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LARGE-DEVIATION PROBABILITIES FOR MAXIMA OF SUMS OF INDEPENDENT RANDOM VARIABLES WITH NEGATIVE MEAN AND SUBEXPONENTIAL DISTRIBUTION*

D. A. KORSHUNOV†

(*Translated by the author*)

Abstract. We consider the sums $S_n = \xi_1 + \cdots + \xi_n$ of independent identically distributed random variables with negative mean value. In the case of subexponential distribution of the summands, the asymptotic behavior is found for the probability of the event that the maximum of sums $\max(S_1, \ldots, S_n)$ exceeds high level x. The asymptotics obtained describe this tail probability uniformly with respect to all values of n.

Key words. maxima of sums of random variables, homogeneous Markov chain, large deviation probabilities, subexponential distribution, integrated tail distribution

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1. Introduction. Let $\xi, \xi_1, \xi_2, \ldots$ be independent random variables with a common distribution F on the real line $\mathbf{R}$; $F((-\infty, 0]) < 1$. We denote $F(x) = F((-\infty, x))$ and $\overline{F}(x) = 1 - F(x)$. In general, for any measure G we denote by $\overline{G}(x) = G([x, \infty))$ the tail of this measure. Put $S_0 = 0$, $S_n = \xi_1 + \cdots + \xi_n$, and $M_n = \max\{S_k,\ 0 \leqq k \leqq n\}$. We assume that $\mathbf{E}\xi$ exists and $\mathbf{E}\xi < 0$. Put $a = |\mathbf{E}\xi|$. In view of the strong law of large numbers, $S_n \to -\infty$ almost surely as $n \to \infty$. Therefore, the family of distributions of maxima M_n, $n \geqq 1$, is tight.

The main goal of the present paper is to investigate the asymptotic behavior of the probability $\mathbf{P}\{M_n \geqq x\}$ as $x \to \infty$ in the case when the summands have distribution of subexponential type. We are interested in fixed values of the time parameter n and unboundedly growing n as well. More precisely, we present results on the asymptotic behavior of the probability $\mathbf{P}\{M_n \geqq x\}$ which are uniform in n.

Let us recall the definitions of some classes of functions and distributions which will be used in what follows.

DEFINITION 1. *The function f is called* long-tailed *if, for any fixed t, the limit of the ratio $f(x+t)/f(x)$ is equal to* 1 *as $x \to \infty$. We say that the distribution G is long-tailed if the function $\overline{G}(x)$ is long-tailed.*

DEFINITION 2. *The distribution G on $\mathbf{R}^+$ with unbounded support belongs to the class $\mathcal{S}$* (*and is called a* subexponential *distribution*) *if the convolution tail $\overline{G{*}G}(x)$ is equivalent to $2\overline{G}(x)$ as $x \to \infty$.*

It is shown in [2] that any subexponential distribution G is long-tailed with necessity. Sufficient conditions for some distribution to belong to the class $\mathcal{S}$ may be found, for example, in [2], [10]. The class $\mathcal{S}$ includes, in particular, the following distributions:

(i) Pareto distribution with the tail $\overline{G}(x) = (\varkappa/x)^\alpha$, $x \geqq \varkappa$, where $\varkappa > 0$, $\alpha > 0$;

(ii) lognormal distribution with the density $e^{-(\log x - \log \alpha)^2/2\sigma^2}/x\sigma\sqrt{2\pi}$, $x > 0$, where $\sigma > 0$, $\alpha > 0$;

(iii) Weibull distribution with the tail $\overline{G}(x) = e^{-x^\alpha}$, $x \geqq 0$, where $\alpha \in (0, 1)$.

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†Sobolev Institute of Mathematics of the Russian Academy of Sciences, Academician Koptiug Pr., 4, 630090 Novosibirsk, Russia (korshunov@math.nsc.ru).

Let G be an arbitrary distribution on $\mathbf{R}$ with support unbounded from above and with finite mean value. For any $t > 0$, let us define the distribution G_t on $\mathbf{R}^+$ with the distribution function

$$\overline{G_t}(x) = \min\biggl(1,\ \int_x^{x+t} \overline{G}(u)\,du\biggr), \qquad x > 0 \tag{1}$$

(hereafter, the integral from x_1 to x_2 is defined as the integral over the domain $[x_1, x_2)$). The family of distributions $\{G_t,\ t > 0\}$ is a stochastically increasing family.

Definition 3. *We say that the subexponential distribution G is* strongly subexponential (*and write $G \in \mathcal{S}_*$*) *if $\overline{G_t{*}G_t}(x)/\overline{G_t}(x) \longrightarrow 2$ as $x \to \infty$ uniformly in $t \in [1, \infty]$.*

By definition, $\mathcal{S}_* \subseteq \mathcal{S}$. It may turn out that the class $\mathcal{S}_*$ coincides with the class of subexponential distributions with finite mean value, but we cannot prove this fact. As is mentioned in [6], it is not even clear whether the included $G \in \mathcal{S}$ implies the subexponentiality of the distribution G_∞. Sufficient conditions for some distribution to belong to the class $\mathcal{S}_*$ are given in section 3. It is also shown there that the Pareto distribution (with $\alpha > 1$), the lognormal distribution, and the Weibull distribution satisfy these sufficient conditions and, therefore, belong to the class of strongly subexponential distributions.

