Convolutions of long-tailed and subexponential distributions

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Convolutions of long-tailed and subexponential distributions play a major role in the analysis of many stochastic systems. We study these convolutions, proving some important new results through a simple and coherent approach, and showing also that the standard properties of such convolutions follow as easy consequences.

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1 Introduction

Heavy-tailed distributions play a major role in the analysis of many stochastic systems. For example, they are frequently necessary to accurately model inputs to computer and communications networks, they are an essential component of the description of many risk processes, and they occur naturally in models of epidemiological spread.

Since the inputs to such systems are frequently cumulative in their effects, the analysis of the corresponding models typically features convolutions of such heavy-tailed distributions. The properties of such convolutions depend on their satisfying certain regularity conditions. From the point of view of applications practically all such distributions may be considered to be long-tailed, and indeed to possess the stronger property of subexponentiality (see below for definitions).

In this paper we study convolutions of long-tailed and subexponential distributions (probability measures), and (in passing) more general finite measures, on the real line. Our aim is to prove some important new results, and to do so through a simple, coherent and systematic approach. It turns out that all the standard properties of such convolutions are then obtained as easy consequences of these results. Thus we also hope to provide further insight into these properties, and to dispel some of the mystery which still seems to surround the phenomenon of subexponentiality in particular.

Our approach is based on a simple decomposition for such convolutions, and on the concept of "h-insensitivity" for a long-tailed distribution or measure with respect to some (slowly) increasing function h. This novel approach and the basic, and very simple, new results we require are given in Section 2. In Section 3 we study convolutions of long-tailed distributions. The key results here are Theorems 5 and 6 which give conditions under which a random shifting preserves tail equivalence; in the remainder of this section we show how other (mostly known) results follow quickly and easily from our approach, and provide some generalisations. In Section 4 we similarly study convolutions of subexponential distributions. The main results here—Theorems 15 and 17—are new, as is Corollary 18; again some classical results are immediate consequences. Finally, in Section 5 we consider closure properties for the class of subexponential distributions. Theorem 20 gives a new necessary and sufficient condition for the convolution of two subexponential distributions to be subexponential, together with a simple demonstration of the equivalence of some existing conditions.

Occasionally we take a few lines to reprove something from the literature. This enables us to give a self-contained treatment of our subject.

Good introductions to the current state of knowledge on long-tailed and subexponential distributions may be found in Asmussen [1, 2], Embrechts et al. [11], and Rolski et al. [15].

2 Basic results

We are concerned primarily with probability distributions. However, we find it convenient to work also with more general finite measures, allowing these to be added, convoluted, etc, as usual. As we shall discuss further, our later results for distributions translate easily into this more general setting, where they then provide additional insight.

Recall that, for any two measures F and G and for any two non-negative numbers p and q, their *mixture* $pF+qG$ is also a finite measure defined by $(pF+qG)(B) = pF(B)+qG(B)$, for any Borel set B.

Recall also that a (finite) measure F on $\mathbb{R}$ is *long-tailed* if and only if $\overline{F}(x) > 0$ for all x and, for all constants a,

$$\overline{F}(x+a) = \overline{F}(x) + o(\overline{F}(x)) \quad \text{as } x \to \infty, \tag{1}$$

where the *tail function* $\overline{F}$ of the measure F is given by $\overline{F}(x) = F(x, \infty)$. It is easy to see that it is sufficient that the relation (1) hold for some $a \neq 0$—an observation due to Landau [13].

We note also that the class of long-tailed measures has the following readily verified closure property (see Proposition 1.3.6(iii) of Bingham et al. [5]), which we henceforth use without comment: if measures $F_1, \ldots, F_n$ are long-tailed and if the measure F is such that $\overline{F}(x) \sim \sum_{k=1}^{n} c_k \overline{F_k}(x)$ as $x \to \infty$, for some $c_1, \ldots, c_k > 0$, then F is also long-tailed. (Here and throughout we use "$\sim$" to mean that the ratio of the quantities on either side of this symbol converges to one; thus, for example, the relation (1) may be written as $\overline{F}(x+a) \sim \overline{F}(x)$ as $x \to \infty$; we further frequently omit, especially in proofs, the qualifier "as $x \to \infty$", as all our limits will be of this form.)

We shall make frequent use of the following construction: given any long-tailed measure F, we may choose a positive function h on $\mathbb{R}^+ = [0, \infty)$ such that $h(x) \to \infty$ as $x \to \infty$ and additionally h is increasing sufficiently slowly that

$$\overline{F}(x \pm h(x)) \sim \overline{F}(x) \quad \text{as } x \to \infty. \tag{2}$$

For example, we may choose a sequence x_n increasing to infinity such that, for all n,

$$|\overline{F}(x \pm n) - \overline{F}(x)| \leq \overline{F}(x)/n \quad \text{for all } x > x_n,$$

and then set $h(x) = n$ for $x \in (x_n, x_{n+1}]$. (The introduction of such a function h will allow us to avoid the general messiness of repeatedly taking limits first as x tends to infinity and then as a further constant a—essentially that featuring in (1)—tends to infinity.) Given a long-tailed measure F and a function h satisfying the above conditions, we shall say that F is *h-insensitive*.

