

The Key Renewal Theorem for a Transient Markov Chain

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We consider a time-homogeneous Markov chain X_n , $n \geq 0$, valued in \mathbf{R} . Suppose that this chain is transient, that is, X_n generates a σ -finite renewal measure. We prove the key renewal theorem under condition that this chain has asymptotically homogeneous at infinity jumps and asymptotically positive drift.

KEY WORDS: transient Markov chain, renewal kernel, renewal measure, key renewal theorem, Green function.

Let ξ_1, ξ_2, \dots be independent identically distributed random variables with a common distribution F on \mathbf{R} . Put $S_n = \xi_1 + \dots + \xi_n$, $S_0 = 0$, and consider the renewal measure generated by sums:

$$U(B) \equiv \mathbf{E} \sum_{n=0}^{\infty} \mathbf{I}\{S_n \in B\} = \sum_{n=0}^{\infty} F^{*n}(B).$$

If F is non-lattice then the celebrated key renewal theorem states that, for every fixed $h > 0$,

$$U(x, x+h) \rightarrow \frac{h}{\mathbf{E}\xi_1} \text{ as } x \rightarrow \infty, \quad (1)$$

provided $\mathbf{E}\xi_1$ is finite and positive (see, for example, Feller and Orey [8], Feller [7, Ch. XI], Woodroffe [25, Appendix]); F is called lattice if it is concentrated on some lattice $\{ka, k \in \mathbf{Z}\}$ with $a > 0$. If F is lattice the same is true when h is a multiple of the span a .

It is proved in Wang and Woodroffe [23] and in Borovkov and Foss [5, Theorem 2.6] that (1) holds uniformly over certain classes of distributions F . Some extensions of the key renewal theorem to the nonidentically distributed case are considered by Williamson [24], Maejima [18]. Another extension is aimed to include random walks perturbed by both a slowly changing sequence and a stationary one, $Z_n = S_n + \eta_n + \zeta_n$ say; see, for example, Lai and Siegmund [16, 17], Woodroffe [25], Horváth [11], Zhang [26], Kim and Woodroffe [13]. All these extensions deal with perturbations depending on time rather than on state space; in most cases summands are independent and have finite variance.

Many papers (see, for example, Kesten [12], Athreya, McDonald and Ney [2], Nummelin [20], Alsmeyer [1], Klüppelberg and Pergamenchtchikov [14], Fuh [9], and also some of their references) are devoted to a Markov modulated random walks. Usually a (Harris-) recurrent Markov chain X_n is considered with an invariant measure, π say. Conditioned on a realisation $\{x_n, n \geq 0\}$, one is given a sequence of independent random variables $\{\xi_n\}$, such that the distribution of ξ_n depends only on x_n . Put $T_n = \xi_0 + \dots + \xi_n$

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and assume $T_n/n \rightarrow \alpha$ with probability 1. Then the typical result states the convergence

$$\mathbf{E} \sum_{n=0}^{\infty} g(X_n, x - T_n) \rightarrow \frac{1}{\alpha} \int \pi(dy) \int_{-\infty}^{\infty} g(y, s) ds \quad \text{as } x \rightarrow \infty,$$

for bounded continuous function g satisfying some conditions. The corresponding proofs use probabilistic arguments, notably the construction of regeneration epochs for $\{X_n, n \geq 0\}$ (for example, visit times to some atom). This approach eventually reduces the problem to Blackwell's renewal theorem for sums of independent identically distributed random variables.

To the best of our knowledge, the only result related to a random walk perturbed in state space is due to Heyde [10]. To be more precise, Heyde discussed the key renewal theorem for maxima $M_n \equiv \max_{0 \leq k \leq n} S_k$ of partial sums: provided $\mathbf{E}\xi_1$ is finite and positive

$$\mathbf{E} \sum_{n=0}^{\infty} \mathbf{I}\{M_n \in (x, x + h)\} \rightarrow \frac{h}{\mathbf{E}\xi_1} \quad \text{as } x \rightarrow \infty. \quad (2)$$

It is well known (see, for example, [7, Ch. VI, Sec. 9]) that M_n has the same distribution as the reflected random walk on \mathbf{R}^+ defined by the recursion

$$W_{n+1} = (W_n + \xi_{n+1})^+, \quad W_0 = 0.$$

So, the renewal function generated by the chain W_n has the same asymptotic behaviour as described in (2).

Both the random walk S_n and the reflected random walk on the positive half-line W_n are particular examples of Markov chains on \mathbf{R} . In the present paper we extend the key renewal theorem from these very important cases onto asymptotically space-homogeneous Markov chains on \mathbf{R} with an asymptotically positive drift. Introduce some relevant definitions.