Define the random sequence $X = \{X_n\}$ by the equality

$$X_{n+1} = (X_n + \xi_{n+1})^+. \tag{2}$$

This sequence is a homogeneous-in-time Markov chain of special type: it is a reflected random walk. It is well known (see, for example, [4, Chap. VI, section 9]), that the distribution of this chain X at time n, given zero initial state $X_0 = 0$, coincides with the distribution of M_n, that is,

$$\mathbf{P}\{M_n \geqq x\} = \mathbf{P}\{X_n \geqq x \mid X_0 = 0\}. \tag{3}$$

Hence, the investigation of the asymptotic behavior of the probability $\mathbf{P}\{M_n \geqq x\}$ is equivalent to the same problem for the probability $\mathbf{P}\{X_n \geqq x \,|\, X_0 = 0\}$. Notice that the finite-dimensional distributions of the sequences $\{M_n\}$ and $\{X_n\}$ do not coincide.

It is shown in [11] that, if the distribution F_∞ on half-line $\mathbf{R}^+$ is subexponential, then the distribution tail of the maximum of sums is equivalent to the integrated tail of the distribution of one summand, that is,

$$\mathbf{P}\Bigl\{\sup_{n\geqq 1} S_n \geqq x\Bigr\} \sim \frac{1}{a}\int_x^\infty \overline{F}(u)\,du \quad \text{as } x \to \infty. \tag{4}$$

It is shown in [7] that the asymptotic (4) holds if and only if F_∞ is subexponential.

In [9], the case of fixed value of time n is considered and it is shown that, if the distribution of the random variable $\xi\mathbf{I}\{\xi \geqq 0\}$ is subexponential, then $\mathbf{P}\{M_n \geqq x\} \sim n\overline{F}(x)$ as $x \to \infty$.

In the present paper we prove the following statement.

Theorem. *Let the distribution of the random variable $\xi\mathbf{I}\{\xi \geqq 0\}$ be strongly subexponential. Then*

$$\mathbf{P}\{M_n \geqq x\} = \frac{1+\varepsilon_n(x)}{a}\int_x^{x+na} \overline{F}(u)\,du,$$

where $\varepsilon_n(x) \to 0$ as $x \to \infty$ uniformly in $n \geqq 1$.

This theorem follows from Lemmas 1 and 9 which will be proved in sections 2 and 6, respectively.

As we have learned, after this paper was prepared for publication, the analogous equivalence was established by other methods in [1, Thm. 6] for the case when $\mathbf{E}\xi^2 < \infty$ and the function $\overline{F}(u)$ is regularly varying at infinity.

Among the previous results, papers [8] and [5] should be mentioned. In [8], the regularly varying at infinity function $\overline{F}(x)$ of index $\alpha \in (-\infty, -2)$ is considered. Putting $g(t) = 1{+}at/\gamma$

and $x = \gamma n$ in Theorem 2 of this paper, it is possible to conclude the following asymptotic, for any fixed $\gamma > 0$:

$$\mathbf{P}\{S_k \geqq \gamma n \text{ for some } k \leqq n\} \sim n\mathbf{P}\{\xi \geqq \gamma n\}\, c_\alpha,$$

where $c_\alpha = \int_0^1 [g(t)]^{-\alpha} dt$. This asymptotic coincides with our result when $x = \gamma n$.

As in [8], the form of the answer in the case $\alpha \in (-2, -1)$ is given in [5].

2. Lower bound for large-deviation probabilities for the maximum of sums. In the following lemma, under minimal conditions on the distribution F, the probability $\mathbf{P}\{M_n \geqq x\}$ is estimated from below for large values of x.

LEMMA 1. *Let the distribution F be long-tailed. Then for any $\varepsilon > 0$, there exists x_1 such that the inequality*

$$\mathbf{P}\{M_n \geqq x\} \geqq \frac{1-\varepsilon}{a} \int_x^{x+na} \overline{F}(u)\, du$$

holds for each $x \geqq x_1$ and $n \geqq 1$.

Proof. In view of equality (3), it is sufficient to prove the corresponding assertion for the Markov chain (2) with zero starting point $X_0 = 0$. As was mentioned above, the family of maxima $\{M_n,\ n \geqq 1\}$ is stochastically bounded. Thus, the family $\{X_n,\ n \geqq 1\}$ is also stochastically bounded, that is, the following convergence holds:

$$\inf_{n \geqq 1} \mathbf{P}\{X_n < x\} \longrightarrow 1 \quad \text{as } x \to \infty. \tag{5}$$

Consider the event A_{in}, $i \in [1, n]$, which occurs if $X_{i-1} < x$ and $X_j \geqq x$ for any $j \in [i, n]$. Since the events A_{in}, $i \in [1, n]$, are disjoint and their union is equal to the event $\{X_n \geqq x\}$, we have by the formula of total probability that

$$\mathbf{P}\{X_n \geqq x\} = \mathbf{P}\{A_{1n}\} + \cdots + \mathbf{P}\{A_{nn}\}. \tag{6}$$

Fix $\delta > 0$ and put $b = a + \delta$. For $v \geqq 0$, introduce the probability $p_i(x+v)$ by

$$p_i(x+v) = \mathbf{P}\{X_j \geqq x \quad \text{for any } j \leqq i \mid X_0 = x + v\}.$$