Note further that, given a finite collection of long-tailed measures $F_1, \ldots, F_n$, we may choose a function h on $\mathbb{R}^+$ such that each F_i is h-insensitive. For example, for each i we may choose h_i such that F_i is h_i-insensitive, and then define h by $h(x) = \min_i h_i(x)$.

The tail function of the convolution of any two measures F and G is given by

$$\overline{F * G}(x) = \int_{-\infty}^{\infty} \overline{F}(x-y) G(dy) = \int_{-\infty}^{\infty} \overline{G}(x-y) F(dy). \tag{3}$$

For any measure F and for any Borel set B we denote by F_B the measure given by the restriction of F to B, that is $F_B(A) = F(A \cap B)$ for all Borel sets A. We also use the shortenings $F_{\leq h} = F_{(-\infty, h]}$ and $F_{>h} = F_{(h,\infty)}$. Finally, we define $m_F = F(\mathbb{R})$ to be the total mass associated with the measure F.

Now let h be any positive function on $\mathbb{R}^+$. Then the tail function of the convolution of any two measures F and G possesses the following decomposition: for $x \geq 0$,

$$\overline{F * G}(x) = \overline{F_{\leq h} * G}(x) + \overline{F_{>h} * G}(x), \tag{4}$$

and upper estimate:

$$\overline{F * G}(x) \leq \overline{F_{\leq h} * G}(x) + \overline{F * G_{\leq h}}(x) + \overline{F_{>h} * G_{>h}}(x), \tag{5}$$

where in (4) and (5) above h stands for $h(x)$, so that, for example, $F_{\leq h}(B) = F_{\leq h(x)}(B) = F(B \cap (-\infty, h(x)])$, again for any Borel set B. If in addition $h(x) \leq x/2$, then

$$\overline{F * G}(x) = \overline{F_{\leq h} * G}(x) + \overline{F * G_{\leq h}}(x) + \overline{F_{>h} * G_{>h}}(x), \tag{6}$$

because in this case $\overline{F_{\leq h} * G}(x) = \overline{F_{\leq h} * G_{>h}}(x)$ and $\overline{F * G_{\leq h}}(x) = \overline{F_{>h} * G_{\leq h}}(x)$. Note that

$$\overline{F_{\leq h} * G}(x) = \int_{-\infty}^{h(x)} \overline{G}(x-y) F(dy), \tag{7}$$

$$\overline{F_{>h} * G}(x) = \int_{-\infty}^{\infty} \overline{F}(\max(h(x), x-y)) G(dy), \tag{8}$$

while $\overline{F_{>h} * G_{>h}}$ is symmetric in F and G and

$$\overline{F_{>h} * G_{>h}}(x) = \int_{h(x)}^{\infty} \overline{F}(\max(h(x), x-y)) G(dy) = \int_{h(x)}^{\infty} \overline{G}(\max(h(x), x-y)) F(dy). \tag{9}$$

Note also that if, on some probability space with probability measure $\mathbf{P}$, ξ and η are independent random variables with respective distributions F and G (with $m_F = m_G = 1$), then for $x \geq 0$,

$$\overline{F_{\leq h} * G}(x) = \mathbf{P}(\xi + \eta > x,\, \xi \leq h(x)),$$
$$\overline{F_{>h} * G_{>h}}(x) = \mathbf{P}(\xi + \eta > x,\, \xi > h(x),\, \eta > h(x)).$$

The following four lemmas are the keys to everything that follows.

Lemma 1. *Suppose that the measure G is long-tailed and that h is such that G is h-insensitive. Then, for any measure F,*

$$\overline{F_{\le h} * G}(x) \sim m_F \overline{G}(x) \qquad \text{as } x \to \infty.$$

Proof. It follows from (7) that $\overline{F_{\le h} * G}(x) \le m_F \overline{G}(x - h(x))$. On the other hand,

$$\begin{aligned} \overline{F_{\le h} * G}(x) &\ge \overline{F_{[-h,h]} * G}(x) \\ &\ge F[-h(x), h(x)]\overline{G}(x + h(x)) \\ &\sim m_F \overline{G}(x + h(x)) \quad \text{ as } x \to \infty, \end{aligned}$$

where the last equivalence follows since $h(x) \to \infty$ as $x \to \infty$. The required result now follows from the h-insensitivity of G. □

We now prove a version of Lemma 1 which is symmetric in F and G and will allow us to get many important results for convolutions—see the further discussion below.

Lemma 2. *Suppose that measures F and G are such that $m_F G + m_G F$ is long-tailed and that h is such that $m_F G + m_G F$ is h-insensitive. Then, as $x \to \infty$,*

$$\overline{F_{\le h} * G}(x) + \overline{F * G_{\le h}}(x) \sim m_F \overline{G}(x) + m_G \overline{F}(x).$$

Proof. It follows from (7) that

$$\overline{F_{\le h} * G}(x) + \overline{F * G_{\le h}}(x) \le m_F \overline{G}(x - h(x)) + m_G \overline{F}(x - h(x)).$$

On the other hand,

$$\begin{aligned} \overline{F_{\le h} * G}(x) + \overline{F * G_{\le h}}(x) &\ge \overline{F_{[-h,h]} * G}(x) + \overline{F * G_{[-h,h]}}(x) \\ &\ge F[-h(x), h(x)]\overline{G}(x + h(x)) + G[-h(x), h(x)]\overline{F}(x + h(x)) \\ &\sim m_F \overline{G}(x + h(x)) + m_G \overline{F}(x + h(x)) \quad \text{ as } x \to \infty, \end{aligned}$$

where the last equivalence follows since $h(x) \to \infty$ as $x \to \infty$. The required result now follows from the h-insensitivity of $m_F G + m_G F$. □