Let $P(x, B)$, $x \in \mathbf{R}$, $B \in \mathcal{B}(\mathbf{R})$, be a transition probability kernel on \mathbf{R} ; hereinafter $\mathcal{B}(\mathbf{R})$ is the Borel σ -algebra on \mathbf{R} . Consider a time-homogeneous Markov chain $X = \{X_n, n = 0, 1, 2, \dots\}$ on \mathbf{R} with transition probabilities $P(\cdot, \cdot)$, that is,

$$\mathbf{P}\{X_{n+1} \in B \mid X_n = x\} = P(x, B).$$

Let $\xi(x)$ be the random variable distributed as the jump of the chain at state x :

$$\mathbf{P}\{x + \xi(x) \in B\} = P(x, B), \quad B \in \mathcal{B}(\mathbf{R}).$$

Let μ_n denote the distribution of X_n ; then equalities $\mu_n = \mu_{n-1}P$ and $\mu_n = \mu_0 P^n$ hold. Formally, define the renewal (or potential) kernel Q by the equality

$$Q(\cdot, \cdot) = \sum_{n=0}^{\infty} P^n(\cdot, \cdot).$$

We assume that the Markov chain X is *transient* (see Meyn and Tweedie [19, Ch. 8]), that is, there exists a countable cover of \mathbf{R} with uniformly transient sets $\{B_k\}$. In its turn a set $B \in \mathcal{B}(\mathbf{R})$ is called *uniformly transient* if

$$\sup_{y \in B} Q(y, B) < \infty. \quad (3)$$

By the Markov property, this is equivalent to

$$\sup_{y \in \mathbf{R}} Q(y, B) < \infty. \quad (4)$$

Indeed, considering the first hitting time of B , we conclude the following inequality, for each state $x \in \mathbf{R}$,

$$Q(x, B) \leq \sup_{y \in B} Q(y, B), \quad (5)$$

which implies (4).

In the present paper we assume that X_n is transient with respect to the collection of sets $B_k = (k, k + 1]$, $k \in \mathbf{Z}$; that is, for any $k \in \mathbf{Z}$,

$$\sup_{y \in \mathbf{R}} Q(y, (k, k + 1]) < \infty. \quad (6)$$

Then $Q(x, B) < \infty$ for all x and bounded set B . Hence the renewal measure generated by the chain X

$$U(B) \equiv \sum_{n=0}^{\infty} \mathbf{P}\{X_n \in B\} = \sum_{n=0}^{\infty} \mu_n(B) = (\mu_0 Q)(B)$$

is well defined for every initial distribution μ_0 and bounded set B ; U is σ -finite with respect to the collection of sets $(k, k + 1]$, $k \in \mathbf{Z}$.

The main goal of our analysis is the local asymptotic behaviour of this renewal measure. Without further restrictions on the chain X_n , the asymptotics of $U(x, x + h]$ as $x \rightarrow \infty$ can be very special. We consider a transient Markov chain X_n as a *perturbation in space* of the random walk S_n with positive drift. To get similar renewal behaviour for X_n as for S_n , it is natural to assume that, being far away from the origin, X_n behaves almost like S_n .

Thus we restrict our attention to the *asymptotically space-homogeneous Markov chain* X , that is, we assume that the distribution of the jump $\xi(x)$ has a weak limit F as $x \rightarrow \infty$. Let ξ be a random variable with distribution F .

The notion of asymptotically space-homogeneous Markov chain is a natural generalisation of both (i) the random walk S_n ; in this case $\xi(x) =_d \xi$ for all x ; (ii) the reflected random walk W_n on the positive half-line; in this

case $\xi(x) =_d (x + \xi)^+ - x$. An asymptotically space-homogeneous Markov chains appear in different areas; in particular, we are motivated by theory of queues when the service rate depends on the current waiting time; and by sequential analysis related to an optimal solutions in a change-point problem (see Borovkov [4]). Some limit theorems for them were obtained by Korshunov [15].

So, let the Markov chain X_n be asymptotically space-homogeneous.

Theorem 1. *Let $\xi(x) \Rightarrow \xi$ as $x \rightarrow \infty$ and $\mathbf{E}\xi > 0$. Let the family of random variables $\{|\xi(x)|, x \in \mathbf{R}\}$ admit an integrable majorant η , that is, $\mathbf{E}\eta < \infty$ and*

$$|\xi(x)| \leq_{\text{st}} \eta \quad \text{for all } x \in \mathbf{R}. \quad (7)$$

Assume that

$$\sup_{k \in \mathbf{Z}} U(k, k+1] < \infty. \quad (8)$$

Assume also that there exists a limit

$$p_0 = \lim_{n \rightarrow \infty} \mathbf{P}\{X_n > 0\}. \quad (9)$$

If the limit distribution F is non-lattice, then $U(x, x+h] \rightarrow h/\mathbf{E}\xi$ as $x \rightarrow \infty$, for every fixed $h > 0$.

