

where

$$c \equiv \sup_{1 \leq i_1, \ldots, i_k \leq n} \mathbf{E}(\xi_{i_1} \cdot \ldots \cdot \xi_{i_k}) < \infty,$$

due to $\mathbf{E}\xi^k < \infty$. Hence, $A_n \leq c^{(p-1)/k}n^{p-1}$ for all n. Therefore, we get $\mathbf{E}\tau A_{\tau-1} \leq c^{(p-1)/k}\mathbf{E}\tau^p < \infty$. All conditions of Theorem 4 are met and the proof is complete.

4. Characterization of heavy-tailed distributions. In the sequel we need the following existence result which strengthens a lemma in Rudin [21, page 989] and Lemma 1 in [15]. Fix any $\delta \in (0, 1]$.

Lemma 2. *If a random variable $\xi \geq 0$ has a heavy-tailed distribution, then there exists a monotone concave function $h : \mathbf{R}^+ \to \mathbf{R}^+$ such that $\mathbf{E}e^{h(\xi)} \leq 1 + \delta$ and $\mathbf{E}\xi e^{h(\xi)} = \infty$.*

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Proof. Without loss of generality assume that $\xi > 0$ a.s., that is, $\overline{F}(0) = 1$. We will construct a piecewise linear function $h(x)$. For that we introduce two positive sequences, $x_n \uparrow \infty$ and $\varepsilon_n \downarrow 0$ as $n \to \infty$, and let

$$h(x) = h(x_{n-1}) + \varepsilon_n(x - x_{n-1}) \quad \text{if } x \in (x_{n-1}, x_n], \ n \geq 1.$$

This function is monotone, since $\varepsilon_n > 0$. Moreover, this function is concave, due to the monotonicity of ε_n.

Put $x_0 = 0$ and $h(0) = 0$. Since ξ is heavy-tailed, we can choose $x_1 \geq 2^1$ so that

$$\mathbf{E}\{e^\xi; \xi \in (x_0, x_1]\} + e^{x_1}\overline{F}(x_1) \quad > \quad 1 + \delta.$$

Choose $\varepsilon_1 > 0$ so that

$$\mathbf{E}\{e^{\varepsilon_1 \xi}; \xi \in (x_0, x_1]\} + e^{\varepsilon_1 x_1}\overline{F}(x_1) \quad = \quad e^{h(x_0)}\overline{F}(0) + \delta/2 = 1 + \delta/2,$$

which is equivalent to say that

$$\mathbf{E}\{e^{h(\xi)}; \xi \in (x_0, x_1]\} + e^{h(x_1)}\overline{F}(x_1) \quad = \quad e^{h(x_0)}\overline{F}(0) + \delta/2,$$

By induction we construct an increasing sequence x_n and a decreasing sequence $\varepsilon_n > 0$ such that $x_n \geq 2^n$ and

$$\mathbf{E}\{e^{h(\xi)}; \xi \in (x_{n-1}, x_n]\} + e^{h(x_n)}\overline{F}(x_n) \quad = \quad e^{h(x_{n-1})}\overline{F}(x_{n-1}) + \delta/2^n$$

for any $n \geq 2$. For $n = 1$ this is already done. Make the induction hypothesis for some $n \geq 2$. Due to heavy-tailedness, there exists $x_{n+1} \geq 2^{n+1}$ so large that

$$\mathbf{E}\{e^{\varepsilon_n(\xi - x_n)}; \xi \in (x_n, x_{n+1}]\} + e^{\varepsilon_n(x_{n+1} - x_n)}\overline{F}(x_{n+1}) \quad > \quad 1 + \delta.$$

Note that

$$\mathbf{E}\{e^{\varepsilon_{n+1}(\xi - x_n)}; \xi \in (x_n, x_{n+1}]\} + e^{\varepsilon_{n+1}(x_{n+1} - x_n)}\overline{F}(x_{n+1})$$

as a function of ε_{n+1} is continuously decreasing to $\overline{F}(x_n)$ as $\varepsilon_{n+1} \downarrow 0$. Therefore, we can choose $\varepsilon_{n+1} \in (0, \varepsilon_n)$ so that

$$\mathbf{E}\{e^{\varepsilon_{n+1}(\xi - x_n)}; \xi \in (x_n, x_{n+1}]\} + e^{\varepsilon_{n+1}(x_{n+1} - x_n)}\overline{F}(x_{n+1})$$
$$= \overline{F}(x_n) + \delta/(2^{n+1}e^{h(x_n)}).$$

By definition of $h(x)$ this is equivalent to the following equality:

$$\mathbf{E}\{e^{h(\xi)}; \xi \in (x_n, x_{n+1}]\} + e^{h(x_{n+1})}\overline{F}(x_{n+1}) \quad = \quad e^{h(x_n)}\overline{F}(x_n) + \delta/2^{n+1}.$$

Our induction hypothesis now holds with $n + 1$ in place of n as required.