Note that special cases under which $m_F G + m_G F$ is long-tailed are (a) F and G are both long-tailed, and (b) F is long-tailed and $\overline{G}(x) = o(\overline{F}(x))$ as $x \to \infty$.[1] The importance of this lemma arises because in many applications, while we may naturally be concerned primarily with long-tailed measures (e.g. service times in queueing models), we nevertheless require to make small corrections arising from other input measures whose tails are relatively lighter (e.g. those of interarrival times); Lemma 2 is then necessary in

[1]There are also other possibilities. For instance, here is an example where F and G are probability distributions such that F is long-tailed and $F + G$ is long-tailed while G does not satisfy (a) or (b). Take $\alpha > 0$ and let $\overline{F}(x) = x^{-\alpha}$ for $x \ge 1$ and $\overline{F}(x) = 1$ for $x \le 1$. Let $\overline{G}(x) = 1$ for $x \le 1$ and construct $\overline{G}$ on $(1, \infty)$ inductively on the intervals $[x_n, x_{n+1}]$, with $x_1 = 1$. For $n = 1, 2, \ldots$, let $y_n = \min\{x > x_n : 1 + \ln(x/x_n) = 2^n\}$ and then, for $x \in [x_n, y_n)$, let $\overline{G}(x) = \overline{F}(x)/(1 + \ln(x/x_n))$. Further, let $\overline{G}(y_n) = \overline{F}(y_n) \cdot 2^{-n-1}$ (this means that $\overline{G}(y_n) = \overline{G}(y_n-)/2$). Finally, let $x_{n+1} = \min\{x > y_n : \overline{F}(x) = \overline{G}(y_n)\}$ and then let $\overline{G}(x) = \overline{G}(y_n)$ for $x \in [y_n, x_{n+1}]$.
As n increases the jumps of $\overline{G}$ at the points y_n become negligible with respect to $\overline{F}$, so $F+G$ is long-tailed. On the other hand, these jumps are not negligible with respect to $\overline{G}$ itself, and G cannot be long-tailed. Also, $\overline{G}(x_n) = \overline{F}(x_n)$, and so condition (a) is violated.

order to obtain asymptotic results such as Theorem 15 below, ensuring that such lighter tails indeed make a negligible contribution. In the case where we are solely concerned with long-tailed measures, there are obvious simplifications to the proofs of our main results.

Finally, the following two simple lemmas will be useful when we come to consider subexponential measures.

Lemma 3. *Let h be any positive function on $\mathbb{R}^+$ such that $h(x) \to \infty$. Then, for any measures F_1, F_2 and G on $\mathbb{R}$,*

$$\limsup_{x\to\infty} \frac{\overline{(F_1)_{>h} * G}(x)}{\overline{(F_2)_{>h} * G}(x)} \le \limsup_{x\to\infty} \frac{\overline{F_1}(x)}{\overline{F_2}(x)}.$$

In particular, in the case where the limit of the ratio $\overline{F_1}(x)/\overline{F_2}(x)$ exists, we have

$$\lim_{x\to\infty} \frac{\overline{(F_1)_{>h} * G}(x)}{\overline{(F_2)_{>h} * G}(x)} = \lim_{x\to\infty} \frac{\overline{F_1}(x)}{\overline{F_2}(x)}.$$

Proof. The results are immediate from (8) and from the first of the integral representations in (9) on noting that $\max(h(x), x-y) \to \infty$ as $x \to \infty$ uniformly in all $y \in \mathbb{R}$. □

Taking into account the symmetry of $\overline{F_{>h} * G_{>h}}$ in F and G (see (9)), we obtain also the following result.

Lemma 4. *Let h be any positive function on $\mathbb{R}^+$ such that $h(x) \to \infty$. Then, for any measures F_1, F_2, G_1 and G_2 on $\mathbb{R}$,*

$$\limsup_{x\to\infty} \frac{\overline{(F_1)_{>h} * (G_1)_{>h}}(x)}{\overline{(F_2)_{>h} * (G_2)_{>h}}(x)} \le \limsup_{x\to\infty} \frac{\overline{F_1}(x)}{\overline{F_2}(x)} \cdot \limsup_{x\to\infty} \frac{\overline{G_1}(x)}{\overline{G_2}(x)}.$$

In particular, in the case where the limits of the ratios $\overline{F_1}(x)/\overline{F_2}(x)$ and $\overline{G_1}(x)/\overline{G_2}(x)$ exist, we have

$$\lim_{x\to\infty} \frac{\overline{(F_1)_{>h} * (G_1)_{>h}}(x)}{\overline{(F_2)_{>h} * (G_2)_{>h}}(x)} = \lim_{x\to\infty} \frac{\overline{F_1}(x)}{\overline{F_2}(x)} \cdot \lim_{x\to\infty} \frac{\overline{G_1}(x)}{\overline{G_2}(x)}.$$

3 Convolutions of long-tailed distributions

For the remainder of this paper we specialise to *distributions*, i.e. to probability measures on $\mathbb{R}$ each of total mass one. This simplifies the statements and proofs of our results, enabling us to dispense with the constants m_F, etc, and is in line with most applications. Of course our results may nevertheless be translated to the case of more general finite measures by renormalising the latter, applying the results, and re-expressing the conclusions in terms of the original measures.