By definition (2), for the initial state $X_0 = x + v$ and for each i, the inequality $X_i \geqq x + v + \xi_1 + \cdots + \xi_i$ is valid. Therefore, $p_i(x+v) \geqq \mathbf{P}\{v + \xi_1 + \cdots + \xi_j \geqq 0$, for any $j \leqq i\}$. Putting $v = U + ib$ here, we obtain by the strong law of large numbers the following convergence, uniformly in x and i:

$$p_i(x + U + ib) \to 1 \quad \text{as } U \to \infty. \tag{7}$$

The intersection of the events $\{X_{i-1} < x\}$, $\{X_i \geqq x + U + (n-i)\, b\}$, and $\{X_j \geqq x$ for $j \in [i+1, n]\}$ implies the event A_{in}. Hence, the following inequality holds:

$$\mathbf{P}\{A_{in}\} \geqq \int_0^x \mathbf{P}\{X_{i-1} \in dy\} \int_{U+(n-i)\, b}^\infty \mathbf{P}\{X_i \in x + du \mid X_{i-1} = y\}\, p_{n-i}(x+u).$$

Since the function $p_i(x+u)$ is nondecreasing in u,

$$\mathbf{P}\{A_{in}\} \geqq p_{n-i}\big(x + U + (n-i)\, b\big) \int_0^x \mathbf{P}\{X_{i-1} \in dy\}\mathbf{P}\big\{X_i \geqq x + U + (n-i)\, b \mid X_{i-1} = y\big\}.$$

Taking into account that, for each $y \in [0, x)$,

$$\mathbf{P}\big\{X_i \geqq x + U + (n-i)\, b \mid X_{i-1} = y\big\} \geqq \mathbf{P}\big\{\xi \geqq x + U + (n-i)\, b\big\},$$

we obtain from the preceding inequality that

$$\mathbf{P}\{A_{in}\} \geqq p_{n-i}\big(x + U + (n-i)\, b\big)\, \mathbf{P}\{X_{i-1} < x\}\mathbf{P}\big\{\xi \geqq x + U + (n-i)\, b\big\}.$$

By virtue of this estimate and convergence (7), there exists sufficiently large U such that, for any x, i, and n, the inequality $\mathbf{P}\{A_{in}\} \geqq (1-\delta/2)\,\mathbf{P}\{X_{i-1} < x\}\,\mathbf{P}\{\xi \geqq x+U+(n-i)\,b\}$ holds. Thus, by convergence (5), for sufficiently large x, we have

$$\mathbf{P}\{A_{in}\} \geqq (1-\delta)\,\overline{F}\big(x+U+(n-i)\,b\big)$$

uniformly in i and n. Using the latter inequality, we deduce from equality (6) the following estimate, for sufficiently large x:

$$\mathbf{P}\{X_n \geqq x\} \geqq (1-\delta)\sum_{i=1}^{n}\overline{F}\big(x+U+(n-i)\,b\big).$$

Because function $\overline{F}(v)$ is long-tailed, we have

$$\sum_{i=1}^{n}\overline{F}\big(x+U+(n-i)\,b\big) \sim \frac{1}{b}\int_x^{x+nb}\overline{F}(v)\,dv$$

as $x\to\infty$ uniformly in n. Since $b = a+\delta$ and $\delta > 0$ is arbitrary, this implies the lemma assertion.

3. Conditions for the membership of some distribution to the class $\mathcal{S}_*$. Let G be a long-tailed distribution on $\mathbf{R}^+$ with finite mean value. For each $U\in(0,x)$ we have the following equalities (the distribution G_t is defined in (1)):

$$\frac{\overline{G_t{*}G_t}(x)}{\overline{G_t}(x)} = \int_0^x \frac{\overline{G_t}(x-u)}{\overline{G_t}(x)}\,dG_t(u)+1 \tag{8}$$

$$= \left(\int_0^U + \int_U^x\right)\frac{\overline{G_t}(x-u)}{\overline{G_t}(x)}\,dG_t(u)+1. \tag{9}$$

Since the function $\overline{G_t}(y)$ is long-tailed, for any fixed U,

$$\int_0^U \frac{\overline{G_t}(x-u)}{\overline{G_t}(x)}\,dG_t(u)\longrightarrow G_t(U)$$

as $x\to\infty$ uniformly in $t\geqq 1$. Therefore, the following lemma holds.

Lemma 2. *Let G be a long-tailed distribution. Then the following three conditions are equivalent*:

(i) $G\in\mathcal{S}_*$;

(ii) *for any $\varepsilon>0$ there exist U and x_0, $0\leqq U\leqq x_0$, such that the following inequality holds, for every $x\geqq x_0$ and $t\geqq 1$*:

$$\int_U^x \overline{G_t}(x-u)dG_t(u)\leqq \varepsilon\overline{G_t}(x);$$

(iii) *there exists a function $U(x)\to\infty$, $0\leqq U(x)\leqq x$, such that $\overline{G}(x-U(x))\sim\overline{G}(x)$ and, uniformly in $t\geqq 1$,*

$$\int_{U(x)}^x \overline{G_t}(x-u)\,dG_t(u) = o(\overline{G_t}(x))\quad \textit{as } x\to\infty;$$

(iv) *for any function $U(x)\to\infty$, $0\leqq U(x)\leqq x$, such that $\overline{G}(x-U(x))\sim\overline{G}(x)$, the following relation holds, uniformly in $t\geqq 1$*:

$$\int_{U(x)}^x \overline{G_t}(x-u)\,dG_t(u) = o(\overline{G_t}(x))\quad \textit{as } x\to\infty.$$

In the following lemma, it is proved that the class $\mathcal{S}_*$ is closed under weak tail equivalence (the terminology is taken from [6]).