Next,

$$\mathbf{E}e^{h(\xi)} \quad = \quad \sum_{n=0}^{\infty} \mathbf{E}\{e^{h(\xi)}; \xi \in (x_n, x_{n+1}]\}$$
$$= \quad \sum_{n=0}^{\infty} \left(e^{h(x_n)}\overline{F}(x_n) - e^{h(x_{n+1})}\overline{F}(x_{n+1}) + \delta/2^{n+1} \right)$$
$$= \quad e^{h(x_0)}\overline{F}(x_0) + \delta = 1 + \delta.$$

On the other hand, since $x_k \geq 2^k$,

$$\mathbf{E}\{\xi e^{h(\xi)}; \xi > x_n\} = \sum_{k=n}^{\infty} \mathbf{E}\{\xi e^{h(\xi)}; \xi \in (x_k, x_{k+1}]\}$$

$$\geq 2^n \sum_{k=n}^{\infty} \mathbf{E}\{e^{h(\xi)}; \xi \in (x_k, x_{k+1}]\}$$

$$\geq 2^n \sum_{k=n}^{\infty} \left(e^{h(x_k)}\overline{F}(x_k) - e^{h(x_{k+1})}\overline{F}(x_{k+1}) + \delta/2^{k+1} \right).$$

Then, for any n,

$$\mathbf{E}\{\xi e^{h(\xi)}; \xi > x_n\} \geq 2^n (e^{h(x_n)}\overline{F}(x_n) + \delta/2^n) \geq \delta,$$

which implies $\mathbf{E}\xi e^{h(\xi)} = \infty$. Note also that necessarily $\lim_{n\to\infty} \varepsilon_n = 0$; otherwise $\liminf_{x\to\infty} h(x)/x > 0$ and ξ is light tailed. The proof of the lemma is complete.

5. Proof of Theorem 2. Since τ has a light-tailed distribution,

$$\mathbf{E}\tau(1 + \varepsilon)^{\tau - 1} < \infty$$

for some sufficiently small $\varepsilon > 0$. By Lemma 2, there exists a concave increasing function h, $h(0) = 0$, such that $\mathbf{E}e^{h(\xi_1)} \leq 1 + \varepsilon$ and $\mathbf{E}\xi_1 e^{h(\xi_1)} = \infty$. Then by concavity

$$A_n \equiv \mathbf{E}e^{h(\xi_1 + \ldots + \xi_n)} \leq \mathbf{E}e^{h(\xi_1) + \ldots + h(\xi_n)} \leq (1 + \varepsilon)^n.$$

Combining altogether, we get $\mathbf{E}\tau A_{\tau-1} < \infty$. All conditions of Theorem 4 are met and the proof is complete.

6. Fractional exponential moments. One can go further and obtain various results on lower limits and equivalencies for heavy-tailed distributions F which have all finite power moments (like Weibull and log-normal distributions). For instance, the following result takes place (see [9] for the proof):

Let there exist α, $0 < \alpha < 1$, such that $\mathbf{E}e^{c\xi^\alpha} = \infty$ for all $c > 0$. If $\mathbf{E}e^{\delta\tau^\alpha} < \infty$ for some $\delta > 0$, then (1) holds.

7. Tail equivalence for randomly stopped sums. The following auxiliary lemma compares the tail behavior of the convolution tail and that of the exponentially transformed distribution.

Lemma 3. *Let the distribution F and the number $\gamma \geq 0$ be such that $\varphi(\gamma) < \infty$. Let the distribution G be the result of the exponential change of measure with parameter γ, i.e., $G(du) = e^{\gamma u}F(du)/\varphi(\gamma)$. Let τ be an independent stopping time such that $\mathbf{E}\varphi^\tau(\gamma) < \infty$ and ν have the distribution $\mathbf{P}\{\nu = k\} = \varphi^k(\gamma)\mathbf{P}\{\tau = k\}/\mathbf{E}\varphi^\tau(\gamma)$. Then*

$$\liminf_{x\to\infty} \frac{\overline{G^{*\nu}}(x)}{\overline{G}(x)} \geq \frac{1}{\mathbf{E}\varphi^{\tau-1}(\gamma)} \liminf_{x\to\infty} \frac{\overline{F^{*\tau}}(x)}{\overline{F}(x)}$$

and

$$\limsup_{x\to\infty} \frac{\overline{G^{*\nu}}(x)}{\overline{G}(x)} \leq \frac{1}{\mathbf{E}\varphi^{\tau-1}(\gamma)} \limsup_{x\to\infty} \frac{\overline{F^{*\tau}}(x)}{\overline{F}(x)}.$$