We denote by $\mathcal{L}$ the class of long-tailed distributions on $\mathbb{R}$. We shall say that two distributions F_1, F_2 on $\mathbb{R}$ are *tail equivalent* if and only if $\overline{F_1}(x) \sim \overline{F_2}(x)$ as $x \to \infty$. The following two theorems, which provide conditions under which a random shifting preserves tail equivalence, turns out to be of key importance in studying convolutions where at least one of the distributions involved belongs to the class $\mathcal{L}$.

Theorem 5. *Suppose that F_1, F_2, and G are distributions on $\mathbb{R}$ such that F_1 and F_2 are tail equivalent. If $G \in \mathcal{L}$ then the distributions $F_1 * G$ and $F_2 * G$ are tail equivalent.*

Proof. Let the function h on $\mathbb{R}^+$ be such that G is h-insensitive. We use the decomposition (4). It follows from Lemma 3 that

$$\overline{(F_2)_{>h} * G}(x) \sim \overline{(F_1)_{>h} * G}(x).$$

Further, by Lemma 1,

$$\overline{(F_2)_{\le h} * G}(x) \sim \overline{G}(x) \sim \overline{(F_1)_{\le h} * G}(x),$$

so that the conclusion of the theorem follows now from (4). □

The next theorem generalises Lemma 2.4(ii) of Cline [7] where the case F_1, F_2, G_1, $G_2 \in \mathcal{L}$ was considered.

Theorem 6. *Suppose that F_1, F_2, G_1 and G_2 are distributions on $\mathbb{R}$ such that $\overline{F}_1(x) \sim \overline{F}_2(x)$ and $\overline{G}_1(x) \sim \overline{G}_2(x)$ as $x \to \infty$. If the measure $F_1 + G_1$ is long-tailed, then the distributions $F_1 * G_1$ and $F_2 * G_2$ are tail equivalent.*

Proof. The conditions imply that $F_2 + G_2$ is long-tailed. Let the function h on $\mathbb{R}^+$ be such that $h(x) \le x/2$ and both $F_1 + G_1$ and $F_2 + G_2$ are h-insensitive. We use the decomposition (6). By Lemma 4,

$$\begin{aligned}\overline{(F_2)_{>h} * (G_2)_{>h}}(x) &= \overline{(F_1)_{>h} * (G_1)_{>h}}(x) + o(\overline{(F_1)_{>h} * (G_1)_{>h}}(x)) \\ &= \overline{(F_1)_{>h} * (G_1)_{>h}}(x) + o(\overline{F_1 * G_1}(x)). \end{aligned} \tag{10}$$

Further, by Lemma 2,

$$\begin{aligned}\overline{(F_1)_{\le h} * G_1}(x) + \overline{F_1 * (G_1)_{\le h}}(x) &= \overline{G}_1(x) + \overline{F_1}(x) + o(\overline{G}_1(x) + \overline{F_1}(x)) \\ &= \overline{G}_2(x) + \overline{F_2}(x) + o(\overline{G}_1(x) + \overline{F_1}(x)) \\ &= \overline{(F_2)_{\le h} * G_2}(x) + \overline{F_2 * (G_2)_{\le h}}(x) + o(\overline{G}_1(x) + \overline{F_1}(x)).\end{aligned}$$

It further follows from Lemma 2 and from (6) that $\overline{F_1}(x) = O(\overline{F_1 * G_1}(x))$ and $\overline{G}_1(x) = O(\overline{F_1 * G_1}(x))$. Thus we obtain

$$\overline{(F_1)_{\le h} * G_1}(x) + \overline{F_1 * (G_1)_{\le h}}(x) = \overline{(F_2)_{\le h} * G_2}(x) + \overline{F_2 * (G_2)_{\le h}}(x) + o(\overline{F_1 * G_1}(x)). \tag{11}$$

Hence, again from the decomposition (6), together with (10) and (11),

$$\overline{F_2 * G_2}(x) = \overline{F_1 * G_1}(x) + o(\overline{F_1 * G_1}(x)),$$

as $x \to \infty$, implying that $F_1 * G_1$ and $F_2 * G_2$ are tail equivalent as required. □

Theorems 5 and 6 have a number of important corollaries, the first of which is well-known from Embrechts and Goldie [8], but of which we may now give a very simple proof.

Corollary 7. *The class $\mathcal{L}$ of long-tailed distributions is closed under convolutions.*

Proof. Suppose that F, $G \in \mathcal{L}$. Fix $y > 0$. Define the distribution F_y to be equal to F shifted by $-y$, that is, $\overline{F_y}(x) = \overline{F}(x + y)$. Then $F_y * G$ is equal to $F * G$ shifted by $-y$. Since $F \in \mathcal{L}$ it follows that F and F_y are tail-equivalent, and since also $G \in \mathcal{L}$ it follows from Theorem 5 that $F * G$ and $F_y * G$ are tail-equivalent. Hence $\overline{F * G}(x) \sim \overline{F * G}(x+y)$, implying that $F * G \in \mathcal{L}$. □

We also have the following, and new, generalisation of Corollary 7, the proof of which is identical, except that we must appeal to the (slightly less straightforward) Theorem 6 in place of Theorem 5.