LEMMA 3. *Let G and H be two long-tailed distributions on $\mathbf{R}^+$. If $G \in \mathcal{S}_*$ and $c_1\overline{G}(x) \leqq \overline{H}(x) \leqq c_2\overline{G}(x)$ for some c_1 and c_2, $0 < c_1 < c_2 < \infty$, then $H \in \mathcal{S}_*$.*

Proof. Since the distribution H is long-tailed and G is strongly subexponential, by Lemma 2, there exists a sequence $U(x) \to \infty$ such that $\overline{G}(x-U(x)) \sim \overline{G}(x)$, $\overline{H}(x-U(x)) \sim \overline{H}(x)$, and

$$\int_{U(x)}^{x} \overline{G_t}(x-u)\, dG_t(u) = o(\overline{G_t}(x)) \tag{10}$$

as $x \to \infty$ uniformly in $t \geqq 1$. Since $\overline{H}(x) \leqq c_2\overline{G}(x)$, the partial integration yields

$$\begin{aligned}\int_{U(x)}^{x} \overline{H_t}(x-u)\, dH_t(u) &\leqq c_2 \int_{U(x)}^{x} \overline{G_t}(x-u)\, dH_t(u) \\ &= -c_2\overline{G_t}(x-u)\overline{H_t}(u)\,\Big|_{U(x)}^{x} + c_2 \int_{U(x)}^{x} \overline{H_t}(u)\, d_u\overline{G_t}(x-u) \\ &\leqq c_2\overline{G_t}(x-U(x))\,\overline{H_t}(U(x)) + c_2^2 \int_{U(x)}^{x} \overline{G_t}(u)\, d_u\overline{G_t}(x-u).\end{aligned}$$

This estimate, in view of (10) and condition $\overline{H}(x) \geqq c_1\overline{G}(x)$, implies the relation

$$\int_{U(x)}^{x} \overline{H_t}(x-u)\, dH_t(u) = o(\overline{G_t}(x)) = o(\overline{H_t}(x)),$$

which completes the proof, by Lemma 2.

LEMMA 4. *Let G be a subexponential distribution with finite mean value and let there exist $c > 0$ such that $\overline{G}(2x) \geqq c\overline{G}(x)$ for each x. Then G is a strongly subexponential distribution.*

Proof. We have

$$\begin{aligned}\overline{G_t}(2x) &= \min\left(1,\ \int_{2x}^{2x+t} \overline{G}(u)\, du\right) \\ &= \min\left(1,\ 2\int_{x}^{x+t/2} \overline{G}(2u)\, du\right) \geqq \min\left(1, \int_{x}^{x+t} \overline{G}(2u)\, du\right).\end{aligned}$$

Since $\overline{G}(2u) \geqq c\overline{G}(u)$, it implies the inequality

$$\overline{G_t}(2x) \geqq \min\left(c,\ c\int_{x}^{x+t} \overline{G}(u)\, du\right) = c\overline{G_t}(x). \tag{11}$$

Further, the function $\overline{G}(y)$ is long-tailed. Thus, for any fixed u, uniformly in $t \geqq 1$,

$$\overline{G_t}(x-u)\,[\overline{G_t}(x)]^{-1} \longrightarrow 1 \quad \text{as } x \to \infty. \tag{12}$$

Integrating by parts and using the continuity of the distribution G_t at points $u > 0$, we obtain the following representations for the tail of the convolution $G_t{*}G_t$:

$$\begin{aligned}\overline{G_t{*}G_t}(x) &= -\left(\int_0^{x/2} + \int_{x/2}^{\infty}\right)\overline{G_t}(x-u)\, d\overline{G_t}(u) \\ &= \int_0^{x/2} \overline{G_t}(x-u)\, dG_t(u) + \int_{x/2}^{\infty} \overline{G_t}(u)\, d_u\overline{G_t}(x-u) + \left(\overline{G_t}\left(\frac{x}{2}\right)\right)^2 \\ &= 2\int_0^{x/2} \overline{G_t}(x-u)\, dG_t(u) + \left(\overline{G_t}\left(\frac{x}{2}\right)\right)^2. \qquad (13)\end{aligned}$$

Let us consider the integral

$$\int_0^{x/2} \frac{\overline{G_t}(x-u)}{\overline{G_t}(x)}\, dG_t(u) = \left(\int_0^U + \int_U^{x/2}\right) \frac{\overline{G_t}(x-u)}{\overline{G_t}(x)}\, dG_t(u).$$

In view of (12), for any fixed U the first summand tends to $G_t(U)$ as $x \to \infty$. By virtue of (11), the second summand does not exceed $\overline{G_\infty}(U)/c$, and may be done as small as we need, by the choice of sufficiently large U. These considerations imply the following convergence of the integrals, as $x \to \infty$ uniformly in $t \geqq 1$:

$$\int_0^{x/2} \frac{\overline{G_t}(x-u)}{\overline{G_t}(x)}\, dG_t(u) \longrightarrow 1.$$

Also, it follows from (11) that $(\overline{G_t}(x/2))^2 \leqq \overline{G_t}(x)\,\overline{G_t}(x/2)/c = o(\overline{G_t}(x))$ as $x \to \infty$. Substituting the latter two relations in (13), we arrive at the desirable asymptotic $\overline{G_t * G_t}(x) \sim 2\overline{G_t}(x)$ as $x \to \infty$ uniformly in $t \geqq 1$.

