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TRANSITION PHENOMENA FOR REAL-VALUED MARKOV CHAINS

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Abstract

We study transition phenomena that appear in investigating stationary, real-valued, ergodic Markov chains which are close in a sense to nonergodic ones and have trajectories drifting to infinity. The approach used in this paper enables us to construct approximations to stationary distributions of Markov chains.

Key words and phrases: Markov chain, invariant distribution, heavy traffic, stability theorem, convergence to gamma, uniform, and normal distributions; test (Lyapunov) function, equilibrium-type identity.

§1. STATEMENT OF RESULTS

Let $\{X_n^{(\varepsilon)}\}_{n=0}^{\infty}$ be a family (indexed by $\varepsilon \geq 0$) of homogeneous, real-valued Markov chains (in n, the discrete time parameter) with transition function $P^{(\varepsilon)}(x, B)$, $x \in \mathbf{R}$, $B \in \mathcal{B}(\mathbf{R})$, where $\mathcal{B}(\mathbf{R})$ is the σ-algebra of Borel sets in $\mathbf{R}$. We shall drop the superscript (0) in the notation for characteristics of the limiting Markov chain $X_n \equiv X_n^{(0)}$. The main object of our study is the invariant measure $\pi^{(\varepsilon)}$ corresponding to the Markov chain $\{X_n^{(\varepsilon)}\}$, i.e., the measure satisfying the equation

$$\pi^{(\varepsilon)}(B) = \int_{\mathbf{R}} P^{(\varepsilon)}(x, B)\, \pi^{(\varepsilon)}(dx), \qquad \pi^{(\varepsilon)}(\mathbf{R}) = 1. \tag{1.1}$$

If the Markov chain $\{X_n^{(\varepsilon)}\}$, $\varepsilon > 0$, is ergodic, then the question will be about the asymptotic behavior of the stationary distribution of $\{X_n^{(\varepsilon)}\}$ as $\varepsilon \downarrow 0$. In what follows, we shall assume that for $\varepsilon > 0$ equation (1.1) has a unique solution. This occurs if ergodicity conditions for the Markov chain $\{X_n^{(\varepsilon)}\}$ are satisfied, including the existence of "mean drift" of the Markov chain to some compact (see (1.6)) and a Doob–Doeblin-type "mixing" condition (see. (1.7)). In this case the distributions $P^{(\varepsilon)}(x, n, \cdot)$ converge in variation to $\pi^{(\varepsilon)}(\cdot)$ as $n \to \infty$, and such an invariant measure $\pi^{(\varepsilon)}(\cdot)$ is unique.

The stationary distribution $\pi^{(\varepsilon)}$ can be found explicitly only in very few particular cases. However, considering Markov chains that are close to nonergodic ones we can prove

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limit theorems for the stationary distribution $\pi^{(\varepsilon)}$ (under certain regularity conditions), which allow us to construct explicit approximations to this distribution.

Let us introduce a family of random variables such that the distribution of the random variable $\xi^{(\varepsilon)}(x)$ coincides with the distribution of the jump of the Markov chain $\{X_n^{(\varepsilon)}\}$, starting from the state x:

$$\mathbf{P}\{ x + \xi^{(\varepsilon)}(x) \in B \} = P^{(\varepsilon)}(x, B) .$$

The regularity conditions are linked to the assumption that the Markov chains $\{X_n^{(\varepsilon)}\}$ are in heavy traffic, which means that the "mean drift" of $\{X_n^{(\varepsilon)}\}$ tends to zero:

$$\lim_{|x| \to \infty, \varepsilon \downarrow 0} \mathbf{E}\, \xi^{(\varepsilon)}(x) = 0 . \tag{1.2}$$

Below we shall consider the asymptotic behavior of the distribution $\pi^{(\varepsilon)}$ as $\varepsilon \downarrow 0$ under certain moment conditions on $\xi^{(\varepsilon)}(x)$. If the Markov chain $\{X_n\}$ has a unique invariant distribution, and if the transition probability is "weakly continuous," i.e., if for each $y \in \mathbf{R}$, the following weak convergence holds:

$$P^{(\varepsilon)}(x, \cdot) \Rightarrow P(y, \cdot) \quad \text{as } x \to y, \ \varepsilon \downarrow 0 , \tag{1.3}$$

then the problem under consideration is that of stability or, in other words, the problem of the continuous dependence $\pi^{(\varepsilon)} \Rightarrow \pi$ as $\varepsilon \downarrow 0$. If the Markov chain $\{X_n\}$ by itself does not have any invariant distribution ($|X_n| \to \infty$ in probability as $n \to \infty$), then the problem of transition phenomena arises. These phenomena describe the asymptotic behavior of the invariant distribution of the Markov chain $\{X_n^{(\varepsilon)}\}$ as $\varepsilon \downarrow 0$.

In Sections 1 and 3–6, we consider in detail transition phenomena for Markov chains taking values on the positive semiaxis. Transition phenomena for Markov chains taking values on the whole axis $\mathbf{R}$ are studied in §7.

Until the end of the present section, we assume $X_n^{(\varepsilon)} \geq 0$. Write

$$m^{(\varepsilon)}(x) = \mathbf{E}\, \xi^{(\varepsilon)}(x) , \qquad b^{(\varepsilon)}(x) = \mathbf{E}\, \left(\xi^{(\varepsilon)}(x)\right)^2 . \tag{1.4}$$

We assume the behavior of $m^{(\varepsilon)}(x)$ and $b^{(\varepsilon)}(x)$, as $x \to \infty, \varepsilon \downarrow 0$, to be sufficiently regular. In particular, we assume

$$\begin{gathered} \lim_{x \to \infty} m(x)\, x = \mu , \qquad -\infty \leq \mu \leq \infty , \\ \lim_{x \to \infty} b(x) = b, \qquad 0 < b < \infty, \qquad \sup_x b(x) < \infty . \end{gathered} \tag{1.5}$$

The parameters μ and b, which characterize the asymptotic behavior of the first two moments of the jumps of the process, play a crucial role in classifying the asymptotic behavior of the distribution $\pi^{(\varepsilon)}$. They are also essential for ergodicity of the Markov chain $\{X_n\}$ (see, e.g., [1]).

Denote $\tau(x) = \min\{ n \geq 1 : X_n \leq N \mid X_0 = x \}$. The following theorem holds.

Theorem A [1]. *Let a function $V(x)$ be nonnegative. If*

$$\sup_{x > N} \mathbf{E}\left(V(x + \xi(x)) - V(x)\right) < 0, \quad \sup_{x} \mathbf{E}\left(V(x + \xi(x)) - V(x)\right) < \infty$$

for some $N \geq 0$, then $\sup_{x \leq N} \mathbf{E}\,\tau(x) < \infty$.

Under the hypotheses of Theorem A, the uniform, positive recurrence of the set $[0, N]$ holds. In particular, taking $V(x) = x^2$, we conclude that if

$$\sup_{x > N} \left(2x\,m(x) + b(x)\right) < 0, \qquad \sup_{x} \left(2x\,m(x) + b(x)\right) < \infty \tag{1.6}$$

for some $N \geq 0$, then $\sup_{x \leq N} \mathbf{E}\,\tau(x) < \infty$.

It is well known that, under certain supplementary conditions, the uniform, positive recurrence of the set $[0, N]$ implies ergodicity of the Markov chain $\{X_n\}$. For example, this is the case when (a) there exist a probability measure φ on $\mathcal{B}$, a real number $p > 0$, and a natural number $m \geq 1$ such that

$$\mathbf{P}\left\{ X_m \in B \mid X_0 = x \right\} \geq p\,\varphi(B) \tag{1.7}$$

for all $x \in [0, N]$ and $B \in \mathcal{B}(\mathbf{R})$, and (b) the Markov chain $\{X_n\}$ is aperiodic (see, e.g., [2, 3]).

In our case, the fulfilment of (1.6) is determined by the ratio $2\mu/b$. It follows from Theorem A that if $2\mu/b < -1$ and inequality (1.7) holds, then an aperiodic Markov chain $\{X_n\}$ will be ergodic.

We shall consider a kind of dependence of $P^{(\varepsilon)}(x, \cdot)$ on the parameter ε, such that $m^{(\varepsilon)}(x)$ is characterized by the negative drift $-\varepsilon$ with respect to $m(x)$, so that $\lim_{x \to \infty} m^{(\varepsilon)}(x) = -\varepsilon$ and the Markov chains $\{X_n^{(\varepsilon)}\}$ are ergodic for $\varepsilon \geq 0$ if condition (1.7) is fulfilled. More precisely, we assume that

$$m^{(\varepsilon)}(x) = -\varepsilon + \mu/x + o(\varepsilon + 1/x), \quad x \to \infty,\ \varepsilon \downarrow 0,\ -\infty \leq \mu \leq \infty,$$

$$\sup_{x,\,\varepsilon} b^{(\varepsilon)}(x) < \infty, \qquad \lim_{x \to \infty,\ \varepsilon \downarrow 0} b^{(\varepsilon)}(x) = b, \quad 0 < b < \infty. \tag{1.8}$$

A particular case of Markov chain with this properties is the random walk $\{X_n^{(\varepsilon)}\}$ defined by the equality $X_{n+1}^{(\varepsilon)} = \max\left(0, X_n^{(\varepsilon)} + \xi_n^{(\varepsilon)}\right)$, where $\xi_n^{(\varepsilon)}$ are independent, identically distributed random variables, $\mathbf{E}\,\xi_n^{(\varepsilon)} = -\varepsilon$, $\mathbf{E}\left(\xi_n^{(\varepsilon)}\right)^2 \to b$. In this case $\mu = 0$. Limit theorems for random walks of this kind have been obtained in [4–9].

While stating the next theorems, we assume the following conditions to be satisfied:

(i) the Markov chains $\{X_n^{(\varepsilon)}\}$, $\varepsilon \geq 0$, are homogeneous with respect to n;

(ii) the Markov chain $\{X_n^{(\varepsilon)}\}$ has a unique invariant distribution for each $\varepsilon > 0$.

Moreover, in Theorems 2 and 5, and partially in Theorems 3 and 4 (namely, in items (b)), we assume that the transition probability satisfies the following condition: for each compact set $K \subset \mathbf{R}^+$, there exists an $\varepsilon_* = \varepsilon_*(K) > 0$ such that

$$\inf_{x \in K,\ \varepsilon \leq \varepsilon_*} \mathbf{P}\left\{ X_n^{(\varepsilon)} \notin K \text{ for some } n \mid X_0^{(\varepsilon)} = x \right\} > 0. \tag{1.9}$$

In order that condition (1.9) be fulfilled, it suffices that for any x, $x_* \in \mathbf{R}^+$, there exists an n such that $\mathbf{P}\{X_n \geq x_* \mid X_0 = x\} > 0$, and that the transition probability satisfies the continuity condition (1.3).

Theorem 1 (stability, $2\mu < -b$). *Suppose that the asymptotic representations* (1.8) *hold and that the random variables* $\left(\max(0, \xi^{(\varepsilon)}(x))\right)^2$ *are uniformly integrable in* $x \geq 0$ *and* $\varepsilon \geq 0$:

$$\sup_{x,\,\varepsilon} \mathbf{E}\left\{(\xi^{(\varepsilon)}(x))^2;\ \xi^{(\varepsilon)}(x) > N\right\} \to 0,\ N \to \infty. \tag{1.10}$$

Assume $2\mu < -b$, *the Markov chain* $\{X_n\}$ *has a unique invariant distribution* π, *and the transition probability satisfies condition* (1.3). *Then the invariant distribution* $\pi^{(\varepsilon)}$ *of the Markov chain* $\{X_n^{(\varepsilon)}\}$ *converges weakly to* π *as* $\varepsilon \downarrow 0$.

Notice that under the hypotheses of Theorem 1 the limiting Markov chain $\{X_n\}$ has a unique invariant distribution if it satisfies the mixing condition (1.7).

In what follows, it is convenient to study the asymptotic behavior of $\pi^{(\varepsilon)}$ in terms of the random variables $X^{(\varepsilon)}$ having the correspondent distribution $\pi^{(\varepsilon)}$. We shall denote by $\mathcal{L}(X)$ the distribution law of a random variable X.

Theorem 2 (convergence to a Γ-distribution, $2\mu > -b$). *Suppose that the asymptotic representations* (1.8) *and condition* (1.9) *are valid, and let the random variables* $(\xi^{(\varepsilon)}(x))^2$ *be uniformly integrable in* x *and* $\varepsilon \geq 0$:

$$\sup_{x,\,\varepsilon} \mathbf{E}\left\{(\xi^{(\varepsilon)}(x))^2;\ |\xi^{(\varepsilon)}(x)| > N\right\} \to 0, \quad \text{as } N \to \infty. \tag{1.11}$$

If $\infty > 2\mu > -b$ *then, as* $\varepsilon \downarrow 0$, *the weak convergence*

$$\mathcal{L}(2\varepsilon X^{(\varepsilon)}) \Rightarrow \Gamma_{1/b,\,1+2\mu/b},$$

holds, where $\Gamma_{\alpha,\,\lambda}$ *is the gamma distribution with parameters* α *and* λ.

It turns out that in the case $2\mu = -b$ it is, in general, impossible to prove a limit theorem for the random variables $X^{(\varepsilon)}$ without specifying the remainder terms in the asymptotic representations (1.8).

Theorem 3 (the critical case, $2\mu = -b$). *Suppose*

$$\begin{aligned} m^{(\varepsilon)}(x) &= -\varepsilon + \mu/x + \alpha/x\log x + o(\varepsilon + 1/x\log x), \\ b^{(\varepsilon)}(x) &= b + \beta/\log x + o(\varepsilon + 1/\log x) \end{aligned} \tag{1.12}$$

as $x \to \infty$, $\varepsilon \downarrow 0$, *and, additionally,*

$$\sup_{x,\,\varepsilon} \mathbf{E}\,|\xi^{(\varepsilon)}(x)|^{2+\delta} < \infty \tag{1.13}$$

for some $\delta > 0$. *Let* $2\mu = -b$. *Then*

(a) *if* $2\alpha + \beta < -b$, *the Markov chain* $\{X_n\}$ *has a unique invariant distribution* π, *and the transition probability satisfies condition* (1.3), *then the invariant distribution* $\pi^{(\varepsilon)}$ *converges weakly to* π *as* $\varepsilon \downarrow 0$;

(b) *if* $2\alpha + \beta > -b$ *and condition* (1.9) *is satisfied, then, as* $\varepsilon \downarrow 0$, *the weak convergence*

$$L\left(\left(\frac{\log X^{(\varepsilon)}}{\log 1/\varepsilon}\right)^{1+(2\alpha+\beta)/b}\right) \Rightarrow U[0,1]$$

holds, where $U[0,1]$ *is the uniform distribution on* $[0,1]$.

Let us introduce the notations for the iterated logarithms and their products:

$$l_0(x) = x, \quad l_{k+1}(x) = \log(l_k(x)), \quad L_k(x) = \prod_{m=1}^{k} l_m(x).$$

Results relating to the case $2\alpha + \beta = -b$ are given by the following theorem.

Theorem 4 (complete classification of the critical case, $2\mu = -b$). *Let* $2 \le k < \infty$. *Suppose that the following asymptotic representations are valid*:

$$\begin{aligned} m^{(\varepsilon)}(x) &= -\varepsilon + \frac{\mu}{x} + \sum_{m=1}^{k} \frac{\alpha_m}{x L_m(x)} + o\left(\varepsilon + \frac{1}{x L_k(x)}\right), \\ b^{(\varepsilon)}(x) &= b + \sum_{m=1}^{k} \frac{\beta_m}{L_m(x)} + o\left(\varepsilon + \frac{1}{L_k(x)}\right) \end{aligned} \tag{1.14}$$

as $x \to \infty$, $\varepsilon \downarrow 0$. *Assume also that condition* (1.13) *is satisfied. Let* $2\mu = -b$, $2\alpha_1 + \beta_1 = -b, \ldots,$ $2\alpha_{k-1} + \beta_{k-1} = -b$. *Then*

(a) *if* $2\alpha_k + \beta_k < -b$, *the Markov chain* $\{X_n\}$ *has a unique invariant distribution* π *and the transition probability satisfies condition* (1.3), *then the invariant distribution* $\pi^{(\varepsilon)}$ *converges weakly to* π *as* $\varepsilon \downarrow 0$;

(b) *if* $2\alpha_k + \beta_k > -b$ *and condition* (1.9) *is satisfied, then, as* $\varepsilon \downarrow 0$, *the weak convergence*

$$L\left(\left(\frac{l_k(X^{(\varepsilon)})}{l_k(1/\varepsilon)}\right)^{1+(2\alpha_k+\beta_k)/b}\right) \Rightarrow U[0,1]$$

holds.

Remark 1. It will follow from the results of §5 (see formulas (5.5), (5.25)) that under the hypotheses of Theorems 3 and 4 (in the cases $2\alpha + \beta < -b$ and $2\alpha_k + \beta_k < -b$ respectively) the limiting Markov chain $\{X_n\}$ has a unique invariant distribution when the mixing condition (1.7) is satisfied.

Theorems 1–4 describe the asymptotic behavior of the random variable $X^{(\varepsilon)}$ for proper values of the parameter μ. The case $\mu = -\infty$ is included in the hypotheses of Theorem 1 on stability (see also Theorem 6). The regular case corresponding to the parameter value $\mu = \infty$ will be studied separately (see Theorem 5). Specifically, consider the situation when

$$m^{(\varepsilon)}(x) = -\varepsilon + h(x) + o(\varepsilon + h(x)), \qquad \sup_{x,\varepsilon} b^{(\varepsilon)}(x) < \infty \tag{1.15}$$

as $x \to \infty$, $\varepsilon \downarrow 0$, where $x h(x) \to \infty$ as $x \to \infty$. Suppose that the function $h(x)$ is regular varying:

$$h(x) = x^{-\lambda} L(x), \tag{1.16}$$

where $0 < \lambda \le 1$ and the function $L(x) > 0$ is slowly varying at infinity, i.e., for each $a > 0$, $L(a\,x) \sim L(x)$ as $x \to \infty$. Thus, the function $h(x)$ is regularly varying at infinity with index $-\lambda$. In addition, assume that

$$1/h(x) \text{ is concave;} \tag{1.17}$$

$$x\,h(x) \text{ is concave.} \tag{1.18}$$

It follows from condition (1.17) that

$$h(x) \text{ is strictly increasing,} \tag{1.19}$$

and we can define a number q_ε (for $\varepsilon \le h(1)$) as a unique solution of the equation

$$-\varepsilon + h(q_\varepsilon) = 0. \tag{1.20}$$

The following theorem is valid.

Theorem 5 (convergence to a normal distribution, $\mu = \infty$). *Suppose that the asymptotic representations* (1.15) *hold and conditions* (1.9), (1.16)–(1.18) *are satisfied. Then, as* $\varepsilon \downarrow 0$,

$$\mathbf{E}\,X^{(\varepsilon)} \sim q_\varepsilon, \tag{1.21}$$

$$X^{(\varepsilon)}/q_\varepsilon \Rightarrow 1. \tag{1.22}$$

If, in addition, the following sharper asymptotic representations hold

$$m^{(\varepsilon)}(x) = -\varepsilon + h(x) + o\left(\sqrt{\varepsilon/q_\varepsilon} + \sqrt{h(x)/x}\right), \qquad b^{(\varepsilon)}(x) = b + o(1), \quad 0 < b < \infty, \tag{1.23}$$

as $x \to \infty, \varepsilon \downarrow 0$ *and condition* (1.11) *is satisfied, then, as* $\varepsilon \downarrow 0$,

$$\mathbf{E}\,X^{(\varepsilon)} = q_\varepsilon + o\left(\sqrt{q_\varepsilon/\varepsilon}\right) \tag{1.24},$$

and the weak convergence

$$L\left(\left(X^{(\varepsilon)} - \mathbf{E}\,X^{(\varepsilon)}\right)\sqrt{\varepsilon/q_\varepsilon}\right) \Rightarrow \mathrm{N}(0, b/2\lambda) \tag{1.25}$$

holds, where $\mathrm{N}(0, \sigma^2)$ *is the normal distribution with parameters* 0 *and* σ^2.

Remark 2. If the remainder terms in (1.23) have the form

$$O\left(\sqrt{\varepsilon/q_\varepsilon}\right) \quad \text{or} \quad O\left(\sqrt{h(x)/x}\right)$$

as $x \to \infty, \varepsilon \downarrow 0$, then the convergence (1.25) may fail.

Remark 3. Conditions (1.17) and (1.18) do not seem to be very restrictive; in particular, they are satisfied for the functions $L(x) \equiv \text{const} > 0$, $\log x$, $1/\log\log x$, etc. It is clear that we may assume these conditions to be fulfilled only for sufficiently large x, not for all x. We can also list sufficient conditions. For example, (1.17) and (1.18) are satisfied when the first and second derivatives of the function $L(x)$ exist and satisfy the relations

$$x\,L'(x)/L(x) \to 0, \quad x^2\,L''(x)/L(x) \to 0 \text{ as } x \to \infty.$$

Notice also that if $L_i(x)$, $i = 1, .., m$, satisfy the preceding relations, then so does the function

$$L(x) = \prod_{i=1}^{m} L_i^{\gamma_i}(x), \quad \gamma_i \in \mathbf{R}.$$

In the important particular case $L(x) \equiv \text{const} > 0$ we have the following corollary.

Corollary. *Suppose*

$$m^{(\varepsilon)}(x) = -\varepsilon + \alpha / x^{\lambda} + o\left(\varepsilon^{(1+\lambda)/2\lambda} + 1/x^{(1+\lambda)/2}\right),$$

$$b^{(\varepsilon)}(x) = b + o(1)$$

as $x \to \infty, \varepsilon \downarrow 0$ *and assume conditions* (1.9) *and* (1.11) *to be satisfied. Then if* $\alpha > 0$, $0 < \lambda < 1$, *then, as* $\varepsilon \downarrow 0$,

$$\mathbf{E} X^{(\varepsilon)} = (\alpha/\varepsilon)^{1/\lambda} + o\left((1/\varepsilon)^{(1+\lambda)/2\lambda}\right),$$

$$X^{(\varepsilon)} \varepsilon^{1/\lambda} \Rightarrow \alpha^{1/\lambda},$$

$$L\left((X^{(\varepsilon)} - \mathbf{E} X^{(\varepsilon)}) \varepsilon^{(1+\lambda)/2\lambda}\right) \Rightarrow \mathrm{N}(0, b\alpha^{1/\lambda}/2\lambda).$$

The next theorem generalizes Theorem 1.