Corollary 8. *Suppose that the distributions F and G are such that $F \in \mathcal{L}$ and $F + G$ is long-tailed. Then the distribution $F * G \in \mathcal{L}$.*

We remark also that a further special case of Corollary 8 arises when we have $F \in \mathcal{L}$ and G is such that $\overline{G}(x) = o(\overline{F}(x))$ as $x \to \infty$, so that here again $F * G \in \mathcal{L}$.

We give two further general, and known, theorems for convolutions of long-tailed distributions.

Theorem 9. *Suppose that the distributions $F_1, \ldots, F_n \in \mathcal{L}$. Then*

$$\overline{F_1 * \ldots * F_n}(x) \quad \geq \quad (1 + o(1)) \sum_{k=1}^{n} \overline{F_k}(x) \qquad \text{as } x \to \infty.$$

Proof. It is sufficient to prove the result for the case $n = 2$, the general result following by induction (since, from Corollary 8 above, the class $\mathcal{L}$ is closed under convolutions). Let the function h be such that $h(x) \leq x/2$ and both F_1 and F_2 are h-insensitive. The required result is now immediate from the inequality

$$\overline{F_1 * F_2}(x) \geq \overline{(F_1)_{\leq h} * F_2}(x) + \overline{F_1 * (F_2)_{\leq h}}(x)$$

and Lemma 1 above. □

In particular we have the following corollary (where, as usual, F^{*n} denotes the n-fold convolution of F with itself).

Corollary 10. *Suppose that $F \in \mathcal{L}$. Then, for any $n \geq 2$,*

$$\liminf_{x \to \infty} \frac{\overline{F^{*n}}(x)}{\overline{F}(x)} \quad \geq \quad n.$$

Theorem 11. *Suppose that $F \in \mathcal{L}$. Then, for any distribution G on $\mathbb{R}$,*

$$\liminf_{x \to \infty} \frac{\overline{F * G}(x)}{\overline{F}(x)} \quad \geq \quad 1.$$

If, furthermore, the function h on $\mathbb{R}^+$ is such that $F \in \mathcal{L}$ is h-insensitive and $\overline{G}(h(x)) = o(\overline{F}(x))$ as $x \to \infty$, then $\overline{F * G}(x) \sim \overline{F}(x)$ as $x \to \infty$. In particular this conclusion holds for $F \in \mathcal{L}$ and any distribution G such that $\overline{G}(a) = 0$ for some a.

Proof. Let the function h be such that F is h-insensitive. We use the decomposition (4) with F and G interchanged. From Lemma 1,

$$\overline{F * G_{\leq h}}(x) \sim \overline{F}(x), \tag{12}$$

and so the first result is immediate. For the second result note that, under the given additional condition and for h as above,

$$\overline{F * G_{>h}}(x) \leq \overline{G}(h(x)) = o(\overline{F}(x)),$$

so that the required result again follows on using (4) and (12). □

4 Convolutions of subexponential distributions on $\mathbb{R}$

We recall that a distribution F on the positive real line $\mathbb{R}^+$ is *subexponential* if and only if $\overline{F}(x) > 0$ for all x and

$$\overline{F * F}(x) = 2\overline{F}(x) + o(\overline{F}(x)) \quad \text{as } x \to \infty; \tag{13}$$

this notion goes back to Chistyakov [6]. By way of interpretation, suppose that, on some probability space with probability measure $\mathbf{P}$, ξ_1 and ξ_2 are independent random variables with common distribution F. Then, since $\mathbf{P}(\max(\xi_1, \xi_2) > x) = 2\overline{F}(x) + o(\overline{F}(x))$, it follows that the subexponentiality of F is equivalent to the condition that $\mathbf{P}(\xi_1 + \xi_2 > x) \sim \mathbf{P}(\max(\xi_1, \xi_2) > x)$ as $x \to \infty$, i.e. that, for large x, the only significant way in which $\xi_1 + \xi_2$ can exceed x is that either ξ_1 or ξ_2 should itself exceed x. This is the well-known "principle of a single big jump" for sums of subexponentially distributed random variables.

It is also well-known [6] (see also [4]) that if F on $\mathbb{R}^+$ is subexponential then $F \in \mathcal{L}$. (To see this, note that, for any distribution F on $\mathbb{R}^+$ such that $\overline{F}(x) > 0$ for all x, for any $a > 0$ and for all $x \ge a$,

$$\begin{aligned}\overline{F * F}(x) &= \overline{F * F_{[0,x-a]}}(x) + \overline{F * F_{(x-a,x]}}(x) + \overline{F * F_{(x,\infty)}}(x) \\ &\ge \overline{F}(x)F(x-a) + \overline{F}(a)F(x-a,x] + \overline{F}(x);\end{aligned}$$

since $F(x-a) \to 1$ and $\overline{F}(a) > 0$, it follows that under the condition (13) we have $F(x-a,x] = o(\overline{F}(x))$ as $x \to \infty$, and so $F \in \mathcal{L}$ as required.)

Following recent practice, we extend the definition of subexponentiality to distributions on the entire real line by saying that a distribution F on $\mathbb{R}$ is subexponential if and only if $F \in \mathcal{L}$ and (13) holds (the latter condition no longer being sufficient to ensure $F \in \mathcal{L}$). We write $\mathcal{S}$ for the class of subexponential distributions on $\mathbb{R}$.