LEMMA 5. *Let G be a subexponential distribution with finite mean value. Let there exist x_0 such that the function $g(x) \equiv -\log \overline{G}(x)$ is concave for $x \geqq x_0$ and, in addition,*

$$\int_0^x \overline{G}(x-u)\overline{G}(u)\, du \sim \overline{G}(x) \int_0^\infty \overline{G}(u)\, du \quad \text{as } x \to \infty. \tag{14}$$

Then G is a strongly subexponential distribution.

Proof. Denote $J_t(x) = \overline{G_t}(x)/\overline{G}(x)$. In particular, if the random variable ζ has distribution G, then $J_\infty(x) = \mathbf{E}\{\zeta \mid \zeta \geqq x\}$ for those values of x for which $\overline{G_\infty}(x) \leqq 1$.

Since the function $g(x)$ is concave, the difference $g(x+u) - g(x)$ is nonincreasing in x and, therefore, the ratio $\overline{G}(x+u)/\overline{G}(x) = e^{-(g(x+u)-g(x))}$ does not decrease with respect to x. Hence, the function

$$J_t(x) = \frac{\overline{G_t}(x)}{\overline{G}(x)} = \int_0^t \frac{\overline{G}(x+u)}{\overline{G}(x)}\, du$$

also does not decrease in x. Because of $\overline{G_t}(x) = J_t(x)\,\overline{G}(x)$, this implies that the value of the second integral in (9) admits the following estimate:

$$\begin{aligned}\int_U^x \frac{J_t(x-u)}{J_t(x)} \frac{\overline{G}(x-u)}{\overline{G}(x)}\, dG_t(u) &\leqq \int_U^x \frac{\overline{G}(x-u)}{\overline{G}(x)}\, dG_t(u) \\ &= \int_U^x \frac{\overline{G}(x-u)}{\overline{G}(x)} \big(\overline{G}(u) - \overline{G}(u+t)\big)\, du \leqq \int_U^x \frac{\overline{G}(x-u)\overline{G}(u)}{\overline{G}(x)}\, du.\end{aligned}$$

In view of condition (14), the value of the latter integral may be done arbitrarily small by the choice of sufficiently large U. According to Lemma 2, this implies that the distribution G is strongly subexponential.

The Pareto distribution with parameter $\alpha > 1$, as well as any distribution with regularly varying at infinity tail and finite mean value, satisfies the conditions of Lemma 4 and, therefore, is strongly subexponential.

The Weibull distribution $\overline{G}(x) = e^{-x^\alpha}$, $\alpha \in (0,1)$, satisfies the conditions of Lemma 5. Indeed, the function $g(x) = x^\alpha$ is concave for $\alpha \in (0,1)$ and it remains to check the fulfillment of condition (14). The function $(x-u)^\alpha + u^\alpha$ reaches its maximum with respect to $u \in [U, x-U]$ at the endpoints of this interval. Therefore,

$$\int_U^{x-U} \overline{G}(x-u)\overline{G}(u)\, du = \int_U^{x-U} e^{-((x-u)^\alpha + u^\alpha)}\, du \leqq x e^{-((x-U)^\alpha + U^\alpha)} = o(e^{-x^\alpha}),$$

for example, for $U = U(x) = \log^{2/\alpha} x$. In addition, under this choice of $U(x)$,

$$\left(\int_0^U + \int_{x-U}^x\right) \overline{G}(x-u)\overline{G}(u)\, du \sim \overline{G}(x) \int_0^\infty \overline{G}(u)\, du.$$

The latter two relations imply (14).

In the same way, it may be verified that the lognormal distribution satisfies the conditions of Lemma 5 and, therefore, is strongly subexponential.

4. Some convolution properties for strongly subexponential distribution. Let G be a strongly subexponential distribution on $\mathbf{R}^+$. In this section we prove that some analogous standard properties of subexponential distributions are valid for the distributions belonging to the class $\mathcal{S}_*$.

LEMMA 6. *For any natural number k, the distribution tail of the kth convolution of the distribution G_t is equivalent to $k\overline{G_t}(x)$ as $x \to \infty$ uniformly in $t \geqq 1$.*

The proof of the theorem follows by induction. Indeed, it is sufficient to use the equality for the tail of the convolution $G_t^{(k+1)*}$ as follows:

$$\overline{G_t^{(k+1)*}}(x) = \left(\int_0^U + \int_U^x\right)\overline{G_t^{k*}}(x-u)\,dG_t(u) + \overline{G_t}(x) \tag{15}$$

and Lemma 2.

The tail of the kth convolution of the measure G_t may be estimated in the following way.