If the chain X_n is integer valued and \mathbf{Z} is the lattice with minimal span for distribution F , then $U\{n\} \rightarrow 1/\mathbf{E}\xi$ as $n \rightarrow \infty$.

Condition (7) and the dominated convergence theorem imply $|\xi| \leq_{\text{st}} \eta$, $\mathbf{E}|\xi| < \infty$ and $\mathbf{E}\xi(x) \rightarrow \mathbf{E}\xi$ as $x \rightarrow \infty$; in particular, the chain X_n has an asymptotically space-homogeneous drift.

Only conditions (8) and (9) of Theorem 1 are not formulated in local terms, i.e., in terms of one-step transition probabilities. Below, in Theorem 2, we give some simple conditions sufficient for (8). Note that the value of p_0 in condition (9) may be very sensitive with respect to the local probabilities. It can be illustrated by the following example. Let X_n be a chain valued on \mathbf{Z} with the following transition probabilities:

$$\begin{aligned} p_{i,i+1} &= 3/4, & p_{i,i-1} &= 1/4 & \text{for } i \geq 1, \\ p_{i,i+1} &= 1/4, & p_{i,i-1} &= 3/4 & \text{for } i \leq -1, \\ p_{0,1} &= p, & p_{0,-1} &= 1-p, \end{aligned}$$

where $p \in [0, 1]$. Given $X_0 = 0$, then $p_0 = p_0(p) = p$ is increasing from 0 to 1 simultaneously with p .

Since the chain is transient, by (4) the convergence $\mu_n(K) \equiv \mathbf{P}\{X_n \in K\} \rightarrow 0$ holds as $n \rightarrow \infty$ for any compact K . Hence, condition (9) is equivalent to the convergence, for every fixed x_0 ,

$$\mathbf{P}\{X_n > x_0\} \rightarrow p_0 \quad \text{as } n \rightarrow \infty. \quad (10)$$

Proof of Theorem 1 follows some ideas of the operator approach proposed by Feller [7, Ch. XI]. First of all, condition (8) allows us to apply Helly's Selection Theorem to the family of measures $\{U(k + \cdot), k \in \mathbf{Z}^+\}$ (see, for example, Theorem 2 in [7, Ch. VIII, Sec. 6]). Hence, there exists a sequence of points $t_n \rightarrow \infty$ such that the sequence of measures $U_n(\cdot) \equiv U(t_n + \cdot)$ converges weakly to some measure λ as $n \rightarrow \infty$. The following two lemmas describe properties of λ .

Lemma 1. *A weak limit λ of the sequence of measures $U(t_n + \cdot)$ satisfies the identity $\lambda = \lambda * F$.*

Proof. The measure λ is non-negative and σ -finite with necessity. Fix any smooth function $f(x)$ with a bounded support; let $A > 0$ be such that $f(x) = 0$ for $x \notin [-A, A]$. The weak convergence of measures means the convergence of integrals

$$\int_{-\infty}^{\infty} f(x)U(t_n + dx) \equiv \int_{-A}^A f(x)U(t_n + dx) \rightarrow \int_{-A}^A f(x)\lambda(dx) \quad (11)$$

as $n \rightarrow \infty$. On the other hand, due to the equality $U = \mu_0 + UP$ we have the following representation for the left side of (11):

$$\int_{-A}^A f(x)\mu_0(t_n + dx) + \int_{-A}^A f(x) \int_{-\infty}^{\infty} P(t_n + y, t_n + dx)U(t_n + dy). \quad (12)$$

Since f is bounded and μ_0 is finite,

$$\int_{-A}^A f(x)\mu_0(t_n + dx) \leq \|f\|_C \mu_0[t_n - A, t_n + A] \rightarrow 0 \quad (13)$$

as $n \rightarrow \infty$. The second term in (12) is equal to

$$\int_{-\infty}^{\infty} U(t_n + dy) \int_{-A}^A f(x)P(t_n + y, t_n + dx). \quad (14)$$