We shall make repeated use of the following observation which follows from the upper estimate (5), Lemma 1 (or Lemma 2) and (13):

Lemma 12 (Asmussen et al. [3])**.** *If $F \in \mathcal{L}$ then the following are equivalent:*

(i) $F \in \mathcal{S}$*;*

(ii) *for every function h such that $h(x) \to \infty$, $\overline{F_{>h} * F_{>h}}(x) = o(\overline{F}(x))$ as $x \to \infty$;*

(iii) *for some function h such that $h(x) \to \infty$ and F is h-insensitive, $\overline{F_{>h} * F_{>h}}(x) = o(\overline{F}(x))$ as $x \to \infty$.*

The following statement provides the foundation for our results on convolutions of subexponential distributions.

Lemma 13. *Suppose that $F \in \mathcal{S}$ and that the function h is such that $h(x) \to \infty$ as $x \to \infty$. Let distributions G_1, G_2 be such that, for $i = 1, 2$, we have $\overline{G_i}(x) = O(\overline{F}(x))$ as $x \to \infty$. Then $\overline{(G_1)_{>h} * (G_2)_{>h}}(x) = o(\overline{F}(x))$ as $x \to \infty$.*

Proof. Since $\overline{G}_i(x) = O(\overline{F}(x))$, it follows from Lemma 4 that

$$\limsup_{x\to\infty} \frac{\overline{(G_1)_{>h} * (G_2)_{>h}}(x)}{\overline{F_{>h} * F_{>h}}(x)} \le \limsup_{x\to\infty} \frac{\overline{G_1}(x)}{\overline{F}(x)} \cdot \limsup_{x\to\infty} \frac{\overline{G_2}(x)}{\overline{F}(x)} < \infty.$$

Hence, it follows from Lemma 12 since $F \in \mathcal{S}$ that

$$\overline{(G_1)_{>h} * (G_2)_{>h}}(x) = O(\overline{F_{>h} * F_{>h}}(x)) = o(\overline{F}(x)).$$

Hence we have the required result. □

We shall say that distributions F and G on $\mathbb{R}$ are *weakly tail equivalent* if both $\overline{F}(x) = O(\overline{G}(x))$ and $\overline{G}(x) = O(\overline{F}(x))$ as $x \to \infty$. We now have the following corollary to Lemma 13.

Corollary 14 (Klüppelberg [12])**.** *Suppose that $F \in \mathcal{S}$, that $G \in \mathcal{L}$, and that F and G are weakly tail-equivalent. Then $G \in \mathcal{S}$.*

Proof. Choose the function h on $\mathbb{R}^+$ so that both F and G are h-insensitive. Then, from Lemma 13 and the given weak tail-equivalence, $\overline{G_{>h} * G_{>h}}(x) = o(\overline{G}(x))$, and so it follows from Lemma 12 that $G \in \mathcal{S}$. □

Remark 1. It follows in particular, as is well known, that subexponentiality is a *tail property* of a distribution, i.e. for any given x_0 it depends only on the restriction of the distribution to the right of x_0. (Indeed this result also follows from Theorem 6: suppose that distributions F_1, F_2 on $\mathbb{R}$ are tail equivalent and that $F_1 \in \mathcal{S}$; since $F_1 \in \mathcal{L}$ we have also $F_2 \in \mathcal{L}$; hence, from Theorem 6, on identifying F_i with G_i for $i = 1, 2$ and on using (13), we have $F_2 \in \mathcal{S}$.) Thus also a distribution F on $\mathbb{R}$ is subexponential if and only if the distribution F^+ on $\mathbb{R}^+$, given by $\overline{F^+}(x) = \overline{F}(x)$ for $x \geq 0$ and $F^+(x) = 0$ for $x < 0$, is subexponential; this provides an alternative definition of subexponentiality on $\mathbb{R}$.

We now have the following theorem.

Theorem 15. *Let (a reference distribution) $F \in \mathcal{S}$. Suppose that distributions G_1, ..., G_n are such that, for each k, the measure $F + G_k$ is long-tailed and $\overline{G_k}(x) = O(\overline{F}(x))$ as $x \to \infty$. Then*

$$\overline{G_1 * \cdots * G_n}(x) = \sum_{i=1}^{n} \overline{G_i}(x) + o(\overline{F}(x)) \qquad \text{as } x \to \infty. \tag{14}$$

Corollary 16. *Let $F \in \mathcal{S}$ and so $F \in \mathcal{L}$. Suppose that distributions G_1, ..., G_n are such that, individually for each k, either* (i) *$G_k \in \mathcal{L}$ and $\overline{G_k}(x) = O(\overline{F}(x))$ as $x \to \infty$ or* (ii) *$\overline{G}_k(x) = o(\overline{F}(x))$ as $x \to \infty$. Then* (14) *holds.*

The latter corollary was proved in [10] for the case $n = 2$, $G_1 = F$, $\overline{G}_2(x) = o(\overline{F}(x))$.