Proof. Put

$$\widehat{c} \;\equiv\; \liminf_{x\to\infty} \frac{\overline{F^{*\tau}}(x)}{\overline{F}(x)}.$$

By Lemma 1, $\widehat{c} \in [\mathbf{E}\tau, \infty]$. For any fixed $c \in (0, \widehat{c})$, there exists $x_0 > 0$ such that, for any $x > x_0$,

$$\overline{F^{*\tau}}(x) \;\geq\; c\overline{F}(x). \tag{14}$$

By the total probability law,

$$\begin{aligned}
\overline{G^{*\nu}}(x) &= \sum_{k=1}^{\infty} \mathbf{P}\{\nu = k\}\overline{G^{*k}}(x) \\
&= \sum_{k=1}^{\infty} \frac{\varphi^k(\gamma)\mathbf{P}\{\tau = k\}}{\mathbf{E}\varphi^\tau(\gamma)} \int_x^\infty e^{\gamma y}\frac{F^{*k}(dy)}{\varphi^k(\gamma)} \\
&= \frac{1}{\mathbf{E}\varphi^\tau(\gamma)}\sum_{k=1}^{\infty}\mathbf{P}\{\tau = k\}\int_x^\infty e^{\gamma y}F^{*k}(dy).
\end{aligned}$$

Integrating by parts, we obtain

$$\sum_{k=1}^{\infty}\mathbf{P}\{\tau = k\}\left[e^{\gamma x}\overline{F^{*k}}(x) + \int_x^\infty \overline{F^{*k}}(y)de^{\gamma y}\right]$$
$$= e^{\gamma x}\overline{F^{*\tau}}(x) + \int_x^\infty \overline{F^{*\tau}}(y)de^{\gamma y}.$$

Using also (14) we get, for $x > x_0$,

$$\begin{aligned}
\overline{G^{*\nu}}(x) &\geq \frac{c}{\mathbf{E}\varphi^\tau(\gamma)}\left[e^{\gamma x}\overline{F}(x) + \int_x^\infty \overline{F}(y)de^{\gamma y}\right] \\
&= \frac{c}{\mathbf{E}\varphi^\tau(\gamma)}\int_x^\infty e^{\gamma y}F(dy) = \frac{c}{\mathbf{E}\varphi^{\tau-1}(\gamma)}\overline{G}(x).
\end{aligned}$$

Letting $c \uparrow \widehat{c}$, we obtain the first conclusion of the lemma. The proof of the second conclusion follows similarly.

Lemma 4. *If* $0 < \widehat{\gamma} < \infty$, $\varphi(\widehat{\gamma}) < \infty$, *and* $\mathbf{E}(\varphi(\widehat{\gamma}) + \varepsilon)^\tau < \infty$ *for some* $\varepsilon > 0$, *then*

$$\liminf_{x\to\infty}\frac{\overline{F^{*\tau}}(x)}{\overline{F}(x)} \;\leq\; \mathbf{E}\tau\varphi^{\tau-1}(\widehat{\gamma})$$

and

$$\limsup_{x\to\infty}\frac{\overline{F^{*\tau}}(x)}{\overline{F}(x)} \;\geq\; \mathbf{E}\tau\varphi^{\tau-1}(\widehat{\gamma}).$$

Proof. We apply the exponential change of measure with parameter $\widehat{\gamma}$ and consider the distribution $G(du) = e^{\widehat{\gamma}u}F(du)/\varphi(\widehat{\gamma})$ and the stopping time ν with the distribution $\mathbf{P}\{\nu = k\} = \varphi^k(\widehat{\gamma})\mathbf{P}\{\tau = k\}/\mathbf{E}\varphi^\tau(\widehat{\gamma})$. ¿From the definition of $\widehat{\gamma}$, the distribution G is heavy-tailed. The distribution of ν is light-tailed, because $\mathbf{E}e^{\kappa\nu} < \infty$ with $\kappa = \ln(\varphi(\widehat{\gamma}) + \varepsilon) - \ln\varphi(\widehat{\gamma}) > 0$. Hence,

$$\limsup_{x\to\infty}\frac{\overline{G^{*\nu}}(x)}{\overline{G}(x)} \geq \liminf_{x\to\infty}\frac{\overline{G^{*\nu}}(x)}{\overline{G}(x)} \;=\; \mathbf{E}\nu,$$

by Theorem 2. The result now follows from Lemma 3 with $\gamma = \widehat{\gamma}$, since $\mathbf{E}\nu = \mathbf{E}\tau\varphi^\tau(\widehat{\gamma})/\mathbf{E}\varphi^\tau(\widehat{\gamma})$.