Theorem 6 (stability). *Suppose*

$$\limsup_{x \to \infty} \left(\sup_{\varepsilon > 0} (2x m^{(\varepsilon)}(x) + b^{(\varepsilon)}(x)) \right) < 0, \qquad \sup_{x, \varepsilon} b^{(\varepsilon)}(x) < \infty. \tag{1.26}$$

Assume that either condition (1.10) *or the following condition is satisfied*:

$$\mathbf{E} \left\{ (\xi^{(\varepsilon)}(x))^{2+\delta}; \xi^{(\varepsilon)}(x) \geq 0 \right\} \leq \Delta(x)(1 + x^{\delta}), \tag{1.27}$$

where $\delta > 0$, $\sup_x \Delta(x) < \infty$, $\Delta(x) \to 0$ *as* $x \to \infty$. *If the Markov chain* $\{X_n\}$ *has a unique invariant distribution* π *and if the transition probability satisfies condition* (1.3), *then the invariant distribution* $\pi^{(\varepsilon)}$ *of* $\{X_n^{(\varepsilon)}\}$ *converges weakly to* π *as* $\varepsilon \downarrow 0$.

Theorems 1–6 yield the desired classification for the limiting behavior of the invariant distributions of the Markov chains $\{X_n^{(\varepsilon)}\}$ on the semiaxis as $\varepsilon \downarrow 0$. They may be applied to describe the behavior of several queueing systems in heavy traffic (see, e.g., [10, 11]).

Analogs of Theorems 1 and 2, as well as Theorem 3 in the case $\alpha = \beta = 0$, for discrete Markov chains have been obtained in [10, 11]. In this connection, condition (1.13) appeared in Theorem 1 instead of condition (1.10). Theorems 3 and 6 for discrete Markov chains have been given without proofs in [12], and, for real-valued Markov chains, in [13]. Other approaches to proving some of the assertions may be found in [10, 11].

As far as Markov chains satisfying the Harris condition are concerned, the derivation of bounds on moments of invariant distributions is given in [14, 15]. In contrast to the above papers, we investigate general Markov chains and obtain bounds that are uniform in the parameter ε.

Recently the author was informed by M.Men'shikov and R.Jasnogorodski about their investigations on the ergodicity, recurrence, and transience properties of Markov chains for which the mean drift admits a representation similar to (1.14).

§2. LEMMA ON THE MEAN DRIFT OF A STATIONARY MARKOV CHAIN (EQUILIBRIUM-TYPE IDENTITIES)

We shall need a lemma whose meaning can be made clear from the following observation. Let X be some space with σ-algebra of sets $B(X)$, $\{\tilde{X}_n\}$ be some stationary Markov chain taking values in X, $\tilde{\pi}$ be its invariant distribution, and $\tilde{X}$ be a random element with distribution $\tilde{\pi}$. Let $V:(X, B(X)) \to \mathbf{R}$ be a measurable function. Denote the mean drift of the test function $V(x)$ by

$$v(x) \equiv \mathbf{E}\{V(x+\xi(x)) - V(x)\}.$$

If $\mathbf{E}V(\tilde{X})$ exists, then the mean drift of the stationary Markov chain (i.e., when $\tilde{X}_n$ has distribution $\tilde{\pi}$ for each n) is equal to zero:

$$0 = \mathbf{E}\left(V(\tilde{X}_{n+1}) - V(\tilde{X}_n)\right) = \int_X v(x)\,\tilde{\pi}(dx).$$

If $\mathbf{E}V(\tilde{X})$ does not exist then the preceding relation can fail. Nevertheless, under certain restrictions on $\xi(x)$, one may prove that the last integral is nonnegative or equal to zero. The following lemma holds.

Lemma 1 (Equilibrium-type identity). *Let $V(x) \geq 0$ be a measurable function such that*

$$\int_X \max(0, v(x))\,\tilde{\pi}(dx) \equiv \mathbf{E}\max(0, v(\tilde{X})) < \infty. \tag{2.1}$$

Then

$$\int_X v(x)\,\tilde{\pi}(dx) \equiv \mathbf{E}v(\tilde{X}) \geq 0. \tag{2.2}$$

If, in addition, there exists a number $c < \infty$ such that for all x

$$\mathbf{E}\,|V(x+\xi(x)) - V(x)| \leq c(1 + |v(x)|), \tag{2.3}$$

then

$$\int_X v(x)\,\tilde{\pi}(dx) \equiv \mathbf{E}v(\tilde{X}) = 0. \tag{2.4}$$

If (2.4) *is satisfied and for a set $B \in B(X)$*

$$\sup_{x \in B} V(x) \leq \inf_{x \notin B} V(x), \tag{2.5}$$

then

$$\int_{X \setminus B} v(x)\,\tilde{\pi}(dx) \leq 0. \tag{2.6}$$

Proof. For an arbitrary $N > 0$, let us consider the bounded measurable function $V_N(x) \equiv \min(V(x), N) \geq 0$. By definition,

$$|V_N(x+y) - V_N(x)| \leq |V(x+y) - V(x)| \tag{2.7}$$

for any $x, y \in X$. We denote the mean drift of the test function $V_N(x)$ as follows:

$$v_N(x) \equiv \mathbf{E}\left\{V_N(x+\xi(x)) - V_N(x)\right\}.$$

Since $0 \le V_N(x) \le N$, $\mathbf{E}\, V_N(\tilde{X})$ exists, and hence the following identity holds:

$$0 = \int_X v_N(x)\, \tilde{\pi}(dx) \equiv \mathbf{E}\, v_N(\tilde{X}) . \tag{2.8}$$

Since

$$0 \le V_N(x) \uparrow V(x) , \qquad N \to \infty , \tag{2.9}$$

for every $x \in X$, we have, by the monotone convergence theorem, the following convergence of moments:

$$v_N(x) \to v(x), \qquad N \to \infty . \tag{2.10}$$

Moreover, by the definition of V_N, for any $N \ge 0$, $x \in X$, the following inequality holds:

$$v_N(x) \le \max(0, v(x)) ; \tag{2.11}$$

i.e., the random variables $v_N(\tilde{X})$ are majorized by the random variable $\eta_1 \equiv \max(0, v(\tilde{X}))$ with $\mathbf{E}\, \eta_1 < \infty$ by virtue of (2.1). Applying successively (2.8), Fatou's lemma, and (2.10), we obtain

$$0 = \limsup_{N \to \infty} \mathbf{E}\, v_N(\tilde{X}) \le \mathbf{E} \limsup_{N \to \infty} v_N(\tilde{X}) = \mathbf{E}\, v(\tilde{X}) .$$

The proof of inequality (2.2) is complete.

Notice that in view of (2.1) and (2.2)

$$\mathbf{E} \mid v(\tilde{X}) \mid < \infty . \tag{2.12}$$

Applying successively (2.7) and (2.3), we now estimate $| v_N(x)|$:

$$| v_N(x)| \le \mathbf{E} \mid V(x + \xi(x)) - V(x) \mid \le c\,(1 + \mid v(x) \mid) .$$

Therefore, the absolute values of the random variables $v_N(\tilde{X})$ are majorized by the random variable η_2 :

$$\mid v_N(\tilde{X}) \mid \le \mathbf{E} \mid V(x + \xi(x)) - V(x) \mid \equiv \eta_2 , \qquad \eta_2 \le c(1 + \mid v(\tilde{X}) \mid)$$

with $\mathbf{E}\, \eta_2 < \infty$ in view of (2.12). Using the Lebesgue dominated convergence theorem, we obtain from (2.8) that

$$0 = \lim_{N \to \infty} \mathbf{E}\, v_N(\tilde{X}) = \mathbf{E} \lim_{N \to \infty} v_N(\tilde{X}) .$$

Thus, $\mathbf{E}\, v(\tilde{X}) = 0$ by virtue of (2.10). The proof of equality (2.4) is complete.

Let us consider the test function

$$V_B(x) = \begin{cases} \sup\limits_{y \in B} V(y) , & \text{if } x \in B, \\ V(x) , & \text{if } x \notin B. \end{cases}$$

Denote the mean drift of the function $V_B(x)$ by

$$v_B(x) \equiv \mathbf{E}\, \{ V_B(x + \xi(x)) - V_B(x) \} .$$

Notice that by the construction of $V_B(x)$ and in view of inequality (2.5) we have

$$v_B(x) = \mathbf{E}\ \{V(x+\xi(x)) - \sup_{y\in B} V(y)\ ;\ x+\xi(x) \in X\setminus B\} \geq 0$$

for $x \in B$, and by virtue of $V_B \geq V$,

$$v_B(x) = \mathbf{E}\ \{V_B(x+\xi(x)) - V(x)\} \geq \mathbf{E}\ \{V(x+\xi(x)) - V(x)\} \equiv v(x)$$

for $x \notin B$. Consequently,

$$\int_{X\setminus B} v(x)\,\tilde{\pi}(dx) \leq \int_{X\setminus B} v_B(x)\,\tilde{\pi}(dx) \leq \int_X v_B(x)\,\tilde{\pi}(dx)\,. \tag{2.13}$$

Write $W(x) = V(x) - V_B(x)$. The function $W(x)$ is bounded: $|W(x)| \leq \sup_{y\in B} V(y)$; therefore,

$$\int_X \mathbf{E}\,\{W(x+\xi(x)) - W(x)\}\,\tilde{\pi}(dx) = 0\,.$$

Since $\mathbf{E}\,W(x+\xi(x)) - W(x) = v(x) - v_B(x)$, we have by virtue of (2.4)

$$\int_X v_B(x)\,\tilde{\pi}(dx) = 0\,. \tag{2.14}$$

Relations (2.13) and (2.14) imply (2.6). The proof of Lemma 1 is complete.

Remark 4. Similar results for discrete Markov chains have been established in [16, 17] under stronger assumptions.

We now cite some corollaries of Lemma 1. In the first we drop the requirement of nonnegativity of the test function $V(x)$, while imposing here a condition on the increments of $\xi(x)$ stronger than conditions (2.1) and (2.3).

Corollary 1. *Let $V: X \to \mathbf{R}$ be a measurable function such that*

$$\int_X \mathbf{E}\,|V(x+\xi(x)) - V(x)|\,\pi(dx) < \infty. \tag{2.15}$$

Then

$$\int_X \mathbf{E}\,\{V(x+\xi(x)) - V(x)\}\,\pi(dx) = 0.$$

Proof. Write

$$V_+(x) = \max(0, V(x))\,, \qquad V_-(x) = -\min(0, V(x)).$$

By definition,

$$V(x) = V_+(x) - V_-(x), \tag{2.16}$$

$$|V_+(x+y) - V_+(x)| \leq |V(x+y) - V(x)|\,, \tag{2.17}$$

$$|V_-(x+y) - V_-(x)| \leq |V(x+y) - V(x)| \tag{2.18}$$

for any $x, y \in X$. In view of (2.15) and (2.17), the nonnegative test function $V_+(x)$ satisfies the condition

$$\int_X \mathbf{E}\,|V_+(x+\xi(x)) - V_+(x)|\,\tilde{\pi}(dx) = 0,$$

and, hence, V_+ satisfies relation (2.12). Therefore, we can repeat the argument following relation (2.12) and conclude that

$$\int_X \mathbf{E}\{V_+(x+\xi(x)) - V_+(x)\}\,\tilde{\pi}(dx) = 0. \tag{2.19}$$

Similarly,

$$\int_X \mathbf{E}\{V_-(x+\xi(x)) - V_-(x)\}\,\tilde{\pi}(dx) = 0. \tag{2.20}$$

Corollary 1 follows from (2.16), (2.19), and (2.20).

Corollary 2. *Let Y be a separable Banach space equipped with some norm $\|\cdot\|$ and the σ-algebra $B(Y)$ induced by this norm. Let $V: X \to Y$ be a measurable mapping such that*

$$\int_X \mathbf{E}\|V(x+\xi(x)) - V(x)\|\,\tilde{\pi}(dx) < \infty.$$

Then

$$\int_X \mathbf{E}\{V(x+\xi(x)) - V(x)\}\,\tilde{\pi}(dx) = 0 \in Y.$$

Proof. The proof of Corollary 1 follows from Corollary 1 applied to the various functions of the form $L(V(x))$, where L is a linear functional on Y.

Corollary 3. *Let a stationary Markov chain $\{\tilde{X}_n\}$ take values in a separable Banach space Y. If*

$$\int_Y \mathbf{E}\|\xi(x)\|\,\tilde{\pi}(dx) < \infty,$$

then

$$\int_Y \mathbf{E}\,\xi(x)\,\tilde{\pi}(dx) = 0 \in Y.$$

Proof. The proof follows from Corollary 2 with $X = Y$ and $V(x) = x$.

§ 3. STABILITY

The main difficulty in proving stability (Theorem 6) consists in showing that the family (parametrized by $\varepsilon \geq 0$) of the random variables $X^{(\varepsilon)}$ is bounded in probability. To this end we shall need the following lemma.

Lemma 2. *Let the hypotheses of Theorem 6 be satisfied. Then there exists an unbounded, nonnegative, increasing function f such that* $\sup_{\varepsilon \geq 0} \mathbf{E} f\left(X^{(\varepsilon)}\right) < \infty$.

Proof. If condition (1.10) of Theorem 6 is satisfied, we construct the desired function $f(x)$ as follows. First of all, observe that by (1.10) there exists an unbounded, increasing function $g(x) \geq 0$ such that

$$\sup_{x,\,\varepsilon} \mathbf{E}\left\{\left(\xi^{(\varepsilon)}(x)\right)^2 g\left(\xi^{(\varepsilon)}(x)\right);\ \xi^{(\varepsilon)}(x) \geq 0\right\} < \infty. \tag{3.1}$$

Without loss of generality, we may assume the function $g(x)$ to be twice differentiable and concave. Set $f(x) \equiv g(\log(1+x))$. By the concavity of g,

$$\sup_{x \geq 0} \; xf'(x) \leq g'(0) < \infty . \tag{3.2}$$

Let us estimate the mean drift of the test function $V_1(x) = x^2 f(x)$ in state x. By Taylor's formula, we have

$$v_1^{(\varepsilon)}(x) \equiv \mathbf{E}\left\{ \left(x + \xi^{(\varepsilon)}(x) \right)^2 f\left(x + \xi^{(\varepsilon)}(x) \right) - x^2 f(x) \right\}$$

$$= \mathbf{E}\left\{ \xi^{(\varepsilon)}(x) \left(2xf(x) + x^2 f'(x) \right) + \left(\xi^{(\varepsilon)}(x) \right)^2 \frac{1}{2} \Big(2f\left(x + \theta \xi^{(\varepsilon)}(x) \right) \right.$$

$$\left. + 4\left(x + \theta \xi^{(\varepsilon)}(x) \right) f'\left(x + \theta \xi^{(\varepsilon)}(x) \right) + \left(x + \theta \xi^{(\varepsilon)}(x) \right)^2 f''\left(x + \theta \xi^{(\varepsilon)}(x) \right) \Big) \right\},$$

where $0 \leq \theta = \theta(x, \xi) \leq 1$. Using (3.2) and taking into account the monotonicity and concavity of the function $f(x)$, we obtain the inequality

$$v_1^{(\varepsilon)}(x) \leq \left(2xf(x) + x^2 f'(x) \right) m^{(\varepsilon)}(x) + \left(2f(x) + c_1 \right) b^{(\varepsilon)}(x)/2$$

$$+ \mathbf{E}\left\{ \left(\xi^{(\varepsilon)}(x) \right)^2 f\left(\theta \xi^{(\varepsilon)}(x) \right) ; \; \xi^{(\varepsilon)}(x) \geq 0 \right\}, \quad c_1 < \infty .$$

Condition (1.26) and inequalities (3.1), (3.2) imply the following estimates, which are uniform in x and ε:

$$v_1^{(\varepsilon)}(x) \leq \left(2x m^{(\varepsilon)}(x) + b^{(\varepsilon)}(x) \right) f(x) + c_2 \leq -c_3 f(x) + c_4 , \tag{3.3}$$

where $c_3 > 0$ and $c_4 < \infty$. By virtue of (3.3), the test function $V_1(x) = x^2 f(x)$ satisfies condition (2.1). Applying (2.2), we obtain

$$0 \leq \int_0^\infty v_1^{(\varepsilon)}(x) \, \pi^{(\varepsilon)}(dx) .$$

By the last inequality and estimate (3.3), the inequality

$$\sup_{\varepsilon} \int_0^\infty f(x) \, \pi^{(\varepsilon)}(dx) \equiv \sup_{\varepsilon} \mathbf{E} f\left(X^{(\varepsilon)} \right) \leq \frac{c_4}{c_3} < \infty .$$

holds uniformly in ε. Thus, the construction of the function $f(x)$ is complete if condition (1.10) is satisfied.

Let now condition (1.27) be fulfilled. We prove that for some $\delta_0 \in (0, \delta)$ the function $f(x) = x^{\delta_0}$ is the desired one. Notice first that, for each $\delta' \in (0, \delta)$, by Hölder's inequality and conditions (1.26) and (1.27), the relation

$$\mathbf{E}\left\{ \left(\xi^{(\varepsilon)}(x) \right)^{2+\delta'} ; \; \xi^{(\varepsilon)}(x) \geq 0 \right\}$$

$$= \mathbf{E}\left\{ \left(\xi^{(\varepsilon)}(x) \right)^{2(\delta - \delta')/\delta} \left(\xi^{(\varepsilon)}(x) \right)^{(\delta\delta' + 2\delta')/\delta} ; \; \xi^{(\varepsilon)}(x) \geq 0 \right\}$$

$$\leq \mathbf{E}\left\{ \left(\left(\xi^{(\varepsilon)}(x) \right)^{2(\delta - \delta')/\delta} \right)^{\delta/(\delta - \delta')} ; \; \xi^{(\varepsilon)}(x) \geq 0 \right\}^{(\delta - \delta')/\delta}$$

$$\times \mathbf{E}\left\{ \left(\left(\xi^{(\varepsilon)}(x) \right)^{(\delta\delta' + 2\delta')/\delta} \right)^{\delta/\delta'} ; \; \xi^{(\varepsilon)}(x) \geq 0 \right\}^{\delta'/\delta} = o\left(x^{\delta'} \right) \quad \text{as } x \to \infty \tag{3.4}$$

holds uniformly in $\varepsilon \geq 0$. It follows from condition (1.26) that there exists a $\delta_0 \in (0, \delta)$ such that

$$\limsup_{x \to \infty} \sup_{\varepsilon \geq 0} \left(x\, m^{(\varepsilon)}(x) + (1+\delta_0)\, b^{(\varepsilon)}(x)/2 \right) = -\, 2\, c_5 < 0\,. \tag{3.5}$$

Using Taylor's formula, let us evaluate the mean drift of the test function $V_2(x) = x^{2+\delta_0}$ in state x:

$$v_2^{(\varepsilon)}(x) \equiv \mathbf{E}\left\{ \left(x + \xi^{(\varepsilon)}(x)\right)^{2+\delta_0} - x^{2+\delta_0} \right\}$$

$$= (2+\delta_0)\, \mathbf{E}\left\{ x^{1+\delta_0} \xi^{(\varepsilon)}(x) + \frac{1+\delta_0}{2} \left(x + \theta\, \xi^{(\varepsilon)}(x)\right)^{\delta_0} \left(\xi^{(\varepsilon)}(x)\right)^2 \right\},$$

where $0 \leq \theta = \theta(x, \xi) < 1$. By the concavity of the function y^{δ_0} $(0 < \delta_0 < 1)$, inequality (3.5), and relation (3.4), the inequalities

$$v_2^{(\varepsilon)}(x)$$

$$\leq (2+\delta_0)\left(m^{(\varepsilon)}(x)\, x^{1+\delta_0} + \frac{1+\delta_0}{2} x^{\delta_0} b^{(\varepsilon)}(x) + \frac{1+\delta_0}{2} \mathbf{E}\left\{ \left(\xi^{(\varepsilon)}(x)\right)^{2+\delta_0}; \xi^{(\varepsilon)}(x) \geq 0 \right\} \right)$$

$$\leq -\, c_5\, x^{\delta_0} + c_6 \tag{3.6}$$

hold uniformly in $\varepsilon \geq 0$; here $c_5 > 0$ and $c_6 < \infty$. As was already pointed out, inequalities of the type (3.6) (see (3.3)) yield the estimate

$$\int_0^\infty x^{\delta_0}\, \pi^{(\varepsilon)}(dx) \equiv \mathbf{E}\left(X^{(\varepsilon)}\right)^{\delta_0} \leq \frac{c_6}{c_5} < \infty\,,$$

which holds uniformly in $\varepsilon \geq 0$. The proof of Lemma 2 is complete.

Proof of Theorem 6. It follows immediately from Lemma 2 that the family of random variables $\{X^{(\varepsilon)}\}_{\varepsilon \geq 0}$ is bounded in probability, since

$$\sup_{\varepsilon} \mathbf{P}\left\{ X^{(\varepsilon)} > N \right\} \leq \sup_{\varepsilon} \mathbf{E} f\left(X^{(\varepsilon)}\right)/f(N) \to 0 \text{ as } N \to \infty\,.$$

Therefore, each limiting distribution $\tilde{\pi}$ (in the sense of weak convergence as $\varepsilon \downarrow 0$, i.e., $\pi^{(\varepsilon_k)} \Rightarrow \tilde{\pi}$ as $\varepsilon_k \downarrow 0$) is proper: $\tilde{\pi}([0, \infty)) = 1$. Let us prove that $\tilde{\pi}$ satisfies equation (1.1) with $\varepsilon = 0$.