LEMMA 7. *For any $\varepsilon > 0$, there exists $c = c(\varepsilon)$ such that for each $x \geqq 0$, $t \geqq 1$, and $k = 1, 2, \dots$, the following inequality holds:*

$$\overline{G_t^{k*}}(x) \leqq c\overline{G_t}(x)(1+\varepsilon)^k.$$

Proof. Fix $\varepsilon > 0$. In view of the strong subexponentiality of G, it follows from (8) that there exists $x_0 = x_0(\varepsilon)$ such that, for each $x \geqq x_0$ and $t \geqq 1$,

$$\int_0^x \frac{\overline{G_t}(x-u)}{\overline{G_t}(x)}\,dG_t(u) \leqq 1+\varepsilon. \tag{16}$$

Put $A_k \equiv \sup_{x\geqq 0,\, t\geqq 1}[\overline{G_t^{k*}}(x)/\overline{G_t}(x)]$. Let us estimate from above the value of A_{k+1} via A_k. By virtue of (15),

$$A_{k+1} \leqq \sup_{x\geqq 0, t\geqq 1} \int_0^x \frac{\overline{G_t^{k*}}(x-u)}{\overline{G_t}(x)}\,dG_t(u) + 1. \tag{17}$$

By the definition of A_k, we have the inequality

$$\int_0^x \frac{\overline{G_t^{k*}}(x-u)}{\overline{G_t}(x)}\,dG_t(u) = \int_0^x \frac{\overline{G_t^{k*}}(x-u)}{\overline{G_t}(x-u)}\frac{\overline{G_t}(x-u)}{\overline{G_t}(x)}\,dG_t(u) \leqq A_k \int_0^x \frac{\overline{G_t}(x-u)}{\overline{G_t}(x)}\,dG_t(u).$$

Substituting (16) here, we obtain for $x \geqq x_0$ the following estimate:

$$\int_0^x \frac{\overline{G_t^{k*}}(x-u)}{\overline{G_t}(x)}\,dG_t(u) \leqq A_k(1+\varepsilon).$$

Also, for $x < x_0$ and $t \geqq 1$, we have the inequality

$$\int_0^x \frac{\overline{G_t^{k*}}(x-u)}{\overline{G_t}(x)}\,dG_t(u) \leqq \frac{1}{\overline{G_t}(x_0)} \leqq \frac{1}{\min(1, \overline{G}(x_0+1))} \equiv c_1(x_0) < \infty.$$

It now follows from the latter two estimates that, for each $x \geqq 0$ and $t \geqq 1$,

$$\int_0^x \frac{\overline{G_t^{k*}}(x-u)}{\overline{G_t}(x)}\,dG_t(u) \leqq A_k(1+\varepsilon) + c_1.$$

Substituting this estimate in (17), we obtain that $A_{k+1} \leqq A_k(1+\varepsilon) + c_1 + 1$. This implies the inequality $A_{k+1} \leqq (c_1+1)(k+1)(1+\varepsilon)^k$, which is equivalent to the lemma estimate.

5. Upper estimate for the distribution tail of the first (up to time n) nonnegative sum. Let $\eta = \min\{k \geqq 1\colon\ S_k \geqq 0\}$ be the index of the first nonnegative sum (or ascending ladder epoch; we put $\min \varnothing = \infty$), let $\eta^{[n]} = \min\{k \in [1, n]\colon\ S_k \geqq 0\}$ be the index of the first (up to time n) nonnegative sum, and let $\chi^{[n]} = S_{\eta^{[n]}}$ be the first (up to time n) nonnegative sum.

Since $\mathbf{E}\xi < 0$, η, $\eta^{[n]}$, and $\chi^{[n]}$ are defective random variables, put $p = \mathbf{P}\{\eta < \infty\}$ and $p^{[n]} = \mathbf{P}\{\eta^{[n]} < \infty\}$. For any n, the following inequalities hold:

$$0 < \mathbf{P}\{\xi \geqq 0\} \leqq p^{[n]} \leqq p < 1. \tag{18}$$

Also,

$$p^{[n]} \uparrow p \quad \text{as } n \to \infty. \tag{19}$$

For sufficiently large values of n and x, the following upper bound is valid for the probability of the event $\{\chi^{[n]} \geqq x\}$.

Lemma 8. *Let F be a long-tailed distribution. Then, for any $\varepsilon > 0$, there exist numbers n_1 and x_1 such that, for $n \geqq n_1$ and $x \geqq x_1$, the following inequality holds*:

$$\mathbf{P}\{\chi^{[n]} \geqq x\} \leqq (1+\varepsilon)\,\frac{1-p}{a}\int_x^{x+na} \overline{F}(u)\,du.$$

Proof. By the formula of total probability, we have

$$\mathbf{P}\{\chi^{[n]} \geqq x\} = \sum_{j=1}^{n} \mathbf{P}\{S_i < 0 \text{ for any } i \leqq j-1,\ S_j \geqq x\}. \tag{20}$$

Put $\psi_j(B) = \mathbf{P}\{S_i < 0 \text{ for any } i \leqq j,\ S_j \in B\}$, $B \subseteq (-\infty, 0)$. By the formula of total probability, the equality

$$\mathbf{P}\{S_i < 0 \text{ for any } i \leqq j-1,\ S_j \geqq x\} = \int_x^{\infty} F(dy)\,\psi_{j-1}([x-y, 0))$$

is valid. Substituting this equality into (20), we obtain

$$\mathbf{P}\{\chi^{[n]} \geqq x\} = \int_x^{\infty} F(dy) \sum_{j=1}^{n} \psi_{j-1}([x-y, 0)). \tag{21}$$