Proof of Theorem 15. Note first that it follows from the conditions of the theorem that, for each i and for any constant a,

$$\begin{aligned}\overline{F}(x+a) + \overline{G_i}(x+a) &= \overline{F}(x) + \overline{G_i}(x) + o(\overline{F}(x) + \overline{G_i}(x)) \\ &= \overline{F}(x) + \overline{G_i}(x) + o(\overline{F}(x)).\end{aligned}$$

Hence from the representation $F + \sum_{i=1}^{k} G_i = \sum_{i=1}^{k}(F + G_i) - (k-1)F$ and since F is also long-tailed, for each k and for any constant a,

$$\overline{F}(x+a) + \sum_{i=1}^{k} \overline{G_i}(x+a) = \overline{F}(x) + \sum_{i=1}^{k} \overline{G_i}(x) + o(\overline{F}(x)),$$

and so the measure $F + \sum_{i=1}^{k} G_i$ is also long-tailed. Note also that for each k we have $\sum_{i=1}^{k} \overline{G_i}(x) = O(\overline{F}(x))$. It now follows that it is sufficient to prove the theorem for case $n = 2$, the general result then following by induction.

Let the function h on $\mathbb{R}^+$ be such that $h(x) \le x/2$ and all F, $F+G_1$ and $F+G_2$ are h-insensitive. It then follows from Lemma 1 that, as $x \to \infty$,

$$\begin{aligned}\overline{G_1 * (G_2)_{\le h}}(x) &= \overline{(G_1+F) * (G_2)_{\le h}}(x) - \overline{F * (G_2)_{\le h}}(x) \\ &= \overline{G_1+F}(x) - \overline{F}(x) + o(\overline{G_1}(x) + \overline{F}(x)) \\ &= \overline{G_1}(x) + o(\overline{F}(x)), \end{aligned} \tag{15}$$

and similarly

$$\overline{(G_1)_{\le h} * G_2}(x) = \overline{G_2}(x) + o(\overline{F}(x)). \tag{16}$$

Further, from Lemma 13,

$$\overline{(G_1)_{>h} * (G_2)_{>h}}(x) = o(\overline{F}(x)). \tag{17}$$

The required result (14) now follows from the decomposition (6) and from (15)–(17). ☐

The following result strengthens the conditions of Theorem 15 to provide a sufficient condition for the convolution obtained there to be subexponential.

Theorem 17. *Suppose again that the conditions of Theorem* 15 *hold, and that additionally* G_1 *satisfies the stronger condition that* $G_1 \in \mathcal{L}$ *and that* G_1 *is weakly tail equivalent to* F*. Then* $G_1 * \cdots * G_n \in \mathcal{S}$*, and additionally* $G_1 * \cdots * G_n$ *is weakly tail equivalent to* F*.*

Proof. It follows from Corollary 14 that $G_1 \in \mathcal{S}$. Further the weak tail equivalence of F and G_1 implies that, for each k, $\overline{G_k}(x) = O(\overline{G_1}(x))$. Hence by Theorem 15 with $F = G_1$, the distribution $G_1 * G_2 * \cdots * G_n$ is long-tailed and weakly tail equivalent to G_1 and so also to F. In particular, again by Corollary 14, $G_1 * \cdots * G_n \in \mathcal{S}$. ☐

We have the following two corollaries of Theorems 15 and 17. The first is new (the version with $G \in \mathcal{L}$ was proved in Embrechts and Goldie [8]), while the other is well-known (and goes back to [9] where the case $n = 2$, $G_1 = G_2$ was considered; some particular results may be found in Teugels [16] and Pakes [14], see also [3]).

Corollary 18. *Suppose that distributions* F *and* G *are such that* $F \in \mathcal{S}$*, that* $F + G$ *is long-tailed and that* $\overline{G}(x) = O(\overline{F}(x))$ *as* $x \to \infty$*. Then* $F * G \in \mathcal{S}$ *and*

$$\overline{F * G}(x) = \overline{F}(x) + \overline{G}(x) + o(\overline{F}(x)) \qquad \text{as } x \to \infty.$$

Proof. The result follows from Theorems 15 and 17 in the case $n = 2$ with G_1 replaced by F and G_2 by G. ☐

Corollary 19. *Suppose that* $F \in \mathcal{S}$*. Let* $G_1, \dots, G_n$ *be distributions such that* $\overline{G_i}(x)/\overline{F}(x) \to c_i$ *as* $x \to \infty$*, for some constants* $c_i \ge 0$*,* $i = 1, \dots, n$*. Then*

$$\frac{\overline{G_1 * \ldots * G_n}(x)}{\overline{F}(x)} \to \sum_{i=1}^{n} c_i \qquad \text{as } x \to \infty.$$

If $c_1 + \ldots + c_n > 0$*, then* $G_1 * \ldots * G_n \in \mathcal{S}$*.*

Proof. The first statement of the corollary is immediate from Theorem 15. If $c_1 + \ldots + c_n > 0$, we may assume without loss of generality that $c_1 > 0$, so that the second statement follows from Theorem 17. ☐

5 Closure properties of subexponential distributions

It is well-known that the class of *regularly varying* distributions, which is a subclass of the class $\mathcal{S}$ of subexponential distributions, is closed under convolution.[2] Indeed if F and G are regularly varying, the result that $F * G$ is also regularly varying is straightforwardly obtained from Theorem 15 by taking the "reference" distribution of that theorem to be $(F + G)/2$. It is also known that the class $\mathcal{S}$ does not possess this closure property. However, if distributions F, $G \in \mathcal{S}$, then it follows from Corollary 18 that a sufficient condition for $F * G \in \mathcal{S}$ is given by $\overline{G}(x) = O(\overline{F}(x))$ as $x \to \infty$. (Indeed, as the corollary shows, G may satisfy weaker conditions than that of being subexponential.) Further it follows that under this condition we have that, for any function h such that both F and G are h-insensitive,

$$\overline{F_{>h} * G_{>h}}(x) = o(\overline{F}(x) + \overline{G}(x)) \quad \text{as } x \to \infty. \tag{18}$$