Let $\tilde{X}$ be a random variable with distribution $\tilde{\pi}$. Denote the characteristic functions of the random variables $X^{(\varepsilon)}$, $\tilde{X}$, $\xi^{(\varepsilon)}(x)$ by

$$\varphi^{(\varepsilon)}(t) = \mathbf{E} \exp\left\{ i\, t\, X^{(\varepsilon)} \right\}, \quad \tilde{\varphi}(t) = \mathbf{E} \exp\left\{ i\, t\, \tilde{X} \right\}, \quad \varphi^{(\varepsilon)}(x, t) = \mathbf{E} \exp\left\{ i\, t\, \xi^{(\varepsilon)}(x) \right\} \tag{3.7}$$

respectively. We prove that for any compact set $K \subset \mathbf{R}^+$, the convergence

$$\sup_{x \in K} \left| \varphi^{(\varepsilon)}(x, t) - \varphi(x, t) \right| \to 0 \tag{3.8}$$

holds uniformly in $x \in K$ for fixed $t \in \mathbf{R}$ as $\varepsilon \downarrow 0$. In fact, assume the contrary. Without loss of generality, we may suppose that there exists a sequence of points $y(\varepsilon) \in K$ such that

$$\liminf_{\varepsilon \downarrow 0} \; \left| \varphi^{(\varepsilon)}(y(\varepsilon), t) - \varphi(y(\varepsilon), t) \right| > 0 .$$

Choose a convergent subsequence $y(\varepsilon_k) \to y_0 \in K$. Then

$$\left| \varphi^{(\varepsilon_k)}(y(\varepsilon_k), t) - \varphi(y(\varepsilon_k), t) \right|$$

$$\le \left| \varphi^{(\varepsilon_k)}(y(\varepsilon_k), t) - \varphi(y_0, t) \right| + \left| \varphi(y_0, t) - \varphi^{(\varepsilon_k)}(y(\varepsilon_k), t) \right| \to 0$$

as $\varepsilon_k \downarrow 0$ by virtue of weak continuity of the transition probability (see (1.3)). As a result, we arrive at a contradiction. Thus, the proof of the uniform convergence (1.8) is complete.

Now we have

$$\left| \tilde{\varphi}(t) - \int_0^\infty e^{itx} \varphi(x, t) \, \tilde{\pi}(dx) \right|$$

$$= \left| \tilde{\varphi}(t) - \varphi^{(\varepsilon_k)}(t) - \int_0^\infty e^{itx} \Big(\varphi(x, t) \, \tilde{\pi}(dx) - \varphi^{(\varepsilon_k)}(x, t) \, \pi^{(\varepsilon_k)}(dx) \Big) \right|$$

$$\le \left| \tilde{\varphi}(t) - \varphi^{(\varepsilon_k)}(t) \right| + \int_0^N \left| \varphi(x, t) - \varphi^{(\varepsilon_k)}(x, t) \right| \, \pi^{(\varepsilon_k)}(dx)$$

$$+ \int_N^\infty \left| \varphi(x, t) - \varphi^{(\varepsilon_k)}(x, t) \right| \, \pi^{(\varepsilon_k)}(dx) + \left| \int_0^\infty e^{itx} \varphi(x, t) \Big(\tilde{\pi}(dx) - \pi^{(\varepsilon_k)}(dx) \Big) \right|$$

$$= I_1 + I_2 + I_3 + I_4 . \tag{3.9}$$

The right-hand side of (3.9) tends to zero as $\varepsilon_k \downarrow 0$. In fact,

$I_1 \to 0$ by virtue of the weak convergence $\pi^{(\varepsilon_k)} \Rightarrow \tilde{\pi}$;

$I_2 \to 0$ for any fixed $N \in \mathbf{R}$ by the uniform convergence (3.8);

$I_3 \to 0$ as $N \to \infty$ uniformly in ε by virtue of the boundedness in probability of the family of random variables $X^{(\varepsilon)}$ (see Lemma 2);

$I_4 \to 0$ as $\varepsilon_k \downarrow 0$, since $\pi^{(\varepsilon_k)} \Rightarrow \tilde{\pi}$ and the function $\psi(x) \equiv e^{itx} \varphi(x, t)$ is continuous (see (1.3)) and bounded for fixed $t \in \mathbf{R}$.

Thus, by (3.9), we have, for each $t \in \mathbf{R}$,

$$\tilde{\varphi}(t) = \int_0^\infty e^{itx} \varphi(x, t) \, \tilde{\pi}(dx) ,$$

i.e., the measure $\tilde{\pi}$ satisfies equation (1.1) with $\varepsilon = 0$. Therefore, the measure $\tilde{\pi}$ coincides with the invariant measure π corresponding to the Markov chain $\{ X_n \}$. Thus, every partial limit (in the sense of weak convergence) of the sequence $\pi^{(\varepsilon)}$ is equal to π. The proof of Theorem 6 is complete.

§4. CONVERGENCE TO A Γ-DISTRIBUTION

In the present section we prove Theorem 2 by the method of characteristic functions. As a preliminary, we prove the following auxiliary result.

Lemma 3. *Let $\tilde{X}_n$ be a stationary Markov chain such that*

$$\limsup_{x\to\infty} \tilde{m}(x) < 0, \qquad \sup_x \tilde{b}(x) < \infty. \tag{4.1}$$

Then, for each γ, $0 \le \gamma \le 2$,

$$0 = \int_0^\infty \mathbf{E}\left\{ (x + \tilde{\xi}(x))^\gamma - x^\gamma \right\} \tilde{\pi}(dx) \tag{4.2}$$

and, for each number x_*,

$$0 \ge \int_{x_*}^\infty \mathbf{E}\left\{ (x + \tilde{\xi}(x))^\gamma - x^\gamma \right\} \tilde{\pi}(dx). \tag{4.3}$$

Proof. The proof of Lemma 3 will be carried out with the help of Lemma 1. By hypothesis, there exist numbers x_1 and $\Delta > 0$ such that, for $x \ge x_1$,

$$\tilde{m}(x) \le -\Delta < 0. \tag{4.4}$$

Since the function $y^{\gamma/2}$ is concave, the inequalities

$$\mathbf{E}(x + \tilde{\xi}(x))^\gamma \le \left(\mathbf{E}(x + \tilde{\xi}(x))^2 \right)^{\gamma/2}$$
$$\le \left((x + \tilde{m}(x))^2 + \tilde{b}(x)\right)^{\gamma/2} \le (x - \Delta)^\gamma + (\tilde{b}(x))^{\gamma/2},$$

hold for $x \ge x_1$. These inequalities imply the following bound on the mean drift of the test function y^γ:

$$\mathbf{E}\left\{ (x + \tilde{\xi}(x))^\gamma - x^\gamma \right\} \le -\gamma\Delta x^{\gamma-1} + c_1, \quad c_1 < \infty, \tag{4.5}$$

for sufficiently large x. Now one can verify that, for every x, y, γ, $x \ge 0$, $y \ge -x$, $0 \le \gamma \le 2$, the inequality

$$(x + y)^\gamma \le x^\gamma + \gamma x^{\gamma-1}|y| + |y|^\gamma \tag{4.6}$$

holds. Therefore,

$$\mathbf{E}\left| (x + \tilde{\xi}(x))^\gamma - x^\gamma \right| \le \gamma x^{\gamma-1} \mathbf{E}\left|\tilde{\xi}(x)\right| + \mathbf{E}\left|\tilde{\xi}(x)\right|^\gamma$$
$$\le c_2\gamma x^{\gamma-1} + c_3 \le c_2\left| -\Delta\gamma x^{\gamma-1} + c_1 \right|/\Delta + (c_3 + c_2 c_1/\Delta). \tag{4.7}$$

Thus, by (4.5) and (4.7), conditions (2.1) and (2.3) of Lemma 2 are satisfied for the increasing test function y^γ with arbitrary γ, $0 \le \gamma \le 2$. Consequently, relations (4.2) and (4.3) hold. The proof of Lemma 3 is complete.

We now describe the plan for the proof of the theorem on convergence to a Γ-distribution. First, we show (Lemma 4) that the mean values of the random variables $2\varepsilon X^{(\varepsilon)}$ are uniformly bounded. Next, we shall overcome the main difficulty, which consists

in proving (Lemma 7) that the random variables $X^{(\varepsilon)}$ drift in distribution to infinity as $\varepsilon \downarrow 0$; i.e., for each fixed $x \geq 0$, $\mathbf{P}\{X^{(\varepsilon)} \leq x\} \to 0$ as $\varepsilon \downarrow 0$. Further, starting from (1.1), we obtain an equation for the characteristic function of the invariant distribution $\pi^{(\varepsilon)}$. Executing a change of variables, we shall show that the characteristic functions of the random variables $2\varepsilon X^{(\varepsilon)}$ converge to the characteristic function of a Γ-distribution.

Lemma 4. *There exists an* $\varepsilon_0 > 0$ *such that the inequality*

$$\sup_{0 < \varepsilon \leq \varepsilon_0} \mathbf{E}\, 2\varepsilon X^{(\varepsilon)} < \infty$$

holds uniformly in ε, $\varepsilon \leq \varepsilon_0$.

Proof. By (1.8), there exist numbers $c_4 < \infty$ and $\varepsilon_0 > 0$ such that for $x \geq 0$ and $\varepsilon \leq \varepsilon_0$,

$$2x m^{(\varepsilon)}(x) + b^{(\varepsilon)}(x) \leq -x\varepsilon + c_4. \tag{4.8}$$

The mean drift of the test function $V(x) = x^2$ in state x can be estimated as follows:

$$\mathbf{E}\left\{V\left(x + \xi^{(\varepsilon)}(x)\right) - V(x)\right\} = 2x m^{(\varepsilon)}(x) + b^{(\varepsilon)}(x) \leq -\varepsilon x + c_4 < \infty \tag{4.9}$$

for $\varepsilon \leq \varepsilon_0$. In particular,

$$\mathbf{E}\left\{V\left(x + \xi^{(\varepsilon)}(x)\right) - V(x)\right\} \leq c_4 < \infty. \tag{4.10}$$

Therefore, the test function y^2 satisfies condition (2.1). Substituting (4.9) into (2.2), we obtain

$$0 \leq \int_0^\infty \left(2x m^{(\varepsilon)}(x) + b^{(\varepsilon)}(x)\right) \pi^{(\varepsilon)}(dx) \leq -\varepsilon \int_0^\infty x\, \pi^{(\varepsilon)}(dx) + c_4. \tag{4.11}$$

These inequalities imply $\mathbf{E}\, 2\varepsilon X^{(\varepsilon)} \leq 2c_4$ for $\varepsilon \leq \varepsilon_0$. The proof of Lemma 4 is complete.

The next lemma yields an estimate of the mean drift of the test function $x^{1+\rho}$.

Lemma 5. *Let the hypotheses of Theorem 2 be satisfied. Then there exist numbers* $0 < \rho < 1$, $x_* \geq 0$, $\varepsilon_* > 0$, *and* c_5, $c_6 > 0$ *such that for* $x \geq x_*$, $\varepsilon \leq \varepsilon_*$,

$$v^{(\varepsilon)}(x) \equiv \mathbf{E}\left\{\left(x + \xi^{(\varepsilon)}(x)\right)^{1+\rho} - x^{1+\rho}\right\} \geq -\varepsilon x^{\rho} c_5 + c_6 x^{\rho - 1}.$$

Proof. Since $\mu + b/2 > 0$, there exists a $\rho \in (0, 1)$ such that $(1+\rho)(\mu + \rho b/2) = 2c_6 > 0$. By Taylor's formula,

$$v^{(\varepsilon)}(x) = \mathbf{E}\left\{(1+\rho)\, x^{\rho} \xi^{(\varepsilon)}(x) + \frac{(1+\rho)\rho}{2} \left(\xi^{(\varepsilon)}(x)\right)^2 \left(x + \theta \xi^{(\varepsilon)}(x)\right)^{\rho - 1}\right\}$$

$$\geq (1+\rho)\, x^{\rho} m^{(\varepsilon)}(x) + \frac{(1+\rho)\rho}{2} \mathbf{E}\left\{\left(\xi^{(\varepsilon)}(x)\right)^2 \left(x + \theta \xi^{(\varepsilon)}(x)\right)^{\rho - 1};\ \xi^{(\varepsilon)}(x) \leq \sqrt{x}\right\},$$

where $0 \leq \theta = \theta(x, \xi) \leq 1$. In view of the inequality $\rho - 1 < 0$ and condition (1.11), we have

$$v^{(\varepsilon)}(x) \geq (1+\rho)x^{\rho}m^{(\varepsilon)}(x)+\frac{(1+\rho)\rho}{2}x^{\rho-1}\,\mathbf{E}\left\{\left(\xi^{(\varepsilon)}(x)\right)^2\left(1+\frac{1}{\sqrt{x}}\right)^{\rho-1};\ \xi^{(\varepsilon)}(x)\leq\sqrt{x}\right\}$$

$$= (1+\rho)\,x^{\rho}m^{(\varepsilon)}(x)+(1+\rho)\,\rho\,x^{\rho-1}\,b^{(\varepsilon)}(x)/2+o\left(x^{\rho-1}\right) \tag{4.12}$$

uniformly in ε as $x\to\infty$. By estimate (4.12), the asymptotic representations (1.8), and the choice of the number ρ, there exist numbers $x_*\geq 1$, $\varepsilon_*>0$ such that the inequality

$$v^{(\varepsilon)}(x)\geq -2(1+\rho)\,\varepsilon\,x^{\rho}+c_6\,x^{\rho-1}.$$

holds for $x\geq x_*$, $0<\varepsilon\leq\varepsilon_*$. The proof of Lemma 5 is complete.

Lemma 6. *Let condition* (1.9) *be satisfied. Suppose that for some numerical sequence* $\psi(\varepsilon)>0$ *and a compact set* K,

$$\limsup_{\varepsilon\downarrow 0}\ \pi^{(\varepsilon)}(K)/\psi(\varepsilon)>0.$$

Then there exists a compact set $K_*\subset\mathbf{R}\setminus K$ *such that*

$$\limsup_{\varepsilon\downarrow 0}\ \pi^{(\varepsilon)}(K_*)/\psi(\varepsilon)>0.$$

Proof. Without loss of generality, we may assume that $K=[0,y]$ for some y. For natural numbers n, N, and a point x, $x\leq y$, we denote

$$p^{(\varepsilon)}(x,n,N)=\mathbf{P}\left\{y+1\leq X_n^{(\varepsilon)}\leq N\ \middle|\ X_0^{(\varepsilon)}=x\right\}.$$

In view of condition (1.9), there exist $\varepsilon_*>0$ and $\delta>0$ such that for $x<y$ and $\varepsilon\leq\varepsilon_*$,

$$\sum_{n\geq 0}\sum_{N\geq y}p^{(\varepsilon)}(x,n,N)\geq\mathbf{P}\left\{X_n^{(\varepsilon)}\geq y\ \text{for some}\ n\ \middle|\ X_0^{(\varepsilon)}=x\right\}\geq\delta>0. \tag{4.13}$$

Consequently, by the hypothesis of the lemma, we have

$$0<\limsup_{\varepsilon\downarrow 0}\int_0^y\frac{\delta}{\psi(\varepsilon)}\,\pi^{(\varepsilon)}(dy)$$

$$\leq\limsup_{\varepsilon\downarrow 0}\sum_n\sum_N\int_0^y\frac{p^{(\varepsilon)}(x,n,N)}{\psi(\varepsilon)}\,\pi^{(\varepsilon)}(dy)$$

$$\leq\sum_n\sum_N\limsup_{\varepsilon\downarrow 0}\int_0^y\frac{p^{(\varepsilon)}(x,n,N)}{\psi(\varepsilon)}\,\pi^{(\varepsilon)}(dy)$$

$$\leq\sum_n\sum_N\limsup_{\varepsilon\downarrow 0}\int_0^{\infty}\frac{p^{(\varepsilon)}(x,n,N)}{\psi(\varepsilon)}\,\pi^{(\varepsilon)}(dy)$$

$$=\sum_n\sum_N\limsup_{\varepsilon\downarrow 0}\frac{\mathbf{P}\{y+1\leq X^{(\varepsilon)}\leq N\}}{\psi(\varepsilon)}. \tag{4.14}$$

Thus, for some N,

$$\limsup_{\varepsilon\downarrow 0} \frac{\mathbf{P}\{y+1 \le X^{(\varepsilon)} \le N\}}{\psi(\varepsilon)} > 0. \tag{4.15}$$

The proof of the lemma is complete.

Corollary 1. *If, for some compact set K,*

$$\limsup_{\varepsilon\downarrow 0} \pi^{(\varepsilon)}(K) > 0,$$

then, given an x_*, there exists $\bar{x}$ such that

$$\limsup_{\varepsilon\downarrow 0} \pi^{(\varepsilon)}([x_*, \bar{x}]) > 0.$$

Proof. The proof follows from Lemma 6 with $\psi(\varepsilon) \equiv 1$.

Corollary 2. *Let x_* be a number and $\psi(\varepsilon) > 0$ be a numerical sequence. If the relation $\pi^{(\varepsilon)}(K) = O(\psi(\varepsilon))$ as $\varepsilon\downarrow 0$ is valid for every compact set $K \subset [x_*, \infty)$, it is also valid for every compact set $K \subset \mathbf{R}^+$.*

Proof. Suppose the contrary, i.e., there exist a compact set K and the sequence $\psi_*(\varepsilon) \uparrow \infty$ as $\varepsilon\downarrow 0$ such that

$$\limsup_{\varepsilon\downarrow 0} \pi^{(\varepsilon)}(K) / (\psi(\varepsilon)\,\psi_*(\varepsilon)) > 0.$$

By Lemma 6, there exists a compact set $K_* \subset [x_*, \infty)$ such that

$$\limsup_{\varepsilon\downarrow 0} \pi^{(\varepsilon)}(K_*) / (\psi(\varepsilon)\,\psi_*(\varepsilon)) = a > 0.$$

Since $\psi_*(\varepsilon) \uparrow \infty$, we have

$$\limsup_{\varepsilon\downarrow 0} \pi^{(\varepsilon)}(K_*) / \psi(\varepsilon) \ge a \lim_{\varepsilon\downarrow 0} \psi_*(\varepsilon) = \infty,$$

which contradicts the hypothesis. The proof of Corollary 2 is complete.

Lemma 7. *Under the hypotheses of Theorem 2, for every fixed $x \ge 0$,*

$$\mathbf{P}\{X^{(\varepsilon)} \le x\} \to 0 \quad \text{as } \varepsilon\downarrow 0.$$

Proof. Let ρ and x_* be as in Lemma 5. Assume that the assertion of Lemma 7 is false, i.e., there exists a number $x_1 \ge 0$ such that

$$\limsup_{\varepsilon\downarrow 0} \mathbf{P}\left\{X^{(\varepsilon)} \le x_1\right\} > 0. \tag{4.16}$$

By Corollary 1 of Lemma 6, there exists $\bar{x}$ such that

$$\limsup_{\varepsilon\downarrow 0} \mathbf{P}\left\{x_* < X^{(\varepsilon)} \le \bar{x}\right\} > 0. \tag{4.17}$$

Using Lemma 4, we obtain the following convergence of moments: as $\varepsilon\downarrow 0$ $(0 < \rho < 1)$,

$$\mathbf{E}\,\varepsilon\,(X^{(\varepsilon)})^{\rho} = \varepsilon^{1-\rho}\,\mathbf{E}\,(\varepsilon X^{(\varepsilon)})^{\rho} \le \varepsilon^{1-\rho}\,(\mathbf{E}\,\varepsilon X^{(\varepsilon)})^{\rho} \to 0. \tag{4.18}$$

By Lemma 5 and estimates (4.17) and (4.18),

$$\limsup_{\varepsilon \downarrow 0} \int_{x_*}^{\infty} \mathbf{E} \left\{ \left(x + \xi^{(\varepsilon)}(x)\right)^{1+\rho} - x^{1+\rho} \right\} \pi^{(\varepsilon)}(dx)$$

$$\geq \limsup_{\varepsilon \downarrow 0} \left(-\varepsilon c_5 \mathbf{E} \left(X^{(\varepsilon)}\right)^{\rho} + c_6 \mathbf{P} \left\{ x_* \leq X^{(\varepsilon)} \leq \bar{x} \right\} \bar{x}^{\rho - 1} \right)$$

$$\geq c_6 \bar{x}^{\rho - 1} \limsup_{\varepsilon \downarrow 0} \mathbf{P} \left\{ x_* \leq X^{(\varepsilon)} \leq \bar{x} \right\} > 0 , \tag{4.19}$$

which contradicts assertion (4.3) of Lemma 3 ($\gamma = 1 + \rho$). The proof of Lemma 7 is complete.