The weak convergence $P(t, t + \cdot) \Rightarrow F(\cdot)$ as $t \rightarrow \infty$ implies the convergence of the inner integral in (14):

$$\int_{-A}^A f(x)P(t_n + y, t_n + dx) \rightarrow \int_{-A}^A f(x)F(dx - y);$$

here the rate of convergence can be estimated in the following way:

$$\begin{aligned} \Delta(n, y) &\equiv \left| \int_{-A}^A f(x)(P(t_n + y, t_n + dx) - F(dx - y)) \right| \\ &= \left| \int_{-A}^A f'(x)(\mathbf{P}\{\xi(t_n + y) \leq x - y\} - F(x - y))dx \right| \\ &\leq \|f'\|_C \int_{-A-y}^{A-y} |\mathbf{P}\{\xi(t_n + y) \leq x\} - F(x)|dx. \end{aligned}$$

Thus, the asymptotic homogeneity of the chain yields for every fixed $C > 0$ the uniform convergence

$$\sup_{y \in [-C, C]} \Delta(n, y) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (15)$$

In addition, by majorisation condition (7), for all x regardless positive or negative,

$$|\mathbf{P}\{\xi(t_n + y) \leq x\} - F(x)| \leq 2\mathbf{P}\{\eta > |x|\}.$$

Hence, for all y ,

$$\begin{aligned} \Delta(n, y) &\leq 2\|f'\|_C \int_{-A-y}^{A-y} \mathbf{P}\{\eta > |x|\} dx \\ &\leq 4A\|f'\|_C \mathbf{P}\{\eta > |y| - A\}. \end{aligned} \quad (16)$$

We have the estimate

$$\begin{aligned} \Delta_n &\equiv \left| \int_{-\infty}^{\infty} U(t_n + dy) \left(\int_{-\infty}^{\infty} f(x) P(t_n + y, t_n + dx) - \int_{-\infty}^{\infty} f(x) F(dx - y) \right) \right| \\ &\leq \int_{-\infty}^{\infty} \Delta(y, n) U(t_n + dy). \end{aligned}$$

For any fixed $C > 0$, uniform convergence (15) implies

$$\begin{aligned} \int_{-C}^C \Delta(y, n) U(t_n + dy) &\leq \sup_{y \in [-C, C]} \Delta(y, n) \cdot \sup_n U[t_n - C, t_n + C] \\ &\rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

The remaining part of the integral can be estimated by (16):

$$\begin{aligned} \limsup_{n \rightarrow \infty} \int_{|y| \geq C} \Delta(y, n) U(t_n + dy) \\ \leq 4A\|f'\|_C \limsup_{n \rightarrow \infty} \int_{|y| \geq C} \mathbf{P}\{\eta > |y| - A\} U(t_n + dy). \end{aligned}$$

Since η has a finite mean, property (8) of the renewal measure U allows us to choose a sufficiently large C in order to make the ‘lim sup’ as small as we please. Therefore, $\Delta_n \rightarrow 0$ as $n \rightarrow \infty$. Hence, (14) has the same limit as the sequence of integrals

$$\int_{-\infty}^{\infty} U(t_n + dy) \int_{-A}^A f(x) F(dx - y).$$

Now the weak convergence to λ implies that (14) has the limit

$$\begin{aligned} \int_{-\infty}^{\infty} \lambda(dy) \int_{-\infty}^{\infty} f(x)F(dx-y) &= \int_{-\infty}^{\infty} f(x) \int_{-\infty}^{\infty} F(dx-y)\lambda(dy) \\ &= \int_{-\infty}^{\infty} f(x)(F * \lambda)(dx). \end{aligned} \quad (17)$$

By (11)–(13) and (17), we conclude the identity

$$\int_{-\infty}^{\infty} f(x)\lambda(dx) = \int_{-\infty}^{\infty} f(x)(F * \lambda)(dx).$$

Since this identity holds for every smooth function f with a bounded support, the measures λ and $F * \lambda$ coincide. The proof is complete.

Further we use the following statement which was proved in [6] (see also [21] or [22, Sec. 5.1]):

Lemma 2. *Let F be a distribution not concentrated at 0. Let λ be a nonnegative measure satisfying the equality $\lambda = F * \lambda$ and the property $\sup_{n \in \mathbf{Z}} \lambda[n, n+1] < \infty$.*

If F is non-lattice, then λ is proportional to Lebesgue measure.

If F is lattice with minimal span 1 and $\lambda(\mathbf{Z}) = 1$, then λ is proportional to the counting measure.

