ISSN 1064-5624, Doklady Mathematics, 2013, Vol. 88, No. 2, pp. 566–568. © Pleiades Publishing, Ltd., 2013.
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On Extremal Behavior of Gaussian Chaos

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Presented by Academician A.N. Shiryaev March 6, 2013

Received March 22, 2013

DOI: 10.1134/S1064562413050220

Let $\boldsymbol{\xi} = (\xi_1, \xi_2, ..., \xi_d)$ be a normally distributed random vector in $\mathbb{R}^d$ with zero mean and covariance matrix B, $B_{ij} := \mathbb{E}\xi_i\xi_j$. A problem of great interest is to analyze the asymptotic behavior of the distribution tail of the product $\prod_{i=1}^{d}\xi_i$. This problem arises in various domains, for example in stochastic geometry, random difference equations, and risk theory.

Consider a more general case of functions of the vector $\boldsymbol{\xi}$, namely, the so-called Gaussian chaos $h(\boldsymbol{\xi})$, where h: $\mathbb{R}^d \to \mathbb{R}$ is a continuous homogeneous function of order $\alpha > 0$; i.e., $h(x\mathbf{t}) = x^\alpha h(\mathbf{t})$ for all $x > 0$ and $\mathbf{t} \in \mathbb{R}^d$. Traditionally, in the literature, the term Gaussian chaos of order $\alpha \in \mathbb{N}$ is referred to the case where g is a homogeneous polynomial of degree α. This concept goes back to Wiener [14], who was the first to consider processes of polynomial chaos. We follow a broader treatment of the concept of Gaussian chaos.

The distribution of $\boldsymbol{\xi}$ is equal to the distribution of $\sqrt{B}\boldsymbol{\eta}$ if the vector $\boldsymbol{\eta} = (\eta_1, \eta_2, ..., \eta_d)$ has independent coordinates with a standard normal distribution. Then

$$\mathbb{P}\{h(\boldsymbol{\xi}) > x\} = \mathbb{P}\{h(\sqrt{B}\boldsymbol{\eta}) > x\} = \mathbb{P}\{g(\boldsymbol{\eta}) > x\},$$

where $g(\mathbf{u}) = h(\sqrt{B}\mathbf{u})$. The continuous function g: $\mathbb{R}^d \to \mathbb{R}$ is also homogeneous of order α like h. Thus, the problem is reduced to the case of a unit covariance matrix. For this reason, in what follows, we study $g(\boldsymbol{\eta})$. By virtue of homogeneity,

$$\mathbb{P}\{g(\boldsymbol{\eta}) > x\} = \mathbb{P}\{(g(x^{-1/\alpha}\boldsymbol{\eta}) > 1\} = \frac{x^{d/\alpha}}{(2\pi)^{d/2}} \int\limits_{\{\mathbf{v}:\, g(\mathbf{v}) > 1\}} e^{-x^{2/\alpha}|\mathbf{v}|^2/2} d\mathbf{v}. \quad (1)$$

Therefore, the asymptotic behavior of probability (1) can be determined using a version of the Laplace asymptotic method (see, for example, [3]). Define

$$c^2 := \min\{|\mathbf{u}|^2 : g(\mathbf{u}) \ge 1\} = \min\{|\mathbf{u}|^2 : g(\mathbf{u}) = 1\},$$

where the last equality follows from the homogeneity of g. Since g is continuous, we have $c^2 > 0$. To apply the Laplace method, we consider the set

$$\mathscr{C} := \arg\min\{|\mathbf{u}| : g(\mathbf{u}) = 1\} = \{\mathbf{u} : |\mathbf{u}| = c \text{ and } g(\mathbf{u}) = 1\},$$

which lies on a sphere of radius c. Assume that this set is a smooth finitely connected manifold of dimension r and the structure of the function g near this manifold is typical of the Laplace method. Define $g(\varphi) := g(\mathbf{u}/|\mathbf{u}|)$, where $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, ..., \varphi_{d-1}) \in \Pi := [0, \pi)^{d-2} \times [0, 2\pi)$ are the spherical coordinates of the vector $\mathbf{u}/|\mathbf{u}|$ on the unit sphere S_{d-1}. The manifold on the parallelepiped Π that corresponds to $\mathscr{C}$ is denoted by $\mathscr{C}_\varphi$. The Jacobian of the transition to spherical coordinates in $\mathbb{R}^d$ is designated as $J(r, \boldsymbol{\varphi})$. Let $g''(\boldsymbol{\varphi})$ denote the Hessian of a function $g(\boldsymbol{\varphi})$, and let $\lambda(A)$ stand for the smallest (in absolute value) nonzero eigenvalue of a symmetric matrix A.