Proof of Theorem 2. Using the method of characteristic functions, we shall prove the weak convergence of the random variables $2\varepsilon X^{(\varepsilon)}$ to a Γ-distribution. Notice that equation (1.1) is equivalent to the following equation for characteristic functions:

$$\varphi^{(\varepsilon)}(t) \equiv \int_0^{\infty} e^{itx} \pi^{(\varepsilon)}(dx) = \int_0^{\infty} e^{itx} \varphi^{(\varepsilon)}(x, t)\, \pi^{(\varepsilon)}(dx) . \tag{4.20}$$

Here we use the notation (3.7). Expanding the characteristic function $\varphi^{(\varepsilon)}(x, t)$ by Taylor's formula with remainder in the integral form, we arrive at the equality

$$\varphi^{(\varepsilon)}(t) = \int_0^{\infty} e^{itx} \left(1 + m^{(\varepsilon)}(x)\, it + \int_0^t \frac{\partial^2 \varphi^{(\varepsilon)}(x, \theta)}{\partial \theta^2} (t - \theta)\, d\theta \right) \pi^{(\varepsilon)}(dx) , \tag{4.21}$$

which implies the relation

$$0 = \int_0^{\infty} e^{itx} \left(i m^{(\varepsilon)}(x) + \frac{1}{t} \int_0^t \frac{\partial^2 \varphi^{(\varepsilon)}(x, \theta)}{\partial \theta^2} (t - \theta)\, d\theta \right) \pi^{(\varepsilon)}(dx) . \tag{4.22}$$

Differentiating the last inequality with respect to t (by Lemma 4, this is always possible for sufficiently small $\varepsilon > 0$) and applying Taylor's formula, we obtain the principal equality

$$0 = \int_0^{\infty} e^{itx}\, ix \left(i m^{(\varepsilon)}(x) + \frac{1}{t} \int_0^t \frac{\partial^2 \varphi^{(\varepsilon)}(x, \theta)}{\partial \theta^2} (t - \theta)\, d\theta \right) \pi^{(\varepsilon)}(dx)$$

$$+ \int_0^{\infty} \frac{e^{itx}}{t^2} \int_0^t \frac{\partial^2 \varphi^{(\varepsilon)}(x, \theta)}{\partial \theta^2}\, \theta\, d\theta\, \pi^{(\varepsilon)}(dx) , \tag{4.23}$$

Set $t = 2\varepsilon s$. Then $\varphi^{(\varepsilon)}(t) = \mathbf{E} \exp \left\{ i 2\varepsilon s X^{(\varepsilon)} \right\} \equiv \beta^{(\varepsilon)}(s)$ represents the characteristic function of the random variable $2\varepsilon X^{(\varepsilon)}$. We have

$$\frac{d\varphi^{(\varepsilon)}(t)}{dt} = \frac{1}{2\varepsilon} \frac{d\beta^{(\varepsilon)}(s)}{ds} .$$

Let $S > 0$ be an arbitrary number. The following estimate holds:

$$\left| \Delta(x, \varepsilon, \theta) \right| \equiv \left| \partial^2 \varphi^{(\varepsilon)}(x, \theta) / \partial \theta^2 + b \right|$$

$$\le \left| b - b^{(\varepsilon)}(x) \right| + \left| \frac{\partial^2 \varphi^{(\varepsilon)}(x,\theta)}{\partial \theta^2}\Big|_{\theta=0} - \frac{\partial^2 \varphi^{(\varepsilon)}(x,\theta)}{\partial \theta^2} \right|$$

$$\le \left| b - b^{(\varepsilon)}(x) \right| + \mathbf{E}\left\{ \left(\xi^{(\varepsilon)}(x)\right)^2 \left| \exp\left\{ i\theta\xi^{(\varepsilon)}(x) \right\} - 1 \right| ; \left| \xi^{(\varepsilon)}(x) \right| \le N \right\}$$

$$+ \mathbf{E}\left\{ \left(\xi^{(\varepsilon)}(x)\right)^2 \left| \exp\left\{ i\theta\xi^{(\varepsilon)}(x) \right\} - 1 \right| ; \left| \xi^{(\varepsilon)}(x) \right| > N \right\}$$

$$\le \left| b - b^{(\varepsilon)}(x) \right| + N^2 \theta N + 2\,\mathbf{E}\left\{ \left(\xi^{(\varepsilon)}(x)\right)^2 ; \left| \xi^{(\varepsilon)}(x) \right| > N \right\} = I_1 + I_2 + I_3 . \quad (4.24)$$

Notice that

$I_1 \to 0$ as $x \to \infty, \varepsilon \downarrow 0$ by virtue of (1.8);

$I_2 \to 0$ as $\varepsilon \downarrow 0$ uniformly in x and θ, $|\theta| < 2\varepsilon S$, for any fixed N;

$I_3 \to 0$ as $N \to \infty$ uniformly in x and ε by virtue of (1.11).

Consequently, as $x \to \infty$ and $\varepsilon \downarrow 0$,

$$\Delta(x, \varepsilon, \theta) \to 0 \quad (4.25)$$

uniformly in θ, $|\theta| < 2\varepsilon S$. Rewrite relation (1.8) in the following form

$$m^{(\varepsilon)}(x) = -\varepsilon + \mu/x + \Delta(x, \varepsilon)\,(\varepsilon + 1/x), \quad (4.26)$$

were $\Delta(x, \varepsilon) \to 0$ as $x \to \infty, \varepsilon \downarrow 0$. Using representations (4.24) and (4.26), we deduce from (4.23) the following equation:

$$0 = \int_0^\infty e^{itx} i x \left(-i\varepsilon + \frac{i\mu}{x} + \Delta(x,\varepsilon)\, i\left(\varepsilon + \frac{1}{x}\right) - \frac{bt}{2} + \frac{1}{t}\int_0^t \Delta(x,\varepsilon,\theta)(t-\theta)\, d\theta \right) \pi^{(\varepsilon)}(dx)$$

$$- \frac{b\,\varphi^{(\varepsilon)}(t)}{2} + \int_0^\infty e^{itx} \frac{1}{t^2} \int_0^t \Delta(x,\varepsilon,\theta)\,\theta\, d\theta\, \pi^{(\varepsilon)}(dx) .$$

Let us single out $d\beta^{(\varepsilon)}(s)/ds$, $\beta^{(\varepsilon)}(s)$, and the remainder terms:

$$\frac{d\beta^{(\varepsilon)}(s)}{ds}\left(\frac{i}{2} + \frac{bs}{2} \right) + \beta^{(\varepsilon)}(s)\left(\mu + \frac{b}{2} \right)$$

$$= \int_0^\infty e^{itx} i x \left(\Delta(x,\varepsilon)\, i\varepsilon + \frac{1}{t}\int_0^t \Delta(x,\varepsilon,\theta)\,(t-\theta)\, d\theta \right) \pi^{(\varepsilon)}(dx)$$

$$+ \int_0^\infty e^{itx} \left(-\Delta(x,\varepsilon) + \frac{1}{t^2}\int_0^t \Delta(x,\varepsilon,\theta)\,\theta\, d\theta \right) \pi^{(\varepsilon)}(dx)$$

$$\equiv \Delta_1(t,\varepsilon) + \Delta_2(t,\varepsilon) . \quad (4.27)$$

As $\varepsilon \downarrow 0$, the summand $\Delta_1(t, \varepsilon)$ tends to zero uniformly in t, $|t| < 2\varepsilon S$, by virtue of (a) the inequality

$$\sup_{\varepsilon \le \varepsilon_0} \left| \Delta_1(t,\varepsilon) \right|$$

$$\le c_7 N\, \pi^{(\varepsilon)}([0,N]) + \sup_{\theta,\, x>N,\, \varepsilon \le \varepsilon_0} \left(\left| \Delta(x,\varepsilon) \right| + \left| \Delta(x,\varepsilon,\theta) \right| \right) \int_N^\infty x\, \pi^{(\varepsilon)}(dx)\,(\varepsilon + \varepsilon S) ,$$

(b) the convergence

$$\sup_{x> N,\, \varepsilon\le \varepsilon_0} \left(|\Delta(x,\varepsilon)| + |\Delta(x,\varepsilon,\theta)| \right) \to 0$$

as $N\to\infty$, $\varepsilon_0\downarrow 0$, which is uniform in θ, $|\theta| < 2\varepsilon S$,

(c) Lemmas 4 and 7.

The summand $\Delta_2(t,\varepsilon)$ tends to zero as $\varepsilon\downarrow 0$ uniformly in t, $|t| < 2\varepsilon S$, by virtue of

(a) the inequality

$$\sup_{\varepsilon\le\varepsilon_0} |\Delta_2(t,\varepsilon)| \le c_7\,\pi^{(\varepsilon)}([0,N]) + \sup_{\theta,\, x> N,\, \varepsilon\le\varepsilon_0} \left(|\Delta(x,\varepsilon)| + |\Delta(x,\varepsilon,\theta)|/2 \right),$$

(b) the convergence

$$\sup_{x> N,\, \varepsilon\le\varepsilon_0} \left(|\Delta(x,\varepsilon)| + |\Delta(x,\varepsilon,\theta)|/2 \right) \to 0$$

as $N\to\infty, \varepsilon_0\downarrow 0$, which uniform in θ, $|\theta| < 2\varepsilon S$,

(c) Lemma 7.

Neglecting the remainder terms in (4.27), we obtain

$$(i + b\,s)\frac{d\beta^{(\varepsilon)}(s)}{ds} = (-2\mu - b)\,\beta^{(\varepsilon)}(s) + \delta(s,\varepsilon), \qquad \beta^{(\varepsilon)}(0) = 1,$$

where $\delta(s,\varepsilon)\to 0$ uniformly in $|s| < S$ as $\varepsilon\downarrow 0$. A solution to this equation can be represented in the form

$$\beta^{(\varepsilon)}(s) = (1 - i\,b\,s)^{-1-2\mu/b} + o(1),\ \varepsilon\downarrow 0.$$

Thus, the characteristic functions do converge as $\varepsilon\downarrow 0$: $\beta^{(\varepsilon)}(t)\to\beta(t)$, where $\beta(t)$ is the characteristic function of the gamma distribution with parameters $1/b$ and $1 + 2\mu/b$. The proof of Theorem 2 is complete.

§5. THE CRITICAL CASE

In this section we prove Theorems 3 and 4.

Case $2\alpha + \beta < -b$. Without loss of generality, we may assume that the Markov chains $\{X_n^{(\varepsilon)}\}$ take values in the set $[e,\infty)$, i.e., for every n, the inequalities $\log X_n^{(\varepsilon)} \ge 1$, $\log X^{(\varepsilon)} \ge 1$ hold with probability one. In the next lemma, we evaluate the mean drift of the test function $x^2\log^r x$.

Lemma 8. *Let the Markov chains $X_n^{(\varepsilon)} \ge e$ satisfy condition* (1.13). *Consider the test function $V_r(x)$, $0\le V_r(x) \equiv x^2\log^r x$, $x\ge e$, $r\ge 0$. Then the mean drift of $V_r(x)$ in state x has the following form*:

$$v_r^{(\varepsilon)}(x) \equiv \mathbf{E}\left\{ V_r(x + \xi^{(\varepsilon)}(x)) - V_r(x) \right\}$$

$$= \left((2x\,m^{(\varepsilon)}(x) + b^{(\varepsilon)}(x))\log x + r\,(x\,m^{(\varepsilon)}(x) + 3\,b^{(\varepsilon)}(x)/2) + o(1) \right)\log^{r-1} x$$

as $x\to\infty$, uniformly in $\varepsilon\ge 0$.

Proof. By Taylor's formula, the following expansion holds:

$$v_r^{(\varepsilon)}(x) = \mathbf{E}\Big\{\xi^{(\varepsilon)}(x)\,(2x\log^r x + rx\log^{r-1}x)$$

$$+ (\xi^{(\varepsilon)}(x))^2\Big(\log^r(x+\theta\xi^{(\varepsilon)}(x)) + \frac{3r}{2}\log^{r-1}(x+\theta\xi^{(\varepsilon)}(x)) + \frac{r(r-1)}{2}\log^{r-2}(x+\theta\,\xi^{(\varepsilon)}(x))\Big)\Big\}$$

$$= \Big((2x\,m^{(\varepsilon)}(x) + b^{(\varepsilon)}(x))\log x + r\,(x\,m^{(\varepsilon)}(x) + 3\,b^{(\varepsilon)}(x)/2)\Big)\log^{r-1}x$$

$$+ \mathbf{E}\Big\{(\xi^{(\varepsilon)}(x))^2\Big(\log^r(x+\theta\,\xi^{(\varepsilon)}(x)) - \log^r x + \frac{3r}{2}\Big(\log^{r-1}(x+\theta\,\xi^{(\varepsilon)}(x)) - \log^{r-1}x\Big)$$

$$+ \frac{r(r-1)}{2}\Big(\log^{r-2}(x+\theta\,\xi^{(\varepsilon)}(x)) - \log^{r-2}x + \log^{r-2}x\Big)\Big)\Big\},$$

where $0 \le \theta = \theta(x,\xi) \le 1$. In order to complete the proof of the lemma, it suffices to establish that for each $s \ge -2$, the following estimate holds uniformly in ε:

$$\mathbf{E}\Big\{(\xi^{(\varepsilon)}(x))^2\Big(\log^s(x+\theta\,\xi^{(\varepsilon)}(x)) - \log^s x\Big)\Big\} = o(\log^{r-1}x), \quad x\to\infty. \tag{5.1}$$

In fact, we split the integration domain into two parts: $|\xi| \le \sqrt{x}$ and $|\xi| > \sqrt{x}$, and then estimate the expectations that are taken over each of them. Since the relation

$$\log^s(x+\xi) - \log^s x = O\Big((\log^{s-1}x)/\sqrt{x}\Big), \quad x\to\infty, \tag{5.2}$$

holds uniformly in $|\xi| \le \sqrt{x}$, the following estimate is uniform in ε:

$$\Big|\mathbf{E}\Big\{(\xi^{(\varepsilon)}(x))^2\Big(\log^s(x+\theta\,\xi^{(\varepsilon)}(x)) - \log^s x\Big);\ |\xi^{(\varepsilon)}(x)| \le \sqrt{x}\Big\}\Big|$$

$$\le c_1(\log^{s-1}x)/\sqrt{x} = o(\log^{r-1}x), \quad x\to\infty. \tag{5.3}$$

Now we estimate the expectation taken over the domain $|\xi| > \sqrt{x}$. Using the inequalities

$$\log^{|s|}x \le c_2 x^{\delta/4}, \quad \log^{|s|}(x+\xi) \le c_2(x^{\delta/4} + \xi^{\delta/4}),$$

that are valid for $x \ge e$, $\xi \ge 0$, $\delta < 4$, and for an appropriate constant $c_2 = c_2(|s|)$, we obtain

$$\Big|\mathbf{E}\Big\{(\xi^{(\varepsilon)}(x))^2\Big(\log^s(x+\theta\,\xi^{(\varepsilon)}(x)) - \log^s x\Big);\ |\xi^{(\varepsilon)}(x)| > \sqrt{x}\Big\}\Big|$$

$$\le \mathbf{E}\Big\{(\xi^{(\varepsilon)}(x))^2\Big(\log^{|s|}(x+|\xi^{(\varepsilon)}(x)|) + \log^{|s|}x\Big);\ |\xi^{(\varepsilon)}(x)| > \sqrt{x}\Big\}$$

$$\le c_2\,\mathbf{E}\Big\{(\xi^{(\varepsilon)}(x))^2\Big(|\xi^{(\varepsilon)}(x)|^{\delta/4} + 2x^{\delta/4}\Big);\ |\xi^{(\varepsilon)}(x)| > \sqrt{x}\Big\}$$

$$\le c_2\,\mathbf{E}\,|\xi^{(\varepsilon)}(x)|^{2+\delta}\Big(1/x^{3\delta/8} + 2/x^{\delta/4}\Big)$$

$$\le c_3/x^{\delta/4} = o(\log^{r-1}x) \tag{5.4}$$

as $x\to\infty$, uniformly in ε by virtue of condition (1.13). Estimates (5.3) and (5.4) yield relation (5.1).

The proof of Lemma 8 is complete.

Case $2\alpha+\beta<-b$. There exists a number $r_*>1$ such that $2\alpha+\beta+r_*b<0$. By Lemma 8 and the asymptotic representations (1.12), the mean drift of the test function $V_{r_*}(x)$ $(x\to\infty,\ \varepsilon\downarrow 0)$ can be expressed as

$$v_{r_*}^{(\varepsilon)}(x) = \bigl(-2\varepsilon(1+o(1))x\log x+2\alpha+\beta+r_*b+o(1)\bigr)\log^{r_*-1}x$$

$$\le (2\alpha+\beta+r_*b)(\log^{r_*-1}x)/2+c_4\,,\quad c_4<\infty\,, \tag{5.5}$$

uniformly in x and $\varepsilon\le\varepsilon_0$, where $\varepsilon_0>0$ is sufficiently small. In view of the choice of r_*, the right-hand side of (5.5) is bounded from above uniformly in $x\ge e$ for each $\varepsilon\le\varepsilon_0$. By Lemma 1 (see (2.2)), we have

$$0\le\int_e^\infty v_{r_*}^{(\varepsilon)}(x)\,\pi^{(\varepsilon)}(dx)\,,\quad \varepsilon\le\varepsilon_0\,.$$

Substituting (5.5), we obtain the estimate

$$\mathbf{E}\log^{r_*-1}X^{(\varepsilon)}\le 2c_4/(-2\alpha-\beta-r_*b)<\infty\,,$$

which is uniform in $\varepsilon\le\varepsilon_0$. Thus, the family of random variables $X^{(\varepsilon)}$, $0\le\varepsilon\le\varepsilon_0$, is bounded in probability. This implies (see §3) that the invariant distribution $\pi^{(\varepsilon)}$ is stable as $\varepsilon\downarrow 0$. The proof of the statement (a) of Theorem 3 is complete.

Case $2\alpha+\beta>-b$. We shall prove the convergence to the uniform distribution in two steps. First, we show that the distribution $\pi^{(\varepsilon)}$ drifts to infinity. Then we shall be able to demonstrate, by using the method of moments, that the distributions of the random variables

$$\left(\frac{\log X^{(\varepsilon)}}{\log 1/\varepsilon}\right)^{1+(2\alpha+\beta)/b}$$

converge to the uniform distribution on $[0,1]$ as $\varepsilon\downarrow 0$. To be exact, we shall prove that

$$\mathbf{E}\left(\frac{\log X^{(\varepsilon)}}{\log 1/\varepsilon}\right)^m\to\frac{b+2\alpha+\beta}{bm+b+2\alpha+\beta}=\mathbf{E}\eta^m\quad\text{as }\varepsilon\downarrow 0\,, \tag{5.6}$$

where $m=1,2,\ldots$, and the random variable $\eta^{1+(2\alpha+\beta)/b}$ is uniformly distributed on $[0,1]$.

By the asymptotic representations (1.12), there exists $\varepsilon_0>0$ such that for each $\varepsilon\in(0,\varepsilon_0]$,

$$\limsup_{x\to\infty} m^{(\varepsilon)}(x)\le-\varepsilon/2\,. \tag{5.7}$$

Lemma 9. *Under the hypotheses of Theorem* 3, *for each* $\varepsilon\in(0,\varepsilon_0]$ *and* $r\ge 0$, *the following equality holds*:

$$0=\int_e^\infty v_r^{(\varepsilon)}(x)\,\pi^{(\varepsilon)}(dx)\,. \tag{5.8}$$

Moreover, for every $x_* \geq e$,

$$0 \geq \int_{x_*}^{\infty} v_r^{(\varepsilon)}(x)\, \pi^{(\varepsilon)}(dx)\,. \tag{5.9}$$

Proof. By Lemma 8 and relation (5.7), the estimate

$$v_r^{(\varepsilon)}(x) \leq -\,\varepsilon\, x \log^r x/2 \tag{5.10}$$

holds for fixed $\varepsilon \in (0, \varepsilon_0]$ and sufficiently large x. Now, in view of Lemma 8, inequality (5.10), and the relation $\sup \mathbf{E} \max\left(0, \xi^{(\varepsilon)}(x)\right) < \infty$ (the supremum is taken over x and ε), we have

$$\begin{aligned} &\mathbf{E}\,\left|\, V_r\left(x + \xi^{(\varepsilon)}(x)\right) - V_r(x)\,\right| \\ &= \mathbf{E}\,\left|\,\xi^{(\varepsilon)}(x)\,\right|\,\left(2\, x \log^r x + r\, x \log^{r-1} x\right) + O\left(\log^r x\right) \\ &= O\left(x \log^r x\right) = O\left(v_r^{(\varepsilon)}(x)\right) \text{ as } x \to \infty\,. \end{aligned} \tag{5.11}$$

Thus, relations (5.10) and (5.11) show that the test function $V_r(x)$ satisfies conditions (2.1) and (2.3). Hence relations (5.8) and (5.9) hold. The proof of Lemma 9 is complete.

Lemma 10. *Let the hypotheses of Theorem* 3 *be satisfied. Then*

$$\limsup_{\varepsilon \downarrow 0} \mathbf{E}\left(\varepsilon\, X^{(\varepsilon)}\right)^{1+\delta} < \infty\,, \tag{5.12}$$

$$\limsup_{\varepsilon \downarrow 0} \varepsilon\, \mathbf{E}\, X^{(\varepsilon)} \log X^{(\varepsilon)} < \infty\,. \tag{5.13}$$

Proof. Let us estimate the mean drift of the test function V, $V(x) = x^{2+\delta}$, from the state x. Since

$$(x+y)^{2+\delta} = x^{2+\delta} + (2+\delta)\left(y\, x^{1+\delta} + (1+\delta)\, y^2 (x + \theta y)^{\delta}/2\right)\,,$$

the inequality $(x+y)^{\delta} \leq x^{\delta} + |y|^{\delta}$ $(0 < \delta < 1)$, condition (1.13), and the asymptotic representations (1.12) imply the relation

$$\begin{aligned} &\mathbf{E}\left\{\left(\xi^{(\varepsilon)}(x)\right)^{2+\delta} - x^{2+\delta}\right\} \\ &\leq (2+\delta)\left(x^{1+\delta} m^{(\varepsilon)}(x) + (1+\delta)\left(b^{(\varepsilon)}(x)\, x^{\delta} + \mathbf{E}\,\left|\,\xi^{(\varepsilon)}(x)\,\right|^{2+\delta}\right)/2\right) \\ &\leq (2+\delta)\left(-\varepsilon\, x^{1+\delta}/2 + c_8\, x^{\delta}\right) \end{aligned} \tag{5.14}$$

for $x \geq x_1$, $\varepsilon \leq \varepsilon_1$, where x_1 is sufficiently large and $\varepsilon_1 > 0$ is sufficiently small. For $\varepsilon \leq \varepsilon_1$, the right-hand side of (5.14) is bounded from above uniformly in x. Therefore, by Lemma 1, we have

$$0 \leq \int_{e}^{\infty} \mathbf{E}\left\{\left(x + \xi^{(\varepsilon)}(x)\right)^{2+\delta} - x^{2+\delta}\right\} \pi^{(\varepsilon)}(dx)\,. \tag{5.15}$$

Substituting (5.15) into (5.14), we obtain, for $\varepsilon \leq \varepsilon_1$, the relation

$$\varepsilon \mathbf{E} \left(X^{(\varepsilon)}\right)^{1+\delta} \leq c_9 \mathbf{E} \left(X^{(\varepsilon)}\right)^{\delta}. \tag{5.16}$$

In particular, setting $\delta = 0$, we have $\varepsilon \mathbf{E} X^{(\varepsilon)} \leq c_9$. Since the function x^{δ} is concave, $\mathbf{E} \left(X^{(\varepsilon)}\right)^{\delta} \leq c_9^{\delta} / \varepsilon^{\delta}$. Substituting this inequality into (5.16), we obtain (5.12).