Let us estimate the sum in the latter representation. For each $N < n$,

$$\begin{aligned}\sum_{j=1}^{n} \psi_{j-1}([x-y,0)) &\leqq N + \sum_{j=N+1}^{n} \mathbf{P}\{S_1 < 0, \dots, S_N < 0, S_j \geqq x-y\} \\ &= N + \mathbf{P}\{S_1 < 0, \dots, S_N < 0\} \sum_{j=N+1}^{n} \mathbf{P}\{S_j \geqq x-y \mid S_1 < 0, \dots, S_N < 0\}.\end{aligned}$$

Since

$$\mathbf{P}\{S_1 < 0, \dots, S_N < 0\} \longrightarrow 1-p \quad \text{as } N \to \infty, \tag{22}$$

for any $\delta > 0$, there exists N such that

$$\begin{aligned}\sum_{j=1}^{n} \psi_{j-1}([x-y,0)) &\leqq N + (1-p+\delta) \sum_{j=N+1}^{n} \mathbf{P}\{S_j \geqq x-y \mid S_1 < 0, \dots, S_N < 0\} \\ &\leqq N + (1-p+\delta) \sum_{j=N+1}^{\infty} \mathbf{P}\{S_j \geqq x-y \mid S_1 < 0, \dots, S_N < 0\}.\end{aligned}$$

The asymptotic behavior of the renewal function does not depend on the first N summands. Therefore, by the renewal theorem, for any fixed N and $\delta > 0$ there exists t such that, for $y - x \geqq t$,

$$\sum_{j=N+1}^{\infty} \mathbf{P}\{S_{j-1} \geqq x - y \mid S_1 < 0, \ldots, S_N < 0\} \leqq (1+\delta)\,\frac{y-x}{a}.$$

Also, for $y \in [x, x+t)$ the estimate

$$\begin{aligned}\sum_{j=N+1}^{\infty} &\mathbf{P}\{S_{j-1} \geqq x - y \mid S_1 < 0, \ldots, S_N < 0\} \\ &\leqq \sum_{j=N+1}^{\infty} \mathbf{P}\{S_{j-1} \geqq -t \mid S_1 < 0, \ldots, S_N < 0\} = \widetilde{c} = \widetilde{c}(-t) < \infty\end{aligned}$$

is valid. The latter two estimates imply the following inequality, which is valid for each $y \geqq x$ ($\widehat{c} = N + \widetilde{c}$):

$$\sum_{j=1}^{n} \psi_{j-1}([x-y, 0)) \leqq (1-p+\delta)(1+\delta)\,\frac{y-x}{a} + \widehat{c}. \tag{23}$$

For each $y \geqq x$ we have the following inequality and asymptotic:

$$\sum_{j=1}^{n} \psi_{j-1}([x-y, 0)) \leqq \sum_{j=1}^{n} \mathbf{P}\{S_1 < 0, S_2 < 0, \ldots, S_j < 0\} \sim n(1-p)$$

as $n \to \infty$, in view of (22). Therefore, for sufficiently large n,

$$\sum_{j=1}^{n} \psi_{j-1}([x-y, 0)) \leqq n(1-p+\delta). \tag{24}$$

Put $g(x,y) \equiv (1-p+\delta)\, \min((1+\delta)\,(y-x)/a, n)$. Substituting estimates (23) and (24) in (21), we arrive at the inequality

$$\begin{aligned}\mathbf{P}\{\chi^{[n]} \geqq x\} &\leqq \widehat{c} \int_x^{\infty} F(dy) + \int_x^{\infty} F(dy)\, g(x,y) \\ &= \widehat{c}\overline{F}(x) - \overline{F}(y)\, g(x,y)\,\big|_x^{\infty} + \int_x^{\infty} \overline{F}(y)\, d_y g(x,y) \\ &= \widehat{c}\overline{F}(x) + (1-p+\delta)\frac{1+\delta}{a} \int_x^{x+na/(1+\delta)} \overline{F}(y)\, dy.\end{aligned} \tag{25}$$

Since the function $\overline{F}(x)$ is long-tailed, $\overline{F}(x) = o(\int_x^{x+na/(1+\varepsilon)} \overline{F}(y)\, dy)$ as $n, x \to \infty$. Hence, (25) implies the assertion of the lemma.

6. Upper bound for large-deviation probabilities for maxima of sums. Let us define the nondefective random variable $\widetilde{\chi}^{[n]}$ with distribution

$$\mathbf{P}\{\widetilde{\chi}^{[n]} \in B\} = \mathbf{P}\{\chi^{[n]} \in B\}(p^{[n]})^{-1}, \qquad B \subseteq [0, \infty).$$

Fix $\varepsilon > 0$. Put $b = (1+\varepsilon)(1-p)/pa$. By Lemma 8 and convergence (19), there exist n_0 and x_0 such that, for any $n \geqq n_0$ and $x \geqq x_0$,

$$\mathbf{P}\{\widetilde{\chi}^{[n]} \geqq x\} \leqq b \int_x^{x+na} \overline{F}(u)\, du. \tag{26}$$