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Proof of Theorem 3. In the case where F is heavy-tailed, we have $\widehat{\gamma} = 0$ and $\varphi(\widehat{\gamma}) = 1$. By Theorem 2, $c = \mathbf{E}\tau$ as required.

In the case $\widehat{\gamma} \in (0, \infty)$ and $\varphi(\widehat{\gamma}) < \infty$, the desired conclusion follows from Lemma 4.

8. Supremum of a random walk. Let $\{\xi_n\}$ be a sequence of independent random variables with a common distribution F on $\mathbf{R}$ and $\mathbf{E}\xi_1 = -m < 0$. Put $S_0 = 0$, $S_n = \xi_1 + \cdots + \xi_n$. By the SLLN, $M = \sup_{n \geq 0} S_n$ is finite with probability 1.

Let F^I be the integrated-tail distribution on $\mathbf{R}^+$, that is,

$$\overline{F^I}(x) \equiv \min\left(1, \int_x^\infty \overline{F}(y)dy\right), \quad x > 0.$$

It is well-known (see, e.g. [1, 12, 13] and references therein) that if $F^I \in \mathscr{S}$, then

$$\mathbf{P}\{M > x\} \sim \frac{1}{m}\overline{F^I}(x) \quad \text{as } x \to \infty. \tag{15}$$

Korshunov [18] proved the converse: (15) implies $F^I \in \mathscr{S}$. Now we accompany this assertion by the following

Theorem 5. *Let F^I be long-tailed, that is, $\overline{F^I}(x+1) \sim \overline{F^I}(x)$ as $x \to \infty$. If, for some $c > 0$,*

$$\mathbf{P}\{M > x\} \sim c\overline{F^I}(x) \quad \text{as } x \to \infty,$$

then $c = 1/m$ and F^I is subexponential.

Proof. Consider the defective stopping time

$$\eta = \inf\{n \geq 1 : S_n > 0\} \leq \infty$$

and let $\{\psi_n\}$ be i.i.d. random variables with common distribution function

$$G(x) \equiv \mathbf{P}\{\psi_n \leq x\} = \mathbf{P}\{S_\eta \leq x \mid \eta < \infty\}.$$

It is well-known (see, e.g. Feller [14, Chapter 12]) that the distribution of the maximum M coincides with the distribution of the randomly stopped sum $\psi_1 + \cdots + \psi_\tau$, where the stopping time τ is independent of the sequence $\{\psi_n\}$ and is geometrically distributed with parameter $p = \mathbf{P}\{M > 0\} < 1$, i.e., $\mathbf{P}\{\tau = k\} = (1-p)p^k$ for $k = 0, 1, \ldots$. Equivalently,

$$\mathbf{P}\{M \in B\} = G^{*\tau}(B).$$

¿From Borovkov [4, Chapter 4, Theorem 10], if F^I is long-tailed, then

$$\overline{G}(x) \sim \frac{1-p}{pm}\overline{F^I}(x). \tag{16}$$

Then it follows from the theorem hypothesis that

$$\overline{G^{*\tau}}(x) \sim \frac{cpm}{1-p}\overline{G}(x) \quad \text{as } x \to \infty.$$

Therefore, by Theorem 3 with $\widehat{\gamma} = 0$, $c = \mathbf{E}\tau(1-p)/pm = 1/m$. Then it follows from [18] that F^I is subexponential. Now the proof is complete.