We now turn to the proof of (5.13). Setting $r = 1$ in Lemma 9, we obtain the equality

$$0 = \int_e^{\infty} v_1^{(\varepsilon)}(x)\, \pi^{(\varepsilon)}(dx).$$

Applying Lemma 8 and substituting the asymptotic representations (1.12), we arrive at the relation

$$0 = \int_e^{\infty} \left(\left(2x m^{(\varepsilon)}(x) + b^{(\varepsilon)}(x) \right) \log x + x m^{(\varepsilon)}(x) + \frac{3}{2} b^{(\varepsilon)}(x) + o(1) \right) \pi^{(\varepsilon)}(dx)$$

$$= 2(-\varepsilon + o(\varepsilon)) \mathbf{E} X^{(\varepsilon)} \log X^{(\varepsilon)} + O(1) \quad \text{as } \varepsilon \downarrow 0.$$

Hence inequality (5.13) is true. The proof of Lemma 10 is complete.

Lemma 11. *Let the hypotheses of Theorem 3 be satisfied and* $2\alpha + \beta > -b$. *Then* $\mathbf{P}\{X^{(\varepsilon)} \leq x\} \to 0$ *as* $\varepsilon \downarrow 0$ *for every fixed* $x \geq 0$.

Proof. Assume the contrary, i.e., there exists a number $x_1 \geq 0$ such that

$$\limsup_{\varepsilon \downarrow 0} \mathbf{P}\left\{ X^{(\varepsilon)} \leq x_1 \right\} > 0.$$

Since $2\alpha + \beta > -b$, there exists $\rho \in (0, 1)$ such that

$$2\alpha + \beta + \rho b > 0. \tag{5.17}$$

By Lemma 8, we can find $x_* \geq e$, $\varepsilon_* > 0$, $c_{10} > 0$, and $c_{11} > 0$ such that the following estimate for the mean drift of the test function $V_\rho(x) = x^2 \log^{\rho} x$ holds: for $x \geq x_*$ and $0 < \varepsilon \leq \varepsilon_*$,

$$v_\rho^{(\varepsilon)}(x) \geq -c_{10} \varepsilon x \log^{\rho} x + c_{11} \log^{\rho - 1} x. \tag{5.18}$$

In view of Corollary 1 of Lemma 6, there exists $\overline{x} \geq e$ such that

$$\limsup_{\varepsilon \downarrow 0} \mathbf{P}\left\{ x_* \leq X^{(\varepsilon)} \leq \overline{x} \right\} > 0. \tag{5.19}$$

Since, for each $N > 1$,

$$\mathbf{E} \varepsilon X^{(\varepsilon)} \log^{\rho} X^{(\varepsilon)} \leq \varepsilon N \log^{\rho} N + \mathbf{E}\left\{ \varepsilon X^{(\varepsilon)} \log X^{(\varepsilon)};\ X^{(\varepsilon)} \geq N \right\} \log^{\rho - 1} N, \tag{5.20}$$

we have, by Lemma 10, the convergence of moments

$$\mathbf{E} \varepsilon X^{(\varepsilon)} \log^{\rho} X^{(\varepsilon)} \to 0 \text{ as } \varepsilon \downarrow 0. \tag{5.21}$$

By (5.18), (5.19), and (5.21),

$$\limsup_{\varepsilon \downarrow 0} \int_{x_*}^{\infty} v_\rho^{(\varepsilon)}(x)\, \pi^{(\varepsilon)}(dx)$$

$$\geq \limsup_{\varepsilon \downarrow 0} \left(- \varepsilon c_{10} \mathbf{E} X^{(\varepsilon)} \log^{\rho} X^{(\varepsilon)} + c_{11} \mathbf{P} \left\{ x_* \leq X^{(\varepsilon)} \leq \overline{x} \right\} \log^{\rho - 1} \overline{x} \right)$$

$$\geq c_{11} \log^{\rho - 1} \overline{x} \limsup_{\varepsilon \downarrow 0} \mathbf{P} \left\{ x_* \leq X^{(\varepsilon)} \leq \overline{x} \right\} > 0 ,$$

which contradicts inequality (5.9). The proof of Lemma 11 is complete.

Lemma 12. *Let the hypotheses of Theorem* 3 *be satisfied and* $2\alpha + \beta > -b$. *Then, as* $\varepsilon \downarrow 0$,

$$2\varepsilon \mathbf{E} X^{(\varepsilon)} \log X^{(\varepsilon)} \to b + 2\alpha + \beta .$$

Proof. While deducing (5.13), we have established the equality

$$0 = 2(-\varepsilon + o(\varepsilon)) \mathbf{E} X^{(\varepsilon)} \log X^{(\varepsilon)} + b + 2\alpha + \beta + \int_e^{\infty} \Delta(x, \varepsilon) \pi^{(\varepsilon)}(dx) ,$$

where $\Delta(x, \varepsilon) \to 0$ as $x \to \infty, \varepsilon \downarrow 0$. Since, by Lemma 11,

$$\int_e^{\infty} \Delta(x, \varepsilon) \pi^{(\varepsilon)}(dx) \to 0 \text{ as } \varepsilon \downarrow 0 ,$$

the proof of Lemma 12 is complete.

Lemma 13. *For each* $m = 1, 2, \ldots,$

$$2\varepsilon \mathbf{E} \{ X^{(\varepsilon)} \log^m X^{(\varepsilon)} \} / \log^{m-1}(1/\varepsilon) \to b + 2\alpha + \beta \quad \text{as } \varepsilon \downarrow 0 .$$

Proof. The proof will be carried out in two steps.

1. Let $\kappa > 0$. Applying the inequality $\log^m y \leq c_{12} y^{\delta \kappa / 2}$ and estimate (5.12), we obtain

$$\varepsilon \mathbf{E} \left\{ X^{(\varepsilon)} \log^m X^{(\varepsilon)} ; \log X^{(\varepsilon)} > (1 + \kappa) \log(1/\varepsilon) \right\}$$

$$\leq c_{12} \varepsilon \mathbf{E} \left\{ (X^{(\varepsilon)})^{1+\delta} / (X^{(\varepsilon)})^{\delta(1-\kappa/2)} ; X^{(\varepsilon)} > 1/\varepsilon^{1+\kappa} \right\}$$

$$\leq c_{13} \varepsilon \varepsilon^{\delta(1+\kappa)(1-\kappa/2)} / \varepsilon^{1+\delta} = c_{13} \varepsilon^{\delta \kappa (1-\kappa)/2} .$$

Consequently,

$$\varepsilon \mathbf{E} X^{(\varepsilon)} \log^m X^{(\varepsilon)} =$$

$$= \varepsilon \mathbf{E} \left\{ X^{(\varepsilon)} \log^m X^{(\varepsilon)} ; \log X^{(\varepsilon)} > (1 + \kappa) \log(1/\varepsilon) \right\}$$

$$+ \varepsilon \mathbf{E} \left\{ X^{(\varepsilon)} \log^m X^{(\varepsilon)} ; \log X^{(\varepsilon)} \leq (1 + \kappa) \log(1/\varepsilon) \right\}$$

$$\leq c_{13} \varepsilon^{\delta \kappa (1-\kappa)/2} + \varepsilon \mathbf{E} \left\{ X^{(\varepsilon)} \log X^{(\varepsilon)} \right\} \log^{m-1}(1/\varepsilon) (1 + \kappa)^{m-1} .$$

Using Lemma 12, we deduce

$$\limsup_{\varepsilon \downarrow 0} 2\varepsilon \mathbf{E} X^{(\varepsilon)} \log^m X^{(\varepsilon)} / \log^{m-1}(1/\varepsilon)$$

$$\leq (b + 2\alpha + \beta)(1 + \kappa)^{m-1} .$$

Since the number $\kappa > 0$ was chosen arbitrarily, we have

$$\limsup_{\varepsilon \downarrow 0} 2\varepsilon \mathbf{E} X^{(\varepsilon)} \log^m X^{(\varepsilon)} / \log^{m-1}(1/\varepsilon) \leq b + 2\alpha + \beta . \tag{5.22}$$

2. Let us carry out the following calculations:

$$X^{(\varepsilon)} \log^m X^{(\varepsilon)} = X^{(\varepsilon)} (\log X^{(\varepsilon)}) \left(\log (X^{(\varepsilon)} \log X^{(\varepsilon)}) - \log \log X^{(\varepsilon)} \right)^{m-1}$$

$$= (X^{(\varepsilon)} \log X^{(\varepsilon)}) \left(\log^{m-1} (X^{(\varepsilon)} \log X^{(\varepsilon)}) \right) (1 - \eta^{(\varepsilon)}) ;$$

here the random variables $\eta^{(\varepsilon)}$ converge in probability to zero. Since "$X^{(\varepsilon)} \Rightarrow \infty$ as $\varepsilon \downarrow 0$" and the function $x \log^{m-1} x$ is convex for $x \geq 1$, we have

$$\mathbf{E} X^{(\varepsilon)} \log^m X^{(\varepsilon)} \sim \mathbf{E} (X^{(\varepsilon)} \log X^{(\varepsilon)}) \log^{m-1} (X^{(\varepsilon)} \log X^{(\varepsilon)})$$

$$\geq (\mathbf{E} X^{(\varepsilon)} \log X^{(\varepsilon)}) \log^{m-1} (\mathbf{E} X^{(\varepsilon)} \log X^{(\varepsilon)}) .$$

The last inequality and Lemma 12 imply

$$\liminf_{\varepsilon \downarrow 0} 2\varepsilon \mathbf{E} X^{(\varepsilon)} \log^m X^{(\varepsilon)} / \log^{m-1}(1/\varepsilon) \geq b + 2\alpha + \beta . \tag{5.23}$$

Estimate (5.22) and (5.23) are equivalent to the assertion of Lemma 13.

Lemma 14. *For each* $m = 1, 2, \ldots,$

$$\mathbf{E} (\log^{m-1} X^{(\varepsilon)}) / \log^{m-1}(1/\varepsilon)$$

$$\to (b + 2\alpha + \beta) / (m\, b + 2\alpha + \beta) \quad \text{as } \varepsilon \downarrow 0 .$$

Proof. By Lemma 9 ($r = m$), the following identity holds for sufficiently small $\varepsilon > 0$:

$$0 = \int_e^\infty v_m^{(\varepsilon)}(x)\, \pi^{(\varepsilon)}(dx) .$$

Applying Lemma 8 and using the asymptotic representations (1.12), we arrive at the equalities

$$0 = \int_e^\infty \left((2xm^{(\varepsilon)}(x) + b^{(\varepsilon)}(x)) \log x + m\left(xm^{(\varepsilon)}(x) + \frac{3}{2} b^{(\varepsilon)}(x)\right) + \Delta(x, \varepsilon) \right) \log^{m-1} x\, \pi^{(\varepsilon)}(dx)$$

$$= \int_e^\infty (2(-\varepsilon + o(\varepsilon)) x \log x + 2\alpha + \beta + m\, b + \Delta(x, \varepsilon)) \log^{m-1} x\, \pi^{(\varepsilon)}(dx)$$

as $\varepsilon \downarrow 0$, where $\Delta(x, \varepsilon) \to 0$ as $x \to \infty, \varepsilon \downarrow 0$. Since, by Lemma 11,

$$\int_e^\infty \Delta(x, \varepsilon) \log^{m-1} x\, \pi^{(\varepsilon)}(dx) = o\left(\mathbf{E} \log^{m-1} X^{(\varepsilon)} \right) \quad \text{as } \varepsilon \downarrow 0 ,$$

we have

$$2(\varepsilon + o(\varepsilon)) \mathbf{E} X^{(\varepsilon)} \log^m X^{(\varepsilon)}$$

$$= \int_e^\infty (2\alpha + \beta + m\,b + \Delta(x,\varepsilon)) \log^{m-1} x\ \pi^{(\varepsilon)}(dx)$$

$$= (2\alpha + \beta + m\,b + o(1))\ \mathbf{E} \log^{m-1} X^{(\varepsilon)} \quad \text{as } \varepsilon \downarrow 0\,.$$

Applying Lemma 13 completes the proof of Lemma 14 and, at the same time, the proof of assertion (b) of Theorem 3.

Sketch of the proof of Theorem 4. The main ideas of the proof are the same as in Theorem 3. However, a detailed proof is rather cumbersome because of the presence of numerous iterated logarithms; therefore, we mention only the main differences in the proofs of Theorems 3 and 4.

Consider the test function

$$V_m(x) = x^2\, l_1(x) \cdot \ldots \cdot l_{k-1}(x) \cdot l_k^m(x) \geq 0\,, \qquad m \geq 1\,,$$

on the set $\{x:\ l_k(x) \geq 0\}$.

Lemma 15. *Let the hypotheses of Theorem* 4 *be satisfied. Then the mean drift of the test function* $V_m(x)$ *from the state* x *has the form*

$$v_m^{(\varepsilon)}(x) \equiv \mathbf{E}\left\{ V_m(x + \xi^{(\varepsilon)}(x)) - V_m(x) \right\}$$

$$= -2\varepsilon\,(1 + o(1))\, x\, l_1(x) \cdot \ldots \cdot l_{k-1}(x)\, l_k^m(x)$$

$$+ (2\alpha_k + \beta_k + mb + o(1))\, l_k^{m-1}(x)\,, \qquad x \to \infty,\ \varepsilon \downarrow 0\,. \tag{5.24}$$

Proof. The proof is similar to that of Lemma 8.

Let $2\alpha_k + \beta_k < -b$. Choose a number $\rho > 1$ such that the inequality $2\alpha_k + \beta_k + \rho\, b < 0$ holds. Set $m = \rho$. By Lemma 15, we have

$$v_\rho^{(\varepsilon)}(x) \leq (2\alpha_k + \beta_k + \rho\, b)\, l_k^{\rho-1}(x)/2 = -\kappa\, l_k^{\rho-1}(x)\,, \quad \kappa > 0\,, \tag{5.25}$$

for sufficiently large x and sufficiently small $\varepsilon > 0$. Applying Lemma 1 (just as in proving Theorem 6) allows us to establish the stability of the invariant distribution.

Let $2\alpha_k + \beta_k > -b$. The following analogs of Lemmas 11 and 9 hold:

(a) the distribution $\pi^{(\varepsilon)}$ converges to infinity as $\varepsilon \downarrow 0$;

(b) for every natural m and sufficiently small $\varepsilon > 0$,

$$0 = \int \mathbf{E}\left\{ V_m(x + \xi^{(\varepsilon)}(x)) - V_m(x) \right\} \pi^{(\varepsilon)}(dx)\,.$$

The last identity, equality (5.24), and the convergence "$\pi^{(\varepsilon)} \Rightarrow \infty$ as $\varepsilon \downarrow 0$" imply the relation

$$2\varepsilon\, \mathbf{E}\, X^{(\varepsilon)}\, l_1(X^{(\varepsilon)}) \cdot \ldots \cdot l_{k-1}(X^{(\varepsilon)})\, l_k^m(X^{(\varepsilon)})$$

$$\sim (2\alpha_k + \beta_k + b\,m)\, \mathbf{E}\, l_k^{m-1}(X^{(\varepsilon)})\,, \quad \varepsilon \downarrow 0\,. \tag{5.26}$$

In particular, setting $m = 1$, we have

$$2\varepsilon\, \mathbf{E}\, X^{(\varepsilon)}\, l_1(X^{(\varepsilon)}) \cdot \ldots \cdot l_k(X^{(\varepsilon)}) \to 2\alpha_k + \beta_k + b \quad \text{as } \varepsilon \downarrow 0\,. \tag{5.27}$$

Just as in Lemma 13, one can show that for every natural m,

$$2\varepsilon \mathbf{E}\, X^{(\varepsilon)}\, l_1(X^{(\varepsilon)}) \cdot \ldots \cdot l_{k-1}(X^{(\varepsilon)})\, l_k^m(X^{(\varepsilon)}) / l_k^{m-1}(1/\varepsilon)$$
$$\to 2\alpha_k + \beta_k + b \quad \text{as } \varepsilon \downarrow 0 .$$

Combining this relation and (5.26), we obtain the desired convergence of moments

$$\lim_{\varepsilon \downarrow 0} \mathbf{E} \left(l_k(X^{(\varepsilon)}) / l_k(1/\varepsilon) \right)^{m-1}$$
$$= (2\alpha_k + \beta_k + b)/(2\alpha_k + \beta_k + m\, b)$$
$$= \mathbf{E}\, \eta^{m/(1 + (2\alpha_k + \beta_k)/b)},$$

where the random variable η has the uniform distribution on $[0, 1]$.

§6. CONVERGENCE TO THE NORMAL DISTRIBUTION

In this section we prove Theorem 5. We first notice that in virtue of Lemma 7, the random variables $X^{(\varepsilon)}$ "drift in distribution to infinity" as $\varepsilon \downarrow 0$. Without loss of generality, assume

$$\limsup_{x \to \infty} m^{(\varepsilon)}(x) \le -\varepsilon/2$$

for each $\varepsilon > 0$.

Lemma 16. *If a numerical sequence $a_\varepsilon \to \infty$ satisfies the relation*

$$\lim_{\varepsilon \downarrow 0} h(a_\varepsilon)/\varepsilon = 1, \tag{6.1}$$

then

$$\lim_{\varepsilon \downarrow 0} a_\varepsilon / q_\varepsilon = 1, \tag{6.2}$$

where the number q_ε is defined by (1.20). *If*

$$\limsup_{\varepsilon \downarrow 0} h(a_\varepsilon)/\varepsilon < \infty, \tag{6.3}$$

then

$$\liminf_{\varepsilon \downarrow 0} a_\varepsilon / q_\varepsilon > 0. \tag{6.4}$$

Proof. Let us establish validity of the first assertion. Let, for definiteness, $a_\varepsilon \le q_\varepsilon$. Assume that (6.2) fails. Without loss of generality, suppose that there exists a number $c < 1$ such that $a_\varepsilon \le c\, q_\varepsilon$. The function $h(x)$ decreases (see (1.19)), and the function $L(x)$ is slowly varying at infinity; therefore,

$$h(a_\varepsilon) \ge h(c\, q_\varepsilon) \equiv 1/(c\, q_\varepsilon)^\lambda L(c\, q_\varepsilon)$$
$$\sim (1/c)^\lambda h(q_\varepsilon) = \varepsilon / c^\lambda. \tag{6.5}$$

Since $c < 1$ and $\lambda > 0$, we have

$$\liminf_{\varepsilon \downarrow 0} h(a_\varepsilon)/\varepsilon \ge 1/c^\lambda > 1,$$

which contradicts condition (6.1).

To verify the second assertion, assume that (6.4) is false, i.e.,

$$\liminf_{\varepsilon\downarrow 0} a_\varepsilon / (c\, q_\varepsilon) < 1$$

for each $c > 0$. Applying (6.5), we obtain, by virtue of condition (6.3), the inequalities

$$1 \le \limsup_{\varepsilon\downarrow 0} h(a_\varepsilon) / h(c\, q_\varepsilon)$$

$$= \limsup_{\varepsilon\downarrow 0} h(a_\varepsilon)\, c^\lambda / \varepsilon \le c_1\, c^\lambda, \quad c_1 < \infty,$$

for each $c > 0$, i.e., we arrive at a contradiction. The proof of Lemma 16 is complete.

Let a function $f(x) > 0$ be convex or concave. Then, for each point x, there exists a support line $D(x)$ of the curve f, i.e., a line that has the single common point $(x, f(x))$ with the graph of f. If such a support line is not unique, we choose any of them and denote its slope by $d_f(x)$. As far as the function $d_f(x)$ is concerned, the following lemma is true.

Lemma 17. *If a function $f(x) > 0$ is regularly varying at infinity with index $\gamma \in \mathbf{R}$, then*

$$d_f(x) = (\gamma + o(1))\, f(x) / x \quad \text{as } x \to \infty .$$

Proof. Assume, for definiteness, that $f(x)$ is concave, and, therefore, $0 \le \gamma \le 1$. Let $\delta > 0$. Since $f(x)$ is concave, the slope of a support line $D(x)$ satisfies the inequalities

$$(f(z) - f(x)) / (z - x) \le d_f(x) \le (f(x) - f(y)) / (x - y)$$

for any points x, y, z, $y < x < z$. Setting $y = x(1 - \delta)$ and $z = x(1 + \delta)$, we obtain

$$(f(x(1 + \delta)) - f(x)) / (\delta x) \le d_f(x) \le (f(x) - f(x(1 - \delta))) / (\delta x).$$

Since the function f is regularly varying at infinity with index γ, the following asymptotic relations hold: as $x \to \infty$,

$$(f(x(1 + \delta)) - f(x)) / (\delta x) \sim ((1 + \delta)^\gamma - 1)\, f(x) / (\delta x),$$

$$(f(x) - f(x(1 - \delta))) / (\delta x) \sim (1 - (1 - \delta)^\gamma)\, f(x) / (\delta x).$$

Using the preceding inequalities, we obtain from these relations that

$$((1 + \delta)^\gamma - 1) / \delta \le \liminf_{x\to\infty} \frac{x\, d_f(x)}{f(x)}$$

$$\le \limsup_{x\to\infty} \frac{x\, d_f(x)}{f(x)} \le (1 - (1 - \delta)^\gamma) / \delta .$$

By the arbitrary choice of the number $\delta > 0$, the assertion of Lemma 17 follows.