Define the probability measure G on the positive half-line $\mathbf{R}^+$ with distribution tail

$$\overline{G}(x) = \min\big(1,\, b\overline{F}(x)\big) \quad \text{for } x \geqq x_0+1, \quad \overline{G}(x_0+1) = 1. \tag{27}$$

Then, by (26) we have the inequality, for $n \geqq n_0$ and $x \geqq 0$,

$$\mathbf{P}\{\widetilde{\chi}^{[n]} \geqq x\} \leqq \overline{G_{na}}(x). \tag{28}$$

Let $\widetilde{\chi}_1^{[n]}, \widetilde{\chi}_2^{[n]}, \dots$ be independent copies of the random variable $\widetilde{\chi}^{[n]}$. Notice that there exists $i \in [1, n]$ such that S_i exceeds a level x if and only if one of the ladder heights exceeds this level. The probability of the event that the ith ladder epoch exists up to the moment of time n does not exceed $(p^{[n]})^i$. Denote by $\widehat{p}_i^{\,[n]}$ the conditional probability of the event that the ith ladder epoch is the last one, given it exists up to the moment of time n. For any fixed i we have the monotone convergence

$$\widehat{p}_i^{[n]} \downarrow 1-p \quad \text{as } n \to \infty. \tag{29}$$

By the formula of total probability, we have the inequality

$$\mathbf{P}\Big\{\max_{0 \leqq i \leqq n} S_i \geqq x\Big\} \leqq \sum_{i=1}^{n} (p^{\,[n]})^i \widehat{p}_i^{[n]} \mathbf{P}\{\widetilde{\chi}_1^{[n]} + \dots + \widetilde{\chi}_i^{[n]} \geqq x\}.$$

Let $N < n$. Splitting the latter sum into two sums and using (18), inequalities $\widehat{p}_i^{[n]} \leqq \widehat{p}_{i+1}^{[n]}$, and (28), we obtain the estimate

$$\mathbf{P}\{M_n \geqq x\} \leqq \widehat{p}_N^{[n]} \sum_{i=1}^{N} p^i \overline{G_{na}^{i*}}(x) + \sum_{i=N+1}^{\infty} p^i \overline{G_{na}^{i*}}(x), \tag{30}$$

which is valid for any long-tailed distribution F.

Now we assume that the random variable ξ has strongly subexponential distribution. Accordingly, by definition (27) and Lemma 3, the distribution G is also strongly subexponential. The following lemma takes place.

LEMMA 9. *Let $F \in \mathcal{S}_*$ and $\varepsilon > 0$. Then there exists x_1 such that, for each $n \geqq 1$ and $x \geqq x_1$, the estimate*

$$\mathbf{P}\{M_n \geqq x\} \leqq \frac{1+\varepsilon}{a} \int_x^{x+na} \overline{F}(u)\, du$$

is valid.

Proof. Since the time parameter n is countable and the function $\overline{F}(u)$ is long-tailed, it is sufficient to prove the following two relations: For any fixed n,

$$\lim_{x\to\infty} \frac{\mathbf{P}\{M_n \geqq x\}}{\overline{F}(x)} = \frac{n}{a}, \qquad \limsup_{n,x\to\infty} \frac{\mathbf{P}\{M_n \geqq x\}}{\int_x^{x+na} \overline{F}(u)\, du} \leqq \frac{1}{a}. \tag{31}$$

Since the distribution F is subexponential, equality (3) allows us to verify the first relation in (31) by induction. Indeed, for $n = 1$, we have $X_1 = \xi_1^+$ and $\mathbf{P}\{X_1 \geqq x\} = \overline{F}(x)$ for $x > 0$. Since $\mathbf{P}\{X_{n+1} \geqq x\} = \mathbf{P}\{X_n + \xi_{n+1} \geqq x\}$ for $x > 0$, the induction step follows from the standard properties of subexponential distributions (see, for example, the proofs of Theorem 1 in [2] and Proposition 1 in [3]).

Now we prove the second relation in (31) by using estimate (30). Distribution G is strongly subexponential, and thus it follows from Lemma 7 that, for any $\delta > 0$, there exists c_1 such that

$$\limsup_{x\to\infty} \frac{1}{\overline{G_{na}}(x)} \sum_{i=N+1}^{\infty} p^i \overline{G_{na}^{i*}}(x) \leqq \frac{c_1[p(1+\delta)]^{N+1}}{1-p(1+\delta)}$$

uniformly in $n \geqq 1$. Substituting this inequality in (30) and using (29) and Lemma 6, we obtain

$$\limsup_{n,x\to\infty} \frac{\mathbf{P}\{M_n \geqq x\}}{\overline{G_{na}}(x)} \leqq (1-p)\sum_{i=1}^{N} p^i i + \frac{c_1[p(1+\delta)]^{N+1}}{1-p(1+\delta)}.$$

By the arbitrary choice of δ and N, this implies the inequality

$$\limsup_{n,x\to\infty} \frac{\mathbf{P}\{M_n \geqq x\}}{\overline{G_{na}}(x)} \leqq \frac{p}{1-p}.$$

Now, by the definition of G,

$$\limsup_{n,x\to\infty} \frac{\mathbf{P}\{M_n \geqq x\}}{\int_x^{x+na} \overline{F}(u)\,du} \leqq \frac{1+\varepsilon}{a}.$$

Since $\varepsilon > 0$ was arbitrary, the second relation in (31) is valid. This completes the proof.

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