9. The compound Poisson distribution. Let F be a distribution on $\mathbf{R}_+$ and t a positive constant. Let G be the compound Poisson distribution

$$G = e^{-t}\sum_{n \geq 0} \frac{t^n}{n!}F^{*n}.$$

Considering τ in Theorem 3 with distribution $\mathbf{P}\{\tau = n\} = t^n e^{-t}/n!$, we get

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Theorem 6. *Let $\varphi(\widehat{\gamma}) < \infty$. If, for some $c > 0$, $\overline{G}(x) \sim c\overline{F}(x)$ as $x \to \infty$, then $c = te^{t(\varphi(\widehat{\gamma})-1)}$.*

Corollary 2. *The following statements are equivalent:*
    (i) *F is subexponential;*
    (ii) *G is subexponential;*
    (iii) *$\overline{G}(x) \sim t\overline{F}(x)$ as $x \to \infty$;*
    (iv) *F is heavy-tailed and $\overline{G}(x) \sim c\overline{F}(x)$ as $x \to \infty$, for some $c > 0$.*

Proof. Equivalence of (i), (ii), and (iii) was proved in [11, Theorem 3]. The implication (iv)⇒(iii) follows from Theorem 3 with $\widehat{\gamma} = 0$.

Some local aspects of this problem for heavy-tailed distributions were discussed in [2, Theorem 6].

10. Infinitely divisible laws. Let H be an infinitely divisible law on $[0, \infty)$. The Laplace transform of an infinitely divisible law F can be expressed as

$$\int_0^\infty e^{-\lambda x} H(dx) \;=\; e^{-a\lambda - \int_0^\infty (1-e^{-\lambda x})\nu(dx)}$$

(see, e.g. [14, Chapter XVII]). Here $a \geq 0$ is a constant and the Lévy measure ν is a Borel measure on $(0, \infty)$ with the properties $\mu = \nu(1, \infty) < \infty$ and $\int_0^1 x\nu(dx) < \infty$. Put $F(B) = \nu(B \cap (1, \infty))/\mu$.

Relations between the tail behaviour of measure H and of the corresponding Lévy measure ν were considered in [11, 19, 22]. The local analogue of that result was proved in [2]. We strengthen the corresponding result of [11] in the following way.

Theorem 7. *The following assertions are equivalent:*
    (i) *H is subexponential;*
    (ii) *F is subexponential;*
    (iii) *$\overline{\nu}(x) \sim \overline{H}(x)$ as $x \to \infty$;*
    (iv) *H is heavy-tailed and $\overline{\nu}(x) \sim c\overline{H}(x)$ as $x \to \infty$, for some $c > 0$.*

Proof. Equivalence of (i), (ii), and (iii) was proved in [11, Theorem 1].

It remains to prove the implication (iv)⇒(iii). It is pointed out in [11] that the distribution H admits the representation $H = G * F^{*\tau}$, where $\overline{G}(x) = O(e^{-\varepsilon x})$ for some $\varepsilon > 0$ and τ has a Poisson distribution with parameter μ. Since H is heavy-tailed and G is light-tailed, F is heavy-tailed with necessity. Then by Theorem 4 we get

$$\liminf_{x\to\infty} \frac{\overline{H}(x)}{\overline{F}(x)} \equiv \liminf_{x\to\infty} \frac{\overline{G * F^{*\tau}}(x)}{\overline{F}(x)} \;=\; \mathbf{E}\tau = \mu.$$

On the other hand, for $x > 1$,

$$\frac{\overline{H}(x)}{\overline{F}(x)} = \mu\frac{\overline{H}(x)}{\overline{\nu}(x)} \;\to\; \mu c \quad \text{as } x \to \infty,$$

by the assumption (iv). Hence, $c = 1$.

11. Branching processes. In this section we consider the limit behaviour of sub-critical, age-dependent branching processes for which the Malthusian parameter does not exist.

Let $h(z)$ be the particle production generating function of an age-dependent branching process with particle lifetime distribution F (see [3, Chapter IV], [16, Chapter VI] for background). We take the process to be sub-critical, i.e. $A \equiv h'(1) < 1$. Let $Z(t)$ denote the number of particles

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at time t. It is known (see, for example, [3, Chapter IV, Section 5] or [5]) that $\mathbf{E}Z(t)$ admits the representation

$$\mathbf{E}Z(t) \;=\; (1-A)\sum_{n=1}^{\infty} A^{n-1}\overline{F^{*n}}(t).$$

It was proved in [5] for sufficiently small values of A and then in [6, 7] for any $A < 1$ that $\mathbf{E}Z(t) \sim \overline{F}(t)/(1-A)$ as $t \to \infty$, provided F is subexponential. The local asymptotics were considered in [2].

Applying Theorem 3 with τ geometrically distributed and $\widehat{\gamma} = 0$, we deduce

Theorem 8. *Let F be heavy-tailed, and, for some $c > 0$, $\mathbf{E}Z(t) \sim c\overline{F}(t)$ as $t \to \infty$. Then $c = 1/(1-A)$ and F is subexponential.*

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