Lemma 18. *The following equality holds*:

$$0 = \int_0^\infty \frac{m^{(\varepsilon)}(x)}{h(x)}\, \pi^{(\varepsilon)}(dx) + o(1), \qquad \varepsilon \downarrow 0 .$$

Proof. Denote

$$V(x) \equiv \int_0^x \frac{dz}{h(z)} .$$

By assumption (1.17), the function $f(x) \equiv 1/h(x)$ is concave; hence, for each x, z, $x \geq 0$, $z \geq -x$,

$$1/h(x+z) - 1/h(x) \leq z\, d_f(x).$$

Integrating the last inequality with respect to z from 0 to y, we obtain

$$\int_x^{x+y} \frac{dz}{h(z)} \leq \frac{y}{h(x)} + d_f(x)\frac{y^2}{2}$$

for $y \geq 0$, and

$$\int_x^{x+y} \frac{dz}{h(z)} \geq \frac{y}{h(x)} + d_f(x)\frac{y^2}{2}$$

for $y \leq 0$. Hence, in virtue of Lemma 17, there exists c, $0 < c < \infty$, such that

$$V(x+y) - V(x) \leq \frac{y}{h(x)} + \frac{c\,y^2}{x\,h(x)} \tag{6.6}$$

for $y \geq 0$, and

$$V(x+y) - V(x) \geq \frac{y}{h(x)} + \frac{c\,y^2}{x\,h(x)}$$

for $y \leq 0$. In particular, for any y, $-x \leq y < \infty$,

$$V(x+y) - V(x) \leq \frac{|y|}{h(x)} + \frac{c\,y^2}{x\,h(x)}. \tag{6.7}$$

Since the function $1/h(x)$ is convex and $1/h(x) \uparrow \infty$, we have the following relation: for $z < 0$

$$\frac{1}{h(x+z)} - \frac{1}{h(x)} \geq z\frac{1/h(x) - 1/h(0)}{x} = \frac{z\,(1 + o(1))}{x\,h(x)}$$

as $x \to \infty$, uniformly in z, $-x \leq z \leq 0$. Integrating the last inequality with respect to z from 0 to y, we obtain for $y < 0$

$$\int_x^{x+y} \frac{dz}{h(z)} \leq \frac{y}{h(x)} + \frac{y^2}{2}\,\frac{1 + o(1)}{x\,h(x)};$$

hence,

$$V(x+y) - V(x) \leq \frac{y}{h(x)} + \frac{c\,y^2}{x\,h(x)}, \qquad 0 < c < \infty,$$

for $y \leq 0$ and x sufficiently large. In view of (6.6) and the last inequality, we have

$$\mathbf{E}\left\{ V\big(x + \xi^{(\varepsilon)}(x)\big) - V(x) \right\} \leq \frac{m^{(\varepsilon)}(x)}{h(x)} + \frac{c\,b^{(\varepsilon)}(x)}{x\,h(x)}, \tag{6.8}$$

and, by (6.7),

$$\mathbf{E}\left| V\big(x + \xi^{(\varepsilon)}(x)\big) - V(x) \right| \leq \frac{\mathbf{E}\,|\xi^{(\varepsilon)}(x)|}{h(x)} + \frac{c\,b^{(\varepsilon)}(x)}{x\,h(x)}.$$

Hence, the test function $V(x)$ satisfies the hypotheses of Lemma 1 and the following identity is valid:

$$0 \equiv \int_0^\infty \mathbf{E}\left\{ V\bigl(x+ \xi^{(\varepsilon)}(x)\bigr) - V(x) \right\} \pi^{(\varepsilon)}(dx).$$

Since the function $V'(x) = 1/h(x)$ increases, the function $V(x)$ is convex. Therefore,

$$\mathbf{E}\left\{ V\bigl(x+ \xi^{(\varepsilon)}(x)\bigr) - V(x) \right\} \geq V'(x)\, m^{(\varepsilon)}(x) = m^{(\varepsilon)}(x)/h(x) .$$

Substituting (6.8) and the last inequality into the identity above and using the convergence "$\pi^{(\varepsilon)} \Rightarrow \infty$ as $\varepsilon \downarrow 0$", we arrive at the conclusion of the lemma.

Lemma 19. *Let conditions* (1.15)–(1.18) *be satisfied. Then, as* $\varepsilon \downarrow 0$, $X^{(\varepsilon)}/q_\varepsilon \Rightarrow 1$; *moreover, the expectations converge*: $\mathbf{E}\, X^{(\varepsilon)}/q_\varepsilon \to 1$.

Proof. Lemma 18, the asymptotic representations (1.15), and the convergence "$\pi^{(\varepsilon)} \Rightarrow \infty$ as $\varepsilon \downarrow 0$" imply the equality

$$\lim_{\varepsilon \downarrow 0} \varepsilon\, \mathbf{E}\, 1/h\left(X^{(\varepsilon)}\right) = 1 . \tag{6.9}$$

Since the function $1/h(x)$ is concave,

$$\limsup_{\varepsilon \downarrow 0} h\left(\mathbf{E}\, X^{(\varepsilon)}\right)/\varepsilon \leq 1 . \tag{6.10}$$

According to Lemma 3 (with $\gamma = 2$), the following equality holds:

$$0 = \int_0^\infty \left(2\, x\, m^{(\varepsilon)}(x) + b^{(\varepsilon)}(x) \right) \pi^{(\varepsilon)}(dx) . \tag{6.11}$$

Substituting the asymptotic representations (1.15) and using the limit relations $xh(x) \to \infty$ as $x \to \infty$ and "$\pi^{(\varepsilon)} \Rightarrow \infty$ as $\varepsilon \downarrow 0$", we obtain

$$(\varepsilon + o(\varepsilon))\ \mathbf{E}\, X^{(\varepsilon)} = \mathbf{E}\, X^{(\varepsilon)}\, h\left(X^{(\varepsilon)}\right), \qquad \varepsilon \downarrow 0 .$$

By assumption (1.18), the function $x\,h(x)$ is concave; therefore,

$$\liminf_{\varepsilon \downarrow 0} h\left(\mathbf{E}\, X^{(\varepsilon)}\right)/\varepsilon \geq 1 . \tag{6.12}$$

Relations (6.10) and (6.12) imply

$$h\left(\mathbf{E}\, X^{(\varepsilon)}\right) \sim \varepsilon \quad \text{as } \varepsilon \downarrow 0, \tag{6.13}$$

whence, by Lemma 16, the convergence of moments follows: $\mathbf{E}\, X^{(\varepsilon)}/q_\varepsilon \to 1$ as $\varepsilon \downarrow 0$. Further, in view of "$\pi^{(\varepsilon)} \Rightarrow \infty$", there exists a sequence $x(\varepsilon) \to \infty$ such that

$$\mathbf{P}\left\{ X^{(\varepsilon)} \geq x(\varepsilon) \right\} \to 1 \quad \text{as } \varepsilon \downarrow 0. \tag{6.14}$$

Setting $\gamma = 1$ in Lemma 3 and substituting the asymptotic representations (1.15), we obtain

$$0 \geq \int_{x(\varepsilon)}^\infty m^{(\varepsilon)}(x)\, \pi^{(\varepsilon)}(dx) \geq -\varepsilon + o(\varepsilon) + \mathbf{E}\left\{ h\left(X^{(\varepsilon)}\right);\ X^{(\varepsilon)} \geq x(\varepsilon) \right\} (1+ o(1)) .$$

Consequently,

$$\limsup_{\varepsilon \downarrow 0} \mathbf{E}\left\{h\left(X^{(\varepsilon)}\right);\ X^{(\varepsilon)} \geq x(\varepsilon)\right\}/\varepsilon \leq 1. \tag{6.15}$$

By (6.9), (6.14), and (6.15), we have

$$0 \leq \mathbf{E}\left\{\left(\sqrt{h\left(X^{(\varepsilon)}\right)/\varepsilon} - \sqrt{\varepsilon/h\left(X^{(\varepsilon)}\right)}\right)^2;\ X^{(\varepsilon)} \geq x(\varepsilon)\right\}$$
$$\leq \mathbf{E}\left\{h\left(X^{(\varepsilon)}\right)/\varepsilon;\ X^{(\varepsilon)} \geq x(\varepsilon)\right\} - 2\mathbf{P}\left\{X^{(\varepsilon)} \geq x(\varepsilon)\right\} + \mathbf{E}\left\{\varepsilon/h\left(X^{(\varepsilon)}\right)\right\} \to 0$$

as $\varepsilon \downarrow 0$. Thus,

$$\sqrt{h\left(X^{(\varepsilon)}\right)/\varepsilon} - \sqrt{\varepsilon/h\left(X^{(\varepsilon)}\right)} \Rightarrow 0 \quad \text{as } \varepsilon \downarrow 0.$$

Hence $h\left(X^{(\varepsilon)}\right)/\varepsilon \Rightarrow 1$, and, by the first assertion of Lemma 16, the following convergence holds: $X^{(\varepsilon)}/q_\varepsilon \Rightarrow 1$ as $\varepsilon \downarrow 0$. The proof of Lemma 19 is complete.

The following lemma will be useful in deriving some estimates.

Lemma 20. *Let conditions* (1.15)–(1.18) *be satisfied. Then the following estimate holds*: *as* $\varepsilon \downarrow 0$,

$$\mathbf{E}\left(h\left(X^{(\varepsilon)}\right)/(1+X^{(\varepsilon)})\right) = O(\varepsilon/q_\varepsilon).$$

Proof. Consider the increasing test function $\log(1+x) \geq 0$. It follows from Lemma 19 that $\mathbf{E}\log(1+X^{(\varepsilon)}) < \infty$; hence

$$0 = \int_0^\infty \mathbf{E}\left\{\log\left(1+x+\xi^{(\varepsilon)}(x)\right) - \log(1+x)\right\}\pi^{(\varepsilon)}(dx).$$

Thus, the test function $\log(1+x)$ satisfies the hypotheses of Lemma 1. Hence, for each $x_* \geq 0$,

$$0 \geq \int_{x_*}^\infty \mathbf{E}\left\{\log\left(1+x+\xi^{(\varepsilon)}(x)\right) - \log(1+x)\right\}\pi^{(\varepsilon)}(dx).$$

Since for any x, y, $x \geq 0$, $y \geq -x$,

$$\log(1+x+y) - \log(1+x) \geq y/(1+x) - y^2 x^{-2}\log(1+x),$$

we have

$$0 \geq \int_{x_*}^\infty \left(\frac{m^{(\varepsilon)}(x)}{1+x} - \frac{b^{(\varepsilon)}(x)}{2x^2}\right)\pi^{(\varepsilon)}(dx). \tag{6.16}$$

By the asymptotic representations (1.15) and the condition $x\,h(x)/\log x \to \infty$ as $x \to \infty$, there exist $x_* > 0$ and $\varepsilon_* > 0$ such that for $x \geq x_*$, $\varepsilon \leq \varepsilon_*$,

$$\frac{m^{(\varepsilon)}(x)}{1+x} - \frac{b^{(\varepsilon)}(x)}{2(1+x)^2} \geq -\frac{2\varepsilon}{1+x} + \frac{h(x)}{2(1+x)}.$$

Substituting this inequality into (6.16), we obtain

$$4\varepsilon\,\mathbf{E}\left\{\,1/(1+X^{(\varepsilon)})\,;\,X^{(\varepsilon)}\ge x_*\right\}$$
$$\ge\ \mathbf{E}\left\{\,h(X^{(\varepsilon)})/(1+X^{(\varepsilon)})\,;\,X^{(\varepsilon)}\ge x_*\right\}\equiv\ \psi(\varepsilon)\,. \tag{6.17}$$

Since $\inf\limits_{x\in K} h(x)/(1+x) > 0$ for each compact set $K\subset[x_*,\infty)$, we have $\pi^{(\varepsilon)}(K) = O(\psi(\varepsilon))$ as $\varepsilon\downarrow 0$, and, by Corollary 2 of Lemma 6, $\pi^{(\varepsilon)}([0,x_*]) = O(\psi(\varepsilon))$. It follows from the last relation and inequality (6.17) that there exists $c_1 > 0$ such that

$$\varepsilon\,\mathbf{E}\ 1/(1+X^{(\varepsilon)})\ \ge\ c_1\,\mathbf{E}\ h(X^{(\varepsilon)})/(1+X^{(\varepsilon)})\,. \tag{6.18}$$

Set

$$V(x)\ \equiv\ (1+\lambda)\int\limits_x^\infty \frac{h(y)}{(1+y)^2}\,dy\,.$$

Since the function $h(y)/(1+y)^2$ is regularly varying at infinity with index $-\lambda-2$, the function $V(x)$ is regularly varying too (this is a well-known fact; see, e.g., [18, Chapter VIII, §9]); moreover,

$$V(x)\ \sim\ h(x)/x\quad\text{ as }x\to\infty\,. \tag{6.19}$$

Consequently, there exists c_2 such that

$$h(x)/(1+x)\ \ge\ c_2\,V(x)\,. \tag{6.20}$$

By construction, the derivative

$$d\,V(1/z-1)/d\,z\ =\ (1+\lambda)\,h(1/z-1)\,,\qquad z<1\,,$$

increases by virtue of (1.19), and hence the function $V(1/z-1)$ is convex. It now follows from inequalities (6.18)–(6.20) that

$$\varepsilon\,\mathbf{E}\,1/(1+X^{(\varepsilon)})\ \ge\ c_1 c_2\,\mathbf{E}\,V\Big(\,1/\big(1/(1+X^{(\varepsilon)})\big)-1\Big)$$
$$\ge\ c_1 c_2\,V\Big(\,1/\mathbf{E}\left\{1/(1+X^{(\varepsilon)})\right\}-1\Big)\ \sim$$
$$\sim\ c_3\,\mathbf{E}\left\{1/(1+X^{(\varepsilon)})\right\}h\Big(\,1/\mathbf{E}\left\{1/(1+X^{(\varepsilon)})\right\}\Big)\,.$$

Therefore, $h\Big(\,1/\mathbf{E}\left\{1/(1+X^{(\varepsilon)})\right\}\Big) = O(\varepsilon)$ as $\varepsilon\downarrow 0$, and hence, by the second assertion of Lemma 16, $\mathbf{E}\,1/(1+X^{(\varepsilon)}) = O(1/q_\varepsilon)$. Substituting the last relation into (6.18), we arrive at the conclusion of Lemma 20.

Lemma 21. *Let a function* $\Delta(x,\varepsilon)$ *satisfy the relations*

$$\sup_{x,\,\varepsilon}\ |\,\Delta(x,\varepsilon)\,|\ <\ \infty\,,\qquad \Delta(x,\varepsilon) = o\,(\,\sqrt{\varepsilon/q_\varepsilon}+\sqrt{h(x)/x}\,)$$

as $x\to\infty,\varepsilon\downarrow 0$. *Then the following assertions hold: as* $\varepsilon\downarrow 0$,

(a) $\mathbf{E}\,\Delta(X^{(\varepsilon)},\varepsilon) = o\,(\sqrt{\varepsilon/q_\varepsilon})$,

(b) $\mathbf{E}\,X^{(\varepsilon)}\,\Delta(X^{(\varepsilon)},\varepsilon) = o\,(\sqrt{\varepsilon\,q_\varepsilon})$.

Proof. By hypothesis, there exists a function $\psi(x, \varepsilon)$ such that

$$\psi(x, \varepsilon) \to 0 \tag{6.21}$$

as $x \to \infty, \varepsilon \downarrow 0$, and, for $x \geq N$,

$$|\Delta(x, \varepsilon)| \leq \psi(N, \varepsilon)\left(\sqrt{\varepsilon / q_\varepsilon} + \sqrt{h(x)/x}\right).$$

Consequently, for every $N > 0$,

$$\mathbf{E}|\Delta(X^{(\varepsilon)}, \varepsilon)| \leq \sup_{x, \varepsilon} |\Delta(x, \varepsilon)| \, \mathbf{P}\{X^{(\varepsilon)} < N\} + \psi(N, \varepsilon)\left(\sqrt{\varepsilon / q_\varepsilon} + \mathbf{E}\sqrt{h(X^{(\varepsilon)})/X^{(\varepsilon)}}\right).$$

It follows from Lemma 20 and Corollary 2 of Lemma 6 that $\mathbf{P}\{X^{(\varepsilon)} < N\} = O(\varepsilon / q_\varepsilon)$ for each $N > 0$. Using Lemma 20, the fact that the function $\sqrt{x}$ is concave, and relation (6.21), we arrive at the assertion (a) of the lemma. Relation (b) follows from the inequality

$$\mathbf{E}\,|X^{(\varepsilon)}\Delta(X^{(\varepsilon)}, \varepsilon)|$$

$$\leq \sup_{x, \varepsilon} |\Delta(x, \varepsilon)| \, N\mathbf{P}\{X^{(\varepsilon)} < N\} + \psi(N, \varepsilon)\left(\mathbf{E}\, X^{(\varepsilon)}\sqrt{\varepsilon / q_\varepsilon} + \mathbf{E}\sqrt{h(X^{(\varepsilon)})\, X^{(\varepsilon)}}\right),$$

Lemma 19, the concavity of the function $\sqrt{h(x)\, x}$, and relation (6.21). The proof of Lemma 21 is complete.

Lemma 22. *Let the asymptotic representations* (1.23) *be valid. Then, as* $\varepsilon \downarrow 0$, *the asymptotic expansion* (1.24) *holds.*

Proof. According to Lemma 3 (with $\gamma = 1$), the equality

$$0 = \int_0^\infty m^{(\varepsilon)}(x)\, \pi^{(\varepsilon)}(dx) \tag{6.22}$$

holds. Substituting the asymptotic representations (1.23) and applying relation (a) of Lemma 21, we obtain

$$\mathbf{E}\, h(X^{(\varepsilon)}) = \varepsilon + o(\sqrt{\varepsilon / q_\varepsilon}) \quad \text{as } \varepsilon \downarrow 0. \tag{6.23}$$

By convexity of the function $h(x)$,

$$h(\mathbf{E}\, X^{(\varepsilon)}) \leq \varepsilon + o(\sqrt{\varepsilon / q_\varepsilon}). \tag{6.24}$$

Substituting the asymptotic representations (1.23) into (6.11) and applying relation (b) of Lemma 21, we obtain: as $\varepsilon \downarrow 0$,

$$0 = -2\varepsilon\, \mathbf{E}\, X^{(\varepsilon)} + 2\, \mathbf{E}\, X^{(\varepsilon)}\, h(X^{(\varepsilon)}) + o(\sqrt{\varepsilon\, q_\varepsilon}). \tag{6.25}$$

Therefore, by assumption (1.18) that the function $x\, h(x)$ is concave and by the asymptotic relation $\mathbf{E}\, X^{(\varepsilon)} \sim q_\varepsilon$, we have

$$h(\mathbf{E}\, X^{(\varepsilon)}) \geq \varepsilon + o(\sqrt{\varepsilon / q_\varepsilon}) \quad \text{as } \varepsilon \downarrow 0. \tag{6.26}$$

Inequalities (6.24) and (6.26) imply the asymptotic expansion: as $\varepsilon \downarrow 0$,

$$h(\mathbf{E}\, X^{(\varepsilon)}) = \varepsilon + o(\sqrt{\varepsilon / q_\varepsilon}). \tag{6.27}$$

Since the function $h(x)$ is convex, Lemma 17 implies that for every x, y, $x \le y$, the relation $h(x) - h(y) \ge (x - y)\, d_h(y) \sim \lambda (y - x)\, h(y)/y$, $y \to \infty$, holds. Set $x \equiv \min(q_\varepsilon, \mathbf{E} X^{(\varepsilon)})$, $y \equiv \max(q_\varepsilon, \mathbf{E} X^{(\varepsilon)})$. Then

$$x \sim y \sim q_\varepsilon, \qquad h(x) \sim h(y) \sim \varepsilon,$$

$$0 \le h(x) - h(y) = |h(\mathbf{E} X^{(\varepsilon)}) - \varepsilon|, \qquad 0 \le y - x = |\mathbf{E} X^{(\varepsilon)} - q_\varepsilon|.$$

Therefore, $|\mathbf{E} X^{(\varepsilon)} - q_\varepsilon| = O\left(|h(\mathbf{E} X^{(\varepsilon)}) - \varepsilon| \, q_\varepsilon/\varepsilon \right)$ as $\varepsilon \downarrow 0$. Applying relation (6.27) completes the proof of Lemma 22.