Theorem 1. *Let* g: $\mathbb{R}^d \to \mathbb{R}$ *be a continuous homogeneous function of order* $\alpha > 0$, *and let* $\dim\mathscr{C}_\varphi = r \in [0, d-1]$. *If the corresponding function* $g(\boldsymbol{\varphi})$: $\Pi \to \mathbb{R}$ *is three times differentiable and*

$$\operatorname{rank} g''(\boldsymbol{\varphi}) \equiv d - 1 - r, \quad \inf_{\boldsymbol{\varphi} \in \mathscr{C}_\varphi} \lambda(g''(\boldsymbol{\varphi})) > 0$$

(*the Hessian is uniformly nonsingular on* $\mathscr{C}_\varphi$), *then*

$$\mathbb{P}\{g(\boldsymbol{\eta}) > x\} = \mathscr{H} x^{(r-1)/\alpha} e^{-c^2 x^{2/\alpha}/2}(1 + O(x^{-2/\alpha})) \quad (2)$$
as $x \to \infty$,

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$$\mathcal{H} := \frac{1}{(2\pi)^{(r+1)/2}} \frac{\alpha^{(d-1-r)/2}}{c^{1-r+\alpha(d-1-r)/2}}$$

$$\times \int_{\mathcal{C}_\varphi} \frac{J(1, \varphi)}{\sqrt{|\det g''_{d-1-r}(\varphi)|}} dV_\varphi,$$

where dV_φ is the volume element of the manifold $\mathcal{C}_\varphi \subset \Pi$ and $\det g''_{d-1-r}(\varphi)$ *is any nonzero minor of the Hessian* $g''(\boldsymbol{\varphi})$ *of order $d - 1 - r$. Relation* (2) *can be differentiated, which gives asymptotics of the distribution density of the Gaussian chaos* $g(\boldsymbol{\eta})$.

Note that, as in the classical case of the Laplace method [3], assuming that g has higher smoothness, we can obtain asymptotic expansions of the considered probability and density in powers of x. In the case $r = 0$, i.e., when $\mathcal{C} = \{\mathbf{t}_1, \mathbf{t}_2, ..., \mathbf{t}_k\}$, where $\mathbf{t}_i$ are isolated absolute minimizers of g in the integration domain, the theorem is proved by directly applying Theorem 4.2 from [3]. The integral in the expression for $\mathcal{H}$ becomes a sum over the points $\boldsymbol{\varphi}_i \in \Pi$ corresponding to the points $\mathbf{t}_i$. In the general case, we apply a version of the Laplace method for parameter-dependent functions, which are used to prove the possibility of integration. On each map of an atlas with sufficiently small maps on the manifold $\mathcal{C}_\varphi$, we construct a coordinate system with the first r coordinates being parameters. When they are fixed, the minimum of the amplitude (of the argument of the exponential) is reached at a unique point of a neighborhood of the map. Next, the standard Laplace method is applied and the maps of the atlas are integrated with respect to these parameters on all neighborhoods in Π.

By Theorem 1, the Gaussian chaos is a subexponential random variable if $\alpha > 2$. The subexponentiality of random variables is an important concept in various applications (see, for example, [4]). The Gaussian chaos is subexponential under rather weak constraints on the function h. For example, let h be nonnegative. The d-dimensional centered Gaussian vector $\boldsymbol{\eta}$ with a unit covariance matrix can be represented as the product $\boldsymbol{\eta} \overset{d}{=} \chi\boldsymbol{\mu}$ of independent values χ and $\boldsymbol{\mu}$, where $\chi^2 = \sum_{i=1}^{d} \eta_i^2$ has a chi-square distribution χ^2 with d degrees of freedom, while $\boldsymbol{\mu}$ has a uniform distribution on the unit sphere $S_{d-1} \subset \mathbb{R}^d$. The Gaussian random vector $\boldsymbol{\xi} = \sqrt{B}\boldsymbol{\eta} = \chi\sqrt{B}\boldsymbol{\mu}$ has the covariance matrix B. Therefore, since h is homogeneous for any $x > 0$, we have

$$\mathbb{P}\{h(\boldsymbol{\xi}) > x\} = \mathbb{P}\{\chi^\alpha h(\sqrt{B}\boldsymbol{\mu}) > x\}. \tag{3}$$

If $h(\sqrt{B}\boldsymbol{\mu})$ is a positive bounded random variable, then, according to [2, Corollary 2.5], the random variable $h(\boldsymbol{\xi})$ is subexponential for $\alpha > 2$, because the distribution χ^α then has a Weibull type density

$$\frac{1}{\alpha \cdot 2^{d/2-1}\Gamma(d/2)} x^{d/\alpha - 1} e^{-x^{2/\alpha}/2}$$

with $2/\alpha < 1$, which means subexponentiality.