(See, for example, the proof of Theorem 15 above.) The following result is therefore not surprising: if F, $G \in \mathcal{S}$, the condition (18) is *necessary and sufficient* for $F * G \in \mathcal{S}$. This is one of the results given by Theorem 20 below. The first three equivalences given by the theorem are known from Embrechts and Goldie [8]; the novelty of the result lies in the fact that each of them is equivalent to the condition (iv) (condition (18) above) and in that we give new and very short proofs of these equivalences.

Theorem 20. *Suppose that the distributions F and G on $\mathbb{R}$ are subexponential. Suppose also that p is any constant such that $0 < p < 1$, and that the function h on $\mathbb{R}^+$ is such that $h(x) \le x/2$ and both F and G are h-insensitive. Then the following conditions are equivalent:*

(i) $\overline{F * G}(x) \sim \overline{F}(x) + \overline{G}(x)$ *as* $x \to \infty$*;*

(ii) $F * G \in \mathcal{S}$*;*

(iii) *the mixture* $pF + (1-p)G \in \mathcal{S}$*;*

(iv) $\overline{F_{>h} * G_{>h}}(x) = o(\overline{F}(x) + \overline{G}(x))$ *as* $x \to \infty$*.*

Proof. We show that each of the conditions (i)–(iii) is equivalent to the condition (iv). First, since F and G are subexponential, and hence long-tailed, it follows from the decomposition (6) and Lemma 1 that

$$\begin{aligned}\overline{F * G}(x) &= \overline{F * G_{\le h}}(x) + \overline{F_{\le h} * G}(x) + \overline{F_{>h} * G_{>h}}(x) \\ &= \overline{F}(x) + o(\overline{F}(x)) + \overline{G}(x) + o(\overline{G}(x)) + \overline{F_{>h} * G_{>h}}(x).\end{aligned} \tag{19}$$

Hence the conditions (i) and (iv) are equivalent.

To show the equivalence of (ii) and (iv) observe first that the subexponentiality of F and G implies that

$$\overline{F^{*2}}(x) \sim 2\overline{F}(x), \qquad \overline{G^{*2}}(x) \sim 2\overline{G}(x), \tag{20}$$

and thus in particular, from Lemma 4, that

$$\overline{(F^{*2})_{>h} * (G^{*2})_{>h}}(x) \sim 4\overline{F_{>h} * G_{>h}}(x). \tag{21}$$

[2]A distribution F is regularly varying if its right tail $\overline{F}$ is a regularly varying function. This means that the latter can be represented as $\overline{F}(x) = x^{-\alpha} l(x)$ for all x positive where $l(x)$ is a *slowly varying* function, i.e. l is strictly positive and $l(cx) \sim l(x)$ as $x \to \infty$, for any positive constant c.

Further, since $(F*G)^{*2} = F^{*2} * G^{*2}$ and since both F^{*2} and G^{*2} are h-insensitive, $\overline{(F*G)^{*2}}(x)$ may be estimated as in (19) with F^{*2} and G^{*2} replacing F and G. Hence, using also (20) and (21),

$$\overline{(F*G)^{*2}}(x) = (2+o(1))(\overline{F}(x)+\overline{G}(x)) + (4+o(1))\overline{F_{>h} * G_{>h}}(x). \tag{22}$$

Now since the subexponentiality of F and G also implies, by Corollary 7, that $F*G \in \mathcal{L}$, the condition (ii) is equivalent to the requirement that

$$\begin{aligned}\overline{(F*G)^{*2}}(x) &= (2+o(1))\overline{F*G}(x) \\ &= (2+o(1))(\overline{F}(x)+\overline{G}(x)) + (2+o(1))\overline{F_{>h} * G_{>h}}(x),\end{aligned} \tag{23}$$

where (23) follows from (19). However, the equalities (22) and (23) hold simultaneously if and only if $\overline{F_{>h} * G_{>h}}(x) = o(\overline{F}(x)+\overline{G}(x))$, i.e. if and only if the condition (iv) holds.

Finally, to show the equivalence of (iii) and (iv), note first that $pF+(1-p)G$ is h-insensitive. Hence, by Lemma 12, the subexponentiality of $pF+(1-p)G$ is equivalent to

$$\overline{(pF+(1-p)G)_{>h} * (pF+(1-p)G)_{>h}}(x) = o(\overline{F}(x)+\overline{G}(x)).$$

The left side is equal to

$$p^2\overline{F_{>h} * F_{>h}}(x) + (1-p)^2\overline{G_{>h} * G_{>h}}(x) + 2p(1-p)\overline{F_{>h} * G_{>h}}(x).$$

By the subexponentiality of F and G and again by Lemma 12, $\overline{F_{>h} * F_{>h}} = o(\overline{F}(x))$ and $\overline{G_{>h} * G_{>h}} = o(\overline{G}(x))$. The equivalence of (iii) and (iv) now follows. □

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