Let $Y^{(\varepsilon)} \equiv X^{(\varepsilon)} - \mathbf{E} X^{(\varepsilon)}$. Define a random variable $r(Y^{(\varepsilon)})$ by the equality

$$h(X^{(\varepsilon)}) \equiv h(\mathbf{E} X^{(\varepsilon)} + Y^{(\varepsilon)}) = h(\mathbf{E} X^{(\varepsilon)}) + d_h(\mathbf{E} X^{(\varepsilon)}) Y^{(\varepsilon)} + r(Y^{(\varepsilon)}). \tag{6.28}$$

Since the function $h(x)$ is convex, the random variables $r(Y^{(\varepsilon)})$ are nonnegative with probability one. Therefore, by equalities (6.23) and (6.27),

$$\mathbf{E} \, | r(Y^{(\varepsilon)}) | = \mathbf{E} r(Y^{(\varepsilon)}) = \mathbf{E} h(X^{(\varepsilon)}) - h(\mathbf{E} X^{(\varepsilon)}) = o(\sqrt{\varepsilon/q_\varepsilon}) \quad \text{as } \varepsilon \downarrow 0. \tag{6.29}$$

Proof of Theorem 5. We shall prove, using the method of characteristic functions, that the distributions of the random variables $(X^{(\varepsilon)} - \mathbf{E} X^{(\varepsilon)}) \sqrt{\varepsilon/q_\varepsilon} \equiv Y^{(\varepsilon)} \sqrt{\varepsilon/q_\varepsilon}$ converge weakly to the normal distribution $N(0, b/2\lambda)$ as $\varepsilon \downarrow 0$. The starting point is equation (4.22). Substituting the asymptotic representations (1.23) both into (4.22) and relation (a) of Lemma 22, we arrive at the equality

$$0 = \int_0^\infty e^{itx} \left(i m^{(\varepsilon)}(x) - (b + \Delta(x, \varepsilon, t))\, t/2 \right) \pi^{(\varepsilon)}(dx)$$

$$= \mathbf{E} \left(i h(X^{(\varepsilon)}) - \varepsilon i + b t/2 \right) \exp\left\{ i t X^{(\varepsilon)} \right\} + o(\sqrt{\varepsilon/q_\varepsilon} + t), \qquad \varepsilon \downarrow 0, \ t \to 0,$$

where $\Delta(x, \varepsilon, t) \to 0$ as $x \to \infty$, $\varepsilon \downarrow 0$, and $t \to 0$. Multiplying this equality by $\exp\{-it \mathbf{E} X^{(\varepsilon)}\}$, we obtain

$$0 = \mathbf{E} \left(i (h(X^{(\varepsilon)}) - \varepsilon) - b t/2 \right) \exp\left\{ it Y^{(\varepsilon)} \right\} + o(\sqrt{\varepsilon/q_\varepsilon} + t)$$

as $\varepsilon \downarrow 0$, $t \to 0$. By (6.28), (6.27), Lemma 17, and the asymptotic relations (6.13) and (1.21),

$$h(X^{(\varepsilon)}) - \varepsilon = d_h(\mathbf{E} X^{(\varepsilon)})\, Y^{(\varepsilon)} + r(Y^{(\varepsilon)}) + o(\sqrt{\varepsilon/q_\varepsilon})$$

$$= (-\lambda + o(1))\, h(\mathbf{E} X^{(\varepsilon)})\, Y^{(\varepsilon)}/\mathbf{E} X^{(\varepsilon)} + r(Y^{(\varepsilon)}) + o(\sqrt{\varepsilon/q_\varepsilon})$$

$$= (-\lambda + o(1))\, Y^{(\varepsilon)} \varepsilon/q_\varepsilon + r(Y^{(\varepsilon)}) + o(\sqrt{\varepsilon/q_\varepsilon})$$

as $\varepsilon \downarrow 0$. Therefore, the following equality holds:

$$0 = (\lambda + o(1))(\varepsilon/q_\varepsilon) \mathbf{E}\, i Y^{(\varepsilon)} \exp\left\{ i t Y^{(\varepsilon)} \right\} + \frac{bt}{2} \mathbf{E} \exp\left\{ i t Y^{(\varepsilon)} \right\}$$

$$- \mathbf{E} r(Y^{(\varepsilon)}) \exp\left\{ i t Y^{(\varepsilon)} \right\} + o(\sqrt{\varepsilon/q_\varepsilon} + t).$$

Using estimate (6.29), setting $t = s\sqrt{\varepsilon / q_\varepsilon}$, and multiplying the last equality by $\sqrt{q_\varepsilon / \varepsilon}$, we conclude that, as $\varepsilon \downarrow 0$,

$$0 = (\lambda + o(1))\, \mathbf{E}\, i\sqrt{\varepsilon/q_\varepsilon}\, Y^{(\varepsilon)} \exp\left\{ i\, s\sqrt{\varepsilon/q_\varepsilon}\, Y^{(\varepsilon)} \right\} + \frac{b\, s}{2}\, \mathbf{E} \exp\left\{ i\, s\sqrt{\varepsilon/q_\varepsilon}\, Y^{(\varepsilon)} \right\} + o(1)$$

uniformly in s in each compact set. Denote by $\varphi^{(\varepsilon)}(s)$ the characteristic function of the random variable $\sqrt{\varepsilon / q_\varepsilon}\, Y^{(\varepsilon)}$, i.e.,

$$\varphi^{(\varepsilon)}(s) \equiv \mathbf{E} \exp\left\{ i\, s\sqrt{\varepsilon / q_\varepsilon}\, Y^{(\varepsilon)} \right\}.$$

Notice that $\varphi^{(\varepsilon)}(s)$ satisfies the equation

$$\frac{d\, \varphi^{(\varepsilon)}(s)}{d\, s} = -\frac{b\, s}{2\, \lambda}\, \varphi^{(\varepsilon)}(s) + \Delta(s, \varepsilon)$$

and the initial condition $\varphi^{(\varepsilon)}(0) = 1$, where $\Delta(s, \varepsilon) \to 0$ as $\varepsilon \downarrow 0$, uniformly in s in an arbitrary compact set. This implies that the function $\varphi^{(\varepsilon)}(s)$ has the form

$$\varphi^{(\varepsilon)}(s) = \left(1 + \int_0^s \Delta(t, \varepsilon)\; e^{\, bt^2/4\lambda}\, dt \right) e^{-bs^2/4\lambda}.$$

Hence

$$\lim_{\varepsilon \downarrow 0} \varphi^{(\varepsilon)}(s) = e^{-b\, s^2/4\, \lambda}$$

for each $s \in \mathbf{R}$. The proof of Theorem 5 is complete.

§7. CLASSIFICATION OF TRANSITION PHENOMENA FOR MARKOV CHAINS TAKING VALUES IN THE WHOLE REAL AXIS

A rather detailed description of transition phenomena for $\mathbf{R}^+$-valued Markov chains allows us to consider Markov chains taking both positive and negative values. Theorems 7–9 give a brief classification of transition phenomena for Markov chains taking values in $\mathbf{R}$. We assume

$$m^{(\varepsilon)}(x) = -\psi_+(\varepsilon) + \mu_+ / x + o\left(\psi_+(\varepsilon) + 1/x\right), \quad -\infty \le \mu_+ \le \infty,$$

$$b^{(\varepsilon)}(x) = b_+ + o(1), \quad 0 < b_+ < \infty, \tag{7.1}$$

as $x \to \infty, \varepsilon \downarrow 0$, where $\psi_+(\varepsilon) \downarrow 0$, and

$$m^{(\varepsilon)}(x) = \psi_-(\varepsilon) + \mu_- / x + o\left(\psi_-(\varepsilon) + 1/x\right), \quad -\infty \le \mu_- \le \infty,$$

$$b^{(\varepsilon)}(x) = b_- + o(1), \quad 0 < b_- < \infty, \tag{7.2}$$

as $x \to -\infty, \varepsilon \downarrow 0$, where $\psi_-(\varepsilon) \downarrow 0$. Let

$$0 \le c_{\inf} \equiv \liminf_{\varepsilon \downarrow 0} \psi_+(\varepsilon) / \psi_-(\varepsilon)$$

$$\leq \limsup_{\varepsilon \downarrow 0} \psi_+(\varepsilon)/\psi_-(\varepsilon) \equiv c_{\sup} \leq \infty .$$

Since the Markov chains now take values in $\mathbf{R}$, condition (1.9) should be supplemented as follows: for each $x_1, x_2 \in \mathbf{R}$, there exist $\varepsilon_* = \varepsilon_*(x_1, x_2) > 0$ and $\delta > 0$ such that for $\varepsilon \leq \varepsilon_*$,

$$\begin{aligned} &\inf_{x_1 \leq x \leq x_2} \mathbf{P}\left\{ X_n^{(\varepsilon)} \geq x_2 \text{ for some } n \mid X_0^{(\varepsilon)} = x \right\} \geq \delta > 0 , \\ &\inf_{x_1 \leq x \leq x_2} \mathbf{P}\left\{ X_n^{(\varepsilon)} \leq x_1 \text{ for some } n \mid X_0^{(\varepsilon)} = x \right\} \geq \delta > 0 . \end{aligned} \tag{7.3}$$

Without loss of generality, we may assume that

$$2\mu_-/b_- \leq 2\mu_+/b_+ . \tag{7.4}$$

Theorem 7. *Suppose that the asymptotic representations* (7.1), (7.2) *and conditions* (1.11), (7.4) *hold. Assume that* $2\mu_+ + b_+ < 0$, *the Markov chain* $\{X_n\}$ *has a unique invariant distribution* π *and the transition probability satisfies condition* (1.3). *Then the invariant distribution* $\pi^{(\varepsilon)}$ *of the Markov chain* $\{X_n^{(\varepsilon)}\}$ *converges weakly to* π *as* $\varepsilon \downarrow 0$.

Theorem 8. *Suppose that the asymptotic representations* (7.1), (7.2) *and conditions* (1.11), (7.3) *hold. If* $2\mu_- + b_- < 0$ *and* $2\mu_+ + b_+ > 0$, *then, as* $\varepsilon \downarrow 0$,

$$L(2\psi_+(\varepsilon) X^{(\varepsilon)}) \Rightarrow \Gamma_{1/b_+,\ 1 + 2\mu_+/b_+} . \tag{7.5}$$

Theorem 9. *Suppose that the asymptotic representations* (7.1), (7.2) *and conditions* (1.11), (7.3) *and* (7.4) *hold. Assume that the random variables* $\xi^{(\varepsilon)}(x)$ *are bounded uniformly in* x *and* ε:

$$\mathbf{P}\left\{ \left| \xi^{(\varepsilon)}(x) \right| \leq L \right\} = 1 , \tag{7.6}$$

where $L < \infty$. *If* $2\mu_- + b_- > 0$, *then the following conditional distributions converge*:

$$\mathbf{P}\left\{ 2\psi_+(\varepsilon) X^{(\varepsilon)} < x \mid X^{(\varepsilon)} \geq 0 \right\} \to \mathbf{P}\left\{ \eta_+ < x \right\}, \ \varepsilon \downarrow 0, \ 0 \leq x < \infty , \tag{7.7}$$

$$\mathbf{P}\left\{ 2\psi_-(\varepsilon) X^{(\varepsilon)} > x \mid X^{(\varepsilon)} \leq 0 \right\} \to \mathbf{P}\left\{ \eta_- < -x \right\}, \ \varepsilon \downarrow 0, \ -\infty < x \leq 0 , \tag{7.8}$$

where the random variables $\eta_\pm$ *have the gamma distribution with parameters* $1/b_\pm$ *and* $1 + 2\mu_\pm/b_\pm$. *Moreover*,

(a) *if* $2\mu_-/b_- < 2\mu_+/b_+$ *and* $c_{\sup} < \infty$, *then for each* $\delta > 0$ *the following estimate holds*: *as* $\varepsilon \downarrow 0$,

$$\mathbf{P}\left\{ X^{(\varepsilon)} \leq 0 \right\} = O\left(\psi_-^{2\mu_+/b_+ - 2\mu_-/b_- - \delta}(\varepsilon)\right) ; \tag{7.9}$$

(b) *if* $2\mu_-/b_- = 2\mu_+/b_+$ *and* $0 < c_{\inf} \leq c_{\sup} < \infty$, *then the set of limit points of the numerical sequence* $\mathbf{P}\{X^{(\varepsilon)} \leq 0\}$ *may coincide with the interval* $[0, 1]$.

Remark 5. In view of assertion (b), it is clear that if $2\mu_-/b_- = 2\mu_+/b_+$, then there can be no limit theorems for the distribution of the random variables $X^{(\varepsilon)}$.

R e m a r k 6. We restrict our considerations to the questions of stability and convergence to a Γ-distribution, since there are too many other combinations of limiting behavior of the invariant distribution on both the negative and positive semiaxes. By the same reason, we do not consider the cases $c_{\inf} = 0$ and $c_{\sup} = \infty$ (e.g., one could examine the situations when $\psi_{\pm}(\varepsilon) = \varepsilon^{\gamma_{\pm}}$, with $\gamma_{\pm} > 0$, $\gamma_{+} \neq \gamma_{-}$).

Proof of Theorem 7. This essentially follows from the next lemma, which is an analog of Lemma 2 for Markov chains taking values in $\mathbf{R}$.

Lemma 23. *If* $2\mu_{+} + b_{+} < 0$, *then the family of random variables* $\{\max(0, X^{(\varepsilon)})\}_{\varepsilon \geq 0}$ *is bounded in probability.*

Proof. Define a new, truncated Markov chain $X_{n*}^{(\varepsilon)}$, whose transition probabilities have the form

$$P_{*}^{(\varepsilon)}(x, B) = P^{(\varepsilon)}(x, B) \quad \text{if } x > 0 \text{ and } B \subset (0, \infty),$$

$$P_{*}^{(\varepsilon)}(x, \{0\}) = P^{(\varepsilon)}(x, (-\infty, 0]) \quad \text{if } x > 0, \tag{7.10}$$

$$P_{*}^{(\varepsilon)}(0, B) = \int_{-\infty}^{0} P^{(\varepsilon)}(y, B)\, \pi^{(\varepsilon)}(dy)/\pi^{(\varepsilon)}((-\infty, 0]) \quad \text{if } B \subset (0, \infty),$$

$$P_{*}^{(\varepsilon)}(0, \{0\}) = \int_{-\infty}^{0} P^{(\varepsilon)}(y, (-\infty, 0])\, \pi^{(\varepsilon)}(dy)/\pi^{(\varepsilon)}((-\infty, 0]).$$

If the Markov chain $\{X_n^{(\varepsilon)}\}$ is in equilibrium (i.e., $X_n^{(\varepsilon)}$ has the distribution $\pi^{(\varepsilon)}$), then $X_{n*}^{(\varepsilon)}$ and $X_n^{(\varepsilon)} \mathbf{I}\{X_n^{(\varepsilon)} \geq 0\}$ have the same distribution. The process $X_{n*}^{(\varepsilon)}$ is a homogeneous Markov chain with values in $\mathbf{R}^{+}$ and invariant measure $\pi_{*}^{(\varepsilon)}$, where $\pi_{*}^{(\varepsilon)}(B) = \pi^{(\varepsilon)}(B)$ for $B \in \mathcal{B}((0, \infty))$ and $\pi_{*}^{(\varepsilon)}(\{0\}) = \pi^{(\varepsilon)}((-\infty, 0])$. Condition (1.11) (the uniform integrability of the jumps squared) ensures that the following relations hold uniformly in $\varepsilon \geq 0$:

$$m_{*}^{(\varepsilon)}(x) = m^{(\varepsilon)}(x) - \mathbf{E}\{x + \xi^{(\varepsilon)}(x);\ \xi^{(\varepsilon)}(x) \leq -x\} = m^{(\varepsilon)}(x) + o(1/x), \tag{7.11}$$

$$b_{*}^{(\varepsilon)}(x) = b^{(\varepsilon)}(x) - \mathbf{E}\{(\xi^{(\varepsilon)}(x))^2 - x^2;\ \xi^{(\varepsilon)}(x) \leq -x\} = b^{(\varepsilon)}(x) + o(1) \tag{7.12}$$

as $x \to \infty$. Moreover, in view of condition (1.11), the family of random variables $(\xi_{*}^{(\varepsilon)}(x))^2$ is uniformly integrable:

$$\sup_{x \geq 0,\, \varepsilon \geq 0} \mathbf{E}\{(\xi_{*}^{(\varepsilon)}(x))^2;\ |\xi_{*}^{(\varepsilon)}(x)| > N\} \to 0 \quad \text{as } N \to \infty. \tag{7.13}$$

It follows from (7.11) and (7.12) that

$$\limsup_{x \to \infty,\, \varepsilon \downarrow 0} (2x\, m_{*}^{(\varepsilon)}(x) + b_{*}^{(\varepsilon)}(x))$$

$$= \limsup_{x \to \infty,\, \varepsilon \downarrow 0} (2x\, m^{(\varepsilon)}(x) + b^{(\varepsilon)}(x)) = 2\mu_{+} + b_{+} < 0. \tag{7.14}$$

Applying Lemma 22 to the Markov chains $\{X^{(\varepsilon)}_{n*}\}$, we arrive at the assertion of Lemma 23.

Lemma 23 and the condition $2\mu_-/b_- \le 2\mu_+/b_+ < -1$ imply that the families of random variables $\{\max(0, X^{(\varepsilon)})\}_{\varepsilon \ge 0}$ and $\{\max(0, -X^{(\varepsilon)})\}_{\varepsilon \ge 0}$ are stochastically bounded. One can show, along the lines of the proof of Theorem 6, that the stochastic boundedness of the family $\{|X^{(\varepsilon)}|\}_{\varepsilon \ge 0}$ implies stability of the invariant distribution as $\varepsilon \downarrow 0$. The proof of Theorem 7 is complete.

Proof of Theorem 8. By condition (7.3) and Corollary 1 of Lemma 6, the following statement is true: if for some compact set $K \subset \mathbf{R}$,

$$\limsup_{\varepsilon \downarrow 0} \pi^{(\varepsilon)}(K) > 0, \tag{7.15}$$

then for each $x_* \in \mathbf{R}^+$, there exists $\bar{x} \in \mathbf{R}^+$ such that

$$\limsup_{\varepsilon \downarrow 0} \pi^{(\varepsilon)}([x_*, \bar{x}]) > 0. \tag{7.16}$$

We now turn to the Markov chains $\{X^{(\varepsilon)}_{n*}\}$ truncated at the level 0 (see (7.10)). By (7.11)–(7.13), they satisfy all conditions of Theorem 2 possibly except for condition (1.9). However, while proving Theorem 22, we used this condition only to prove the relation

$$\text{“}\pi^{(\varepsilon)} \Rightarrow \infty \text{ as } \varepsilon \downarrow 0\text{”}. \tag{7.17}$$

Notice that for each number $A > 0$, the following equality holds:

$$\mathbf{P}\{X^{(\varepsilon)} \le A\} = \mathbf{P}\{X^{(\varepsilon)} < -A\} + \mathbf{P}\{-A \le X^{(\varepsilon)} \le A\}, \tag{7.18}$$

where the first summand in the right-hand side tends to zero as $A \to \infty$ uniformly in ε (this follows from Lemma 23 and the condition $2\mu_- + b_- < 0$). Therefore, to prove (7.17), it suffices to show that for each compact set $K \subset \mathbf{R}$ the following relation holds: as $\varepsilon \downarrow 0$,

$$\pi^{(\varepsilon)}(K) \to 0. \tag{7.19}$$

Assume the contrary, i.e., relation (7.15) is true for some compact set $K \subset \mathbf{R}$, and hence so is relation (7.16). Now following the argument of the proof of Lemma 8 applied to the Markov chains $\{X^{(\varepsilon)}_{n*}\}$ truncated at the level 0, we obtain, as a result, relation (7.19) and hence relation (7.17). Thus, applying Theorem 2 to the Markov chains $\{X^{(\varepsilon)}_{n*}\}$ is justified, and we have

$$\mathcal{L}(2\psi_+(\varepsilon) X^{(\varepsilon)}_{n*}) \equiv \mathcal{L}(2\psi_+(\varepsilon) \max(0, X^{(\varepsilon)})) \Rightarrow \Gamma_{1/b_+,\, 1 + 2\mu_+/b_+}$$

as $\varepsilon \downarrow 0$. The proof of Theorem 8 is complete.

Proof of Theorem 9. The proof differs from the preceding argument in that we cannot refer to Theorem 2, since at least one of the relations “$\pi^{(\varepsilon)} \Rightarrow \infty$ as $\varepsilon \downarrow 0$” or “$\pi^{(\varepsilon)} \Rightarrow -\infty$ as $\varepsilon \downarrow 0$” fails. In this connection, we shall need the following refined version of Lemma 7 for Markov chains in $\mathbf{R}$.

Lemma 24. *Suppose that* $2\mu_+ + b_+ > 0$ *and condition* (7.6) *is satisfied. Then for each compact set* $K \subset \mathbf{R}$, *the following estimate holds*:

$$\pi^{(\varepsilon)}(K) = o\left(\mathbf{P}\{X^{(\varepsilon)} \ge 0\}\right) \quad \text{as } \varepsilon \downarrow 0.$$

Proof. Assume the contrary, i.e., assume

$$\limsup_{\varepsilon \downarrow 0} \pi^{(\varepsilon)}(K) \,/\mathbf{P}\{ X^{(\varepsilon)} \geq 0 \} > 0$$

for some compact set K. Setting $\psi(\varepsilon) \equiv \mathbf{P}\{X^{(\varepsilon)} \geq 0\}$ in Lemma 6, we obtain that for each x_* there exists $\bar{x}$ such that

$$\limsup_{\varepsilon \downarrow 0} \pi^{(\varepsilon)}([x_*, \bar{x}]) \,/\mathbf{P}\left\{ X^{(\varepsilon)} \geq 0 \right\} > 0 . \tag{7.20}$$

Now applying Lemma 3 (with $\gamma = 2$) to the Markov chains $\{X^{(\varepsilon)}_{n*}\}$, we deduce the equality

$$0 = \int_{+0}^{\infty} \left(2\, x\, m^{(\varepsilon)}_*(x) + b^{(\varepsilon)}_*(x)\right) \pi^{(\varepsilon)}(dx) + b^{(\varepsilon)}_*(0)\, \pi^{(\varepsilon)}_*(\{0\}) . \tag{7.21}$$

By definition (7.10) and condition (7.6), we have

$$\begin{aligned} b^{(\varepsilon)}_*(0)\, \pi^{(\varepsilon)}_*(\{0\}) &\equiv \mathbf{E}\left(\xi^{(\varepsilon)}_*(0)\right)^2 \pi^{(\varepsilon)}_*(\{0\}) \\ &\equiv \int_{-L}^{0} \mathbf{E}\left\{\left(\xi^{(\varepsilon)}(y) + y\right)^2; \xi^{(\varepsilon)}(x) + y > 0\right\} \pi^{(\varepsilon)}(d\, y) \\ &\leq \sup_{x,\, \varepsilon} b^{(\varepsilon)}(x)\, \pi^{(\varepsilon)}([-L, 0]) . \end{aligned} \tag{7.22}$$

By (7.11) and (7.12), there exist $c_1 < \infty$ and $\varepsilon_0 > 0$ such that for $x > 0$ and $\varepsilon \leq \varepsilon_0$

$$2\, x\, m^{(\varepsilon)}_*(x) + b^{(\varepsilon)}_*(x) \leq -\psi_+(\varepsilon)\, x + c_1 . \tag{7.23}$$

Substituting (7.22) and (7.23) into (7.21), we conclude that for $\varepsilon \leq \varepsilon_0$

$$\begin{aligned} \psi_+(\varepsilon)\, \mathbf{E}\left\{ X^{(\varepsilon)};\, X^{(\varepsilon)} > 0 \right\} &\leq c_2\, \pi^{(\varepsilon)}([-L, 0]) + c_1\, \mathbf{P}\left\{ X^{(\varepsilon)} > 0 \right\} \\ &\leq \max(c_1, c_2)\, \mathbf{P}\left\{ X^{(\varepsilon)} \geq -L \right\} \leq c_3\, \mathbf{P}\left\{ X^{(\varepsilon)} \geq 0 \right\}, \quad c_3 < \infty . \end{aligned} \tag{7.24}$$

Now following the proof of Lemma 7 and using here assumption (7.6), Lemma 5, and inequality (7.24), we complete the proof of Lemma 24.

The proof of the conditional convergence (7.7) will be carried out with the help of the method of moments. Namely, we shall show that for every natural $m \geq 1$ the moments converge as $\varepsilon \downarrow 0$:

$$\begin{aligned} &\mathbf{E}\left\{\left(2\, \psi_+(\varepsilon)\, X^{(\varepsilon)}\right)^m; X^{(\varepsilon)} \geq 0\right\} / \mathbf{P}^m\left\{ X^{(\varepsilon)} \geq 0 \right\} \\ &\qquad \to \Gamma(\lambda + m) / \alpha^m\, \Gamma(\lambda) = \mathbf{E}\, \xi^m , \end{aligned} \tag{7.25}$$

where the random variable ξ has the $\Gamma_{\alpha, \lambda}$-distribution with parameters $\alpha = = 1/b_+$ and $\lambda = 1 + 2\, \mu_+ / b_+$.