It follows from (3) that, if h is bounded on the unit sphere S_{d-1}, i.e., $h^* := \max\{h(\mathbf{u}): |\mathbf{u}| = 1\} < \infty$, then estimates

$$\mathbb{P}\{h(\boldsymbol{\xi}) > x\} \le \mathbb{P}\{\chi^\alpha > x/h^*\}$$

$$\le \frac{1}{\alpha \cdot 2^{d/2-1}\Gamma(d/2)} \int_{x/h^*}^{\infty} y^{d/\alpha - 1} e^{-y^{2/\alpha}/2} dy.$$

This explicit upper bound improves the one obtained in [10, Corollary 1]. In our conditions, it is better than the bound that can be derived from [1, Theorem 4.3].

Theorem 1 underlies a unified approach to different problems. Below are some examples.

Example 1. (Product of independent $N(0, 1)$ random variables) Let $\boldsymbol{\eta} = (\eta_1, \eta_2, ..., \eta_d)$ be a standard Gaussian vector and $g(\mathbf{u}) = u_1u_2...u_d$. We have $\alpha = d$, $c^2 = d$, and $\mathcal{C} = \{(\pm 1, ..., \pm 1)$ with an even number of negative coordinates$\}$ consists of 2^{d-1} points. Applying Theorem 1 yields the asymptotics

$$p_{\eta_1 \dots \eta_d}(x) = \frac{2^{(d-1)/2}}{\sqrt{2\pi d}} x^{1/d - 1} e^{-dx^{2/d}/2}(1 + O(x^{-2/d}))$$

$$\text{as} \quad x \to \infty.$$

This asymptotic relation can be intuitively interpreted as follows (see, e.g., [13]): the product takes the most probable large value when all the multipliers are roughly identical; therefore, $p_{\eta_1 \dots \eta_d}(x)$ asymptotically resembles the product of d densities at the same point $x^{1/d}$.

For the product of the coordinates of an arbitrary Gaussian vector $\boldsymbol{\xi}$ with a covariance matrix B, we have a similar formula based on the representation $\xi = \sqrt{B}\boldsymbol{\eta}$, but the computation of the constants encounters certain difficulties.

Example 2. (Quadratic forms of independent $N(0, 1)$ random variables.) Let $g(\boldsymbol{\eta}) = \sum_{i=1}^{d} a_i\eta_i^2$, where the constants $a_i \in \mathbb{R}$ are such that $a_1 \le a_2 \le ... \le a_{d-r} < a_{d-r+1} = \dots = a_d = a$, $a > 0$.

Since

$$g(\mathbf{u}) = \sum_{i=1}^{d-r} a_i u_i^2 + a \sum_{i=d-r+1}^{d} u_i^2$$

and $a_i < a$ for $i \le d - r$, the minimum of $|\mathbf{u}|^2$ on the set $g(\mathbf{u}) = 1$ is reached at points $\mathbf{u}$ satisfying $u_{d-r+1}^2 + \dots + u_d^2 = \frac{1}{a}$ and $u_1 = u_2 = ... = u_{d-r} = 0$, so that $c^2 = \frac{1}{a}$. If

$r = 1$, the set $\mathcal{C}_\varphi$ consists of two points $\left(\frac{\pi}{2}, \ldots, \frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\left(\frac{\pi}{2}, \ldots, \frac{\pi}{2}, \frac{3\pi}{2}\right)$. By using Theorem 1, we can find that

$$\mathbb{P}\left\{\sum_{i=1}^{d} a_i\eta_i^2 > x\right\}$$

$$= \frac{1}{2^{r/2-1}\Gamma(r/2)}\prod_{i=1}^{d-r}\frac{1}{\sqrt{1-a_i/a}}(x/a)^{r/2-1}e^{-x/2a}(1+O(1/x))$$

as $x \to \infty$, which agrees (up to the first-order asymptotics) with the results of [6] (see also [11, 12] or [7, Theorem 1]). This also supplements the upper bounds obtained in [5, 9].

Example 3. (Scalar product) The quadratic forms in Example 2 are closely related to $g(\boldsymbol{\eta}, \boldsymbol{\eta}^*) = \sum_{i=1}^{d} a_i\eta_i\eta_i^*$, where η_i and η_i^*, $i \le d$, are independent $N(0, 1)$ random variables and $a_i \in \mathbb{R}^+$. Indeed, since $\eta_i\eta_i^*$ coincides in distribution with

$$\frac{\eta_i + \eta_i^*}{\sqrt{2}}\frac{\eta_i - \eta_i^*}{\sqrt{2}} = \frac{\eta_i^2 - \eta_i^{*2}}{2},$$

we have the distribution equality

$$g(\boldsymbol{\eta}, \boldsymbol{\eta}^*) \stackrel{d}{=} \frac{1}{2}\left(\sum_{i=1}^{d} a_i\eta_i^2 - \sum_{i=1}^{d} a_i\eta_i^{*2}\right),$$

and, to the quadratic form on the right, we can apply the result of Example 2, with the dimension replaced by $2d$ and with the parameter r replaced by the number of maximal a_i. Some results for scalar products can be found in [8].

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Translated by I. Ruzanova