Using Lemma 3, one can verify that under assumption (7.6),

$$0 = \int_{0}^{\infty} \mathbf{E}\left\{\left(x + \xi^{(\varepsilon)}_*(x)\right)^k - x^k\right\} \pi^{(\varepsilon)}_*(dx) \tag{7.26}$$

for every natural $k \geq 2$ and sufficiently small $\varepsilon > 0$. By the boundedness of the jumps (see (7.6)), we have

$$\mathbf{E}\left\{x + \xi_*^{(\varepsilon)}(x)\right\}^k - x^k$$
$$= k\, m_*^{(\varepsilon)}(x)\, x^{k-1} + k\,(k-1)\, b_*^{(\varepsilon)}(x)\, x^{k-2}/2 + O\left(x^{k-3}\right).$$

uniformly in ε as $x \to \infty$. Now using relations (7.11), (7.12), and the asymptotic representations (7.1), we arrive at the equalities

$$\mathbf{E}\left\{x + \xi_*^{(\varepsilon)}(x)\right\}^k - x^k$$
$$= k\, x^{k-2}\left(x\, m^{(\varepsilon)}(x) + (k-1)\, b^{(\varepsilon)}(x)/2 + o(1)\right)$$
$$= k\, x^{k-2}\left(-x\,\psi_+(\varepsilon)\,(1 + o(1)) + \mu_+ + (k-1)\, b_+/2 + o(1)\right)$$

as $x \to \infty, \varepsilon \downarrow 0$. Combining this equality and (7.26), we conclude by Lemma 24 that, as $\varepsilon \downarrow 0$,

$$2\,\psi_+(\varepsilon)\,(1 + o(1))\,\mathbf{E}\left\{\left(X^{(\varepsilon)}\right)^{k-1};\ X^{(\varepsilon)} \geq 0\right\}$$
$$= \left(2\mu_+ + (k-1)\, b_+ + o(1)\right)\mathbf{E}\left\{\left(X^{(\varepsilon)}\right)^{k-2};\ X^{(\varepsilon)} \geq 0\right\} + o\left(\mathbf{P}\left\{X^{(\varepsilon)} \geq 0\right\}\right).$$

Multiply the last equality by $(2\,\psi_+(\varepsilon))^{k-2}$. Then

$$\mathbf{E}\left\{\left(2\,\psi_+(\varepsilon)\, X^{(\varepsilon)}\right)^{k-1};\ X^{(\varepsilon)} \geq 0\right\}$$
$$= \left(2\mu_+ + (k-1)\, b_+ + o(1)\right)\mathbf{E}\left\{\left(2\,\psi_+(\varepsilon)\, X^{(\varepsilon)}\right)^{k-2};\ X^{(\varepsilon)} \geq 0\right\}$$
$$+ o\left(\psi_+^{k-2}(\varepsilon)\,\mathbf{P}\left\{X^{(\varepsilon)} \geq 0\right\}\right) \quad \text{as } \varepsilon \downarrow 0.$$

In particular, for $k = 2$,

$$\mathbf{E}\left\{2\,\psi_+(\varepsilon)\, X^{(\varepsilon)};\ X^{(\varepsilon)} \geq 0\right\}$$
$$= \left(2\mu_+ + b_+ + o(1)\right)\mathbf{P}\left\{X^{(\varepsilon)} \geq 0\right\} \quad \text{as } \varepsilon \downarrow 0,$$

which is equivalent to relation (7.25) with $m = 1$. Proceeding by induction, we conclude the proof of (7.25) for every $m \geq 1$. The proof of the conditional convergence (7.7) is complete.

We turn to the proof of estimate (7.9). By Taylor's formula, condition (7.6), and relations (7.11)–(7.13), we conclude that for each $\gamma \in \mathbf{R}$, as $x \to \infty$,

$$v_\gamma^{(\varepsilon)}(x) \equiv \mathbf{E}\left\{\left(x + \xi_*^{(\varepsilon)}(x)\right)^\gamma - x^\gamma\right\}$$
$$\equiv \gamma\left(x^{\gamma-1} m^{(\varepsilon)}(x) + (\gamma-1)\, x^{\gamma-2} b^{(\varepsilon)}(x)/2 + o\left(x^{\gamma-2}\right)\right) \tag{7.27}$$

uniformly in ε. The asymptotic representations (7.1) imply that for each γ there exist numbers $x_* = x_*(\gamma) \geq 1 + L$ and $\varepsilon_* = \varepsilon_*(\gamma) > 0$ such that for $x \geq x_*$ and $0 < \varepsilon \leq \varepsilon_*$, the following assertions hold:

(a) if $\gamma > 0$ and $\mu_+ + (\gamma - 1)\, b_+/2 > 0$, then

$$v_\gamma^{(\varepsilon)}(x) \geq \gamma\left(- 2\,\psi_+(\varepsilon)\, x^{\gamma-1} + \left(\mu_+ + (\gamma-1)\, b_+/2\right) x^{\gamma-2}/2 \right); \tag{7.28}$$

(b) if $\gamma > 0$ and $\mu_+ + (\gamma - 1)\, b_+/2 < 0$, then

$$v_\gamma^{(\varepsilon)}(x) \leq - \left| \gamma\left(\mu_+ + (\gamma-1)\, b_+/2\right) x^{\gamma-2}/2 \right| ; \tag{7.29}$$

(c) if $\gamma < 0$ and $\mu_+ + (\gamma - 1)\, b_+/2 > 0$, then

$$v_\gamma^{(\varepsilon)}(x) \leq \gamma\left(- 2\,\psi_+(\varepsilon)\, x^{\gamma-1} + \left(\mu_+ + (\gamma-1)\, b_+/2\right) x^{\gamma-2}/2 \right); \tag{7.30}$$

(d) if $\gamma < 0$ and $\mu_+ + (\gamma - 1)\, b_+/2 < 0$, then

$$v_\gamma^{(\varepsilon)}(x) \geq \left| \gamma\left(\mu_+ + (\gamma-1)\, b_+/2\right) x^{\gamma-2}/2 \right| . \tag{7.31}$$

Lemma 25. *For each compact set K and every $\delta > 0$, the following estimate holds*:

$$\pi^{(\varepsilon)}(K) = O\left(\psi_+^{1 + 2\mu_+/b_+ - \delta}(\varepsilon) \right)$$

as $\varepsilon \downarrow 0$.

Proof. For each γ, $0 \leq \gamma \leq 2$, and sufficiently small ε, Lemma 3 implies the equality

$$0 = \int_0^\infty v_\gamma^{(\varepsilon)}(x)\, \pi_*^{(\varepsilon)}(dx) \tag{7.32}$$

and the inequality

$$0 \geq \int_{x_*}^\infty v_\gamma^{(\varepsilon)}(x)\, \pi_*^{(\varepsilon)}(dx) . \tag{7.33}$$

The function $V_\gamma(x) \equiv \max(0, 1 - x^\gamma) \geq 0$ increases when $\gamma < 0$. Therefore, according to Lemma 1, the following inequality holds:

$$0 \geq \int_{x_*}^\infty \mathbf{E}\left\{ V_\gamma\left(x + \xi_*^{(\varepsilon)}(x)\right) - V_\gamma(x) \right\} \pi_*^{(\varepsilon)}(dx) .$$

Since $x_* \geq 1 + L$, we have

$$\int_{x_*}^\infty \mathbf{E}\left\{ V_\gamma\left(x + \xi_*^{(\varepsilon)}(x)\right) - V_\gamma(x) \right\} \pi_*^{(\varepsilon)}(dx)$$

$$= \int_{x_*}^\infty \mathbf{E}\left\{ V_\gamma\left(x + \xi^{(\varepsilon)}(x)\right) - V_\gamma(x) \right\} \pi^{(\varepsilon)}(dx) = \int_{x_*}^\infty \mathbf{E}\left\{ x^\gamma - \left(x + \xi^{(\varepsilon)}(x)\right)^\gamma \right\} \pi^{(\varepsilon)}(dx) .$$

Thus, for $\gamma < 0$,

$$0 \leq \int_{x_*}^\infty v_\gamma^{(\varepsilon)}(x)\, \pi_*^{(\varepsilon)}(dx) . \tag{7.34}$$

Substituting (7.28) into (7.33) and (7.30) into (7.34), we obtain, for each $\gamma \in (1-2\mu_+/b_+, 2]$, the inequality

$$2\,\psi_+(\varepsilon)\,\mathbf{E}\left\{\left(X^{(\varepsilon)}\right)^{\gamma-1};\ X^{(\varepsilon)} > x_*\right\}$$

$$\geq \left(\mu_+ + (\gamma-1)\,b_+/2\right)\mathbf{E}\left\{\left(X^{(\varepsilon)}\right)^{\gamma-2};\ X^{(\varepsilon)} > x_*\right\}/2\,. \tag{7.35}$$

If $\gamma < 1$ then $0 < (\gamma-1)/(\gamma-2) < 1$, and hence

$$\mathbf{E}\left\{\left(X^{(\varepsilon)}\right)^{\gamma-1};\ X^{(\varepsilon)} > x_*\right\} \leq \mathbf{E}\left\{\left(X^{(\varepsilon)}\right)^{\gamma-2};\ X^{(\varepsilon)} > x_*\right\}^{(\gamma-1)/(\gamma-2)}. \tag{7.36}$$

If $1 < \gamma < 2$ then $0 < \gamma - 1 < 1$ and, by (7.24),

$$\mathbf{E}\left\{\left(X^{(\varepsilon)}\right)^{\gamma-1};\ X^{(\varepsilon)} > x_*\right\} \leq \left(\mathbf{E}\left\{X^{(\varepsilon)};\ X^{(\varepsilon)} > x_*\right\}\right)^{\gamma-1} \leq c_4^{\gamma-1}\,\psi_+^{1-\gamma}(\varepsilon)\,. \tag{7.37}$$

By inequalities (7.36) and (7.37), relation (7.35) implies that for each $\gamma \in (1-2\mu_+/b_+, 2]$,

$$\mathbf{E}\left\{\left(X^{(\varepsilon)}\right)^{\gamma-2};\ X^{(\varepsilon)} > x_*\right\} \leq c_5(\gamma)\,\psi_+^{2-\gamma}(\varepsilon), \qquad c_5(\gamma) < \infty\,. \tag{7.38}$$

Set $\gamma = \gamma(\delta) = 1 - 2\mu_+/b_+ + \delta$, $\delta > 0$. Then

$$\mathbf{E}\left\{\left(X^{(\varepsilon)}\right)^{\gamma(\delta)-2};\ X^{(\varepsilon)} > x_*\right\} \leq c_5(\gamma(\delta))\,\psi_+^{1+2\mu_+/b_+-\delta}(\varepsilon)\,. \tag{7.39}$$

The assertion of Lemma 25 follows from this estimate and Corollary 2 of Lemma 6.

We have

$$\left|\,v_\gamma^{(\varepsilon)}(x_*)\,\pi_*^{(\varepsilon)}(\{x_*\})\,\right| = \left|\int_{-L}^{0}\mathbf{E}\left\{\left(x+\xi^{(\varepsilon)}(x)\right)^{\gamma};\ x+\xi^{(\varepsilon)}(x) > x_*\right\}\pi^{(\varepsilon)}(dx)\right|$$

$$\leq c_6\,\pi^{(\varepsilon)}([-L, 0]), \qquad c_6 < \infty\,, \tag{7.40}$$

$$\sup_{0\leq x\leq x_*}\left|\,v_\gamma^{(\varepsilon)}(x)\,\right| \leq c_7 < \infty\,. \tag{7.41}$$

Therefore, in view of (7.32), the following inequalities hold: for each $0 \leq \gamma \leq 2$,

$$0 = \pi_*^{(\varepsilon)}(\{0\})\,v_\gamma^{(\varepsilon)}(0) + \int_{+0}^{x_*} v_\gamma^{(\varepsilon)}(x)\,\pi^{(\varepsilon)}(dx) + \int_{x_*+0}^{\infty} v_\gamma^{(\varepsilon)}(x)\,\pi^{(\varepsilon)}(dx)$$

$$\leq c_8\,\pi^{(\varepsilon)}([-L, x_*]) + \int_{x_*}^{\infty} v_\gamma^{(\varepsilon)}(x)\,\pi^{(\varepsilon)}(dx), \qquad c_8 < \infty\,. \tag{7.42}$$

Since the function $0 \leq V_\gamma(x) = \max(0, 1-x^\gamma) \leq 1$ is bounded for $\gamma < 0$, we have

$$0 = \int_0^{\infty}\mathbf{E}\left\{V_\gamma\left(x+\xi_*^{(\varepsilon)}(x)\right) - V_\gamma(x)\right\}\pi_*^{(\varepsilon)}(dx)\,. \tag{7.43}$$

By analogy with the preceding calculation, one can deduce from the last equality that

$$0 \geq - c_9 \pi^{(\varepsilon)}([-L, x_*]) + \int_{x_*}^{\infty} v_\gamma^{(\varepsilon)}(x)\, \pi^{(\varepsilon)}(dx), \qquad c_9 < \infty,\ \gamma < 0. \tag{7.44}$$

Substitute estimate (7.29) into (7.42) and (7.31) into (7.44). Then, for each $\gamma \in (-\infty, 1 - 2\mu_+ / b_+)$,

$$\mathbf{E}\left\{ \left(X^{(\varepsilon)}\right)^{\gamma - 2}; X^{(\varepsilon)} \geq x_* \right\} \leq c_{10}\, \pi^{(\varepsilon)}([-L, x_*]), \qquad c_{10} = c_{10}(\gamma) < \infty. \tag{7.45}$$

By the conditional convergence (7.7), we have the following inequality for sufficiently small $\varepsilon > 0$:

$$\mathbf{E}\left\{ \left(X^{(\varepsilon)}\right)^{\gamma - 2}; X^{(\varepsilon)} \geq x_* \right\} \geq c_{11}\, \mathbf{P}\left\{ X^{(\varepsilon)} > 0 \right\} \psi_+^{2 - \gamma}(\varepsilon), \qquad c_{11} > 0, \tag{7.46}$$

Substituting this inequality into (7.45), we obtain

$$\mathbf{P}\left\{ X^{(\varepsilon)} > 0 \right\} \leq c_{12}\, \psi_+^{\gamma - 2}(\varepsilon)\, \pi^{(\varepsilon)}([-L, x_*]), \qquad c_{12} < \infty. \tag{7.47}$$

Replacing the Markov chain $\{X_n^{(\varepsilon)}\}$ with $\{-X_n^{(\varepsilon)}\}$, we arrive at the similar inequality

$$\mathbf{P}\left\{ X^{(\varepsilon)} < 0 \right\} \leq c_{12}\, \psi_-^{\gamma - 2}(\varepsilon)\, \pi^{(\varepsilon)}(K), \tag{7.48}$$

where K is some compact set and $\gamma \in (-\infty, 1 - 2\mu_- / b_-)$. Set

$$\gamma = \gamma(\delta) = 1 - 2\mu_- / b_- - \delta, \quad \delta > 0.$$

By Lemma 25 and inequality (7.48), we have: as $\varepsilon \downarrow 0$,

$$\mathbf{P}\left\{ X^{(\varepsilon)} < 0 \right\} = O\left(\psi_-^{-1 - 2\mu_- / b_- - \delta}(\varepsilon)\, \psi_+^{1 + 2\mu_+ / b_+ - \delta}(\varepsilon) \right). \tag{7.49}$$

By the definition of $c_{\inf}$ and $c_{\sup}$ and in view of the condition $c_{\sup} < \infty$, we have $\psi_+(\varepsilon) = O(\psi_-(\varepsilon))$ as $\varepsilon \downarrow 0$. Therefore, the following estimate holds:

$$\mathbf{P}\left\{ X^{(\varepsilon)} < 0 \right\} = O\left(\psi_-^{2\mu_+ / b_+ - 2\mu_- / b_- - 2\delta}(\varepsilon) \right)$$

for each $\delta > 0$. The proof of estimate (7.9) is complete.

We now turn to the case $2\mu_- / b_- = 2\mu_+ / b_+$. Construct a sequence of Markov chains $\{X_n^{(\varepsilon)}\}$ with the following property: for each $p \in [0, 1]$, there exists a subsequence of superscripts $\varepsilon_k \downarrow 0$ such that

$$\lim_{\varepsilon_k \downarrow 0} \mathbf{P}\left\{ X^{(\varepsilon_k)} \leq 0 \right\} = p. \tag{7.50}$$

Let $a(\varepsilon)$ be a sequence such that $a(\varepsilon) \to 0$ as $\varepsilon \downarrow 0$ and that the set of limit points of the sequence $a(\varepsilon) \log(1/\varepsilon)$, $\varepsilon \downarrow 0$, coincides with $\mathbf{R}$. Consider the Markov chain with the countable set of states $\{0, \pm 1, \pm 2, \dots\}$ and transition probabilities

$$p_{i, i+1}^{(\varepsilon)} = 1 - p_{i, i-1}^{(\varepsilon)} = \frac{1 - \varepsilon}{2} + \frac{a(\varepsilon)}{4i}, \qquad i \geq 1,$$

$$p^{(\varepsilon)}_{i,\,i+1} = 1 - p^{(\varepsilon)}_{i,\,i-1} = \frac{1+\varepsilon}{2}, \qquad i \le 0.$$

Then

$$m^{(\varepsilon)}(i) = -\varepsilon + a(\varepsilon)/(2i) = -\varepsilon + o\left(1/i\right), \qquad b^{(\varepsilon)}(i) \equiv 1 \equiv b_+, \qquad \psi_+(\varepsilon) = \varepsilon, \qquad \mu_+ = 0$$

for $i \ge 1$, $i \to \infty, \varepsilon \downarrow 0$ and

$$m^{(\varepsilon)}(i) = \varepsilon, \qquad \psi_-(\varepsilon) = \varepsilon, \qquad \mu_- = 0, \qquad b^{(\varepsilon)}(i) \equiv 1 \equiv b_-$$

for $i \le 0$. The limiting Markov chain $\{X_n\}$ is the symmetric random walk with jumps ± 1. Let us write out the stationary probabilities $\pi^{(\varepsilon)}(i) = \mathbf{P}\{X^{(\varepsilon)} = i\}$ for the Markov chain $\{X^{(\varepsilon)}_n\}$:

$$\pi^{(\varepsilon)}(i) = \pi^{(\varepsilon)}(0) \prod_{k=1}^{i} \frac{p^{(\varepsilon)}_{k-1,\,k}}{p^{(\varepsilon)}_{k,\,k-1}}, \qquad i \ge 1,$$

$$\pi^{(\varepsilon)}(i) = \pi^{(\varepsilon)}(0) \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{i}, \qquad i \le 0.$$

Summing over i, $-\infty < i \le 0$, we obtain the equality

$$\mathbf{P}\left\{X^{(\varepsilon)} \le 0\right\} = \pi^{(\varepsilon)}(0)\,\frac{1+\varepsilon}{2\varepsilon}. \tag{7.51}$$

Let us evaluate $\pi^{(\varepsilon)}(i)$ for $i \ge 1$:

$$\log \pi^{(\varepsilon)}(i) = \log \pi^{(\varepsilon)}(0) + \log \frac{p^{(\varepsilon)}_{0,1}}{p^{(\varepsilon)}_{i,i-1}} + \sum_{k=1}^{i-1} \left(\log p^{(\varepsilon)}_{k,k+1} - \log p^{(\varepsilon)}_{k,k-1}\right)$$

$$= \log \pi^{(\varepsilon)}(0) + o(1) + \sum_{k=1}^{i-1} \left(-2\varepsilon + \frac{a(\varepsilon)}{k} + O\left(\varepsilon^2 + \varepsilon\, a(\varepsilon)/k + a^2(\varepsilon)/k^2\right)\right)$$

$$= \log \pi^{(\varepsilon)}(0) - 2\varepsilon i + a(\varepsilon) \log i + O\left(i\varepsilon^2\right) + o(\varepsilon \log i) + o(1)$$

as $\varepsilon \downarrow 0$ uniformly in i. Consequently,

$$\pi^{(\varepsilon)}(i) = \pi^{(\varepsilon)}(0)\, e^{-2\varepsilon i + O\left(i\varepsilon^2\right) + o(1)}\, i^{a(\varepsilon) + o(\varepsilon)}$$

as $\varepsilon \downarrow 0$ uniformly in i. Summing over i, $0 < i < \infty$, we arrive at the equality

$$\mathbf{P}\left\{X^{(\varepsilon)} > 0\right\}$$

$$= (1 + o(1))\, \pi^{(\varepsilon)}(0) \int_1^{\infty} e^{-2\varepsilon x + O\left(x\varepsilon^2\right) + o(1)}\, x^{a(\varepsilon) + o(\varepsilon)}\, dx$$

$$= \frac{\pi^{(\varepsilon)}(0)}{2\varepsilon}\, (2\varepsilon)^{-a(\varepsilon) + o(\varepsilon)}\, (1 + o(1)) \int_{2\varepsilon}^{\infty} e^{-y + O(\varepsilon y)}\, y^{a(\varepsilon) + o(\varepsilon)}\, dy$$

$$= \pi^{(\varepsilon)}(0)\, (2\varepsilon)^{-1 - a(\varepsilon)}\, (1 + o(1)) \quad \text{as } \varepsilon \downarrow 0. \tag{7.52}$$

Equalities (7.51) and (7.52) imply the asymptotic relation

$$\pi^{(\varepsilon)}(0) \sim \ 2\varepsilon\,(1+\exp\{\,a(\varepsilon)\log(1/\varepsilon)\})^{-1},$$

and, by (7.51), we have

$$\mathbf{P}\left\{X^{(\varepsilon)} \le 0\right\} \sim \ \left(1+\exp\{a(\varepsilon)\log(1/\varepsilon)\}\right)^{-1} \quad \text{as } \varepsilon \downarrow 0\,.$$

The set of limit points of the sequence $a(\varepsilon)\log(1/\varepsilon)$, as $\varepsilon\downarrow 0$, coincides with $\mathbf{R}$, and the function $(1+e^{x})^{-1}$ maps the line $(-\infty,\infty)$ onto the interval $(0,1)$; therefore, the Markov chain $\{X_n^{(\varepsilon)}\}$ satisfies (7.50). The proof of Theorem 9 is complete.

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