The concluding part of the proof of Theorem 1 will be carried out for the non-lattice case. Choose any sequence of points $t_n \rightarrow \infty$ such that the measure $U(t_n + \cdot)$ converges weakly to some measure λ as $n \rightarrow \infty$. It follows from Lemmas 1 and 2 that then $\lambda(dx) = \alpha \cdot dx$ with some α , i.e.,

$$U(t_n + dx) \Rightarrow \alpha \cdot dx \text{ as } n \rightarrow \infty.$$

Now it suffices to prove that $\alpha = p_0/\mathbf{E}\xi$.

Fix some $k \in \mathbf{N}$. Put $U_k \equiv UP^k = \sum_{j=k}^{\infty} \mu_j$. Then

$$U_k(t_n + dx) \Rightarrow \alpha \cdot dx \text{ as } n \rightarrow \infty. \quad (18)$$

Consider the measure $U_k - U_{k+1} = U_k(I - P)$; by the definition of the renewal measure it is equal to μ_k , that is, for any *bounded* Borel set B , $U_k(B) - U_{k+1}(B) = \mu_k(B)$ (the equality may fail for unbounded sets, say, for $(-\infty, x]$). In particular,

$$(U_k - U_{k+1})(0, x] = \mu_k(0, x] \rightarrow \mu_k(0, \infty) \text{ as } x \rightarrow \infty. \quad (19)$$

On the other hand,

$$\begin{aligned} (U_k - U_{k+1})(0, x] &= \int_{-\infty}^{\infty} (I - P)(y, (0, x])U_k(dy) \\ &= - \int_{-\infty}^0 P(y, (0, x])U_k(dy) + \int_0^x P(y, (-\infty, 0])U_k(dy) \\ &\quad + \int_0^x P(y, (x, \infty))U_k(dy) - \int_x^{\infty} P(y, (0, x])U_k(dy). \end{aligned} \quad (20)$$

The asymptotic homogeneity of the chain and weak convergence (18) imply the following convergences of the integrals, for any fixed $A > 0$:

$$\int_{t_n-A}^{t_n} P(y, (t_n, \infty)) U_k(dy) \rightarrow \alpha \int_0^A \mathbf{P}\{\xi > z\} dz \quad (21)$$

as $n \rightarrow \infty$, and

$$\int_{t_n}^{t_n+A} P(y, (0, t_n]) U_k(dy) \rightarrow \alpha \int_0^A \mathbf{P}\{\xi \leq -z\} dz. \quad (22)$$

Majorisation condition (7) allows us to estimate the tails of the integrals:

$$\int_0^{t_n-A} P(y, (t_n, \infty)) U_k(dy) \leq - \int_A^\infty \mathbf{P}\{\eta > z\} U(t_n - dz) \quad (23)$$

and

$$\int_{t_n+A}^\infty P(y, (0, t_n]) U_k(dy) \leq \int_A^\infty \mathbf{P}\{\eta \geq z\} U(t_n + dz). \quad (24)$$

Since the majorant η is integrable, condition (8) guarantees that the right sides of inequalities (23) and (24) can be made as small as we please by the choice of sufficiently large A . By these reasons we conclude from (20)–(22) that, as $n \rightarrow \infty$,

$$\begin{aligned} & (U_k - U_{k+1})(0, t_n] \\ & \rightarrow - \int_{-\infty}^0 P(y, (0, \infty)) U_k(dy) + \int_0^\infty P(y, (-\infty, 0]) U_k(dy) \\ & \quad + \alpha \int_0^\infty \mathbf{P}\{\xi > z\} dz - \alpha \int_0^\infty \mathbf{P}\{\xi \leq -z\} dz. \end{aligned}$$

Together with (19) it implies the following equality, for any fixed k :

$$\mu_k(0, \infty) = - \int_{-\infty}^0 P(y, (0, \infty)) U_k(dy) + \int_0^\infty P(y, (-\infty, 0]) U_k(dy) + \alpha \mathbf{E}\xi. \quad (25)$$

Now let $k \rightarrow \infty$, then both integrals go to zero. For example, the first integral can be estimated in the following way, for every $A > 0$:

$$\int_{-\infty}^0 P(y, (0, \infty)) U_k(dy) \leq \int_{-\infty}^{-A} \mathbf{P}\{\eta > -y\} U(dy) + U_k(-A, 0].$$

Here, for any fixed A , $U_k(-A, 0] \rightarrow 0$ as $k \rightarrow \infty$ by (4). Therefore, it follows from (25) and (9) that $p_0 = \alpha \mathbf{E}\xi$. The proof of Theorem 1 is complete.

In the next theorem we provide some simple conditions sufficient for condition (8), that is, for local compactness of the renewal measure. Denote $a \wedge b = \min\{a, b\}$.

Theorem 2. *Suppose that there exists $A > 0$ such that*

$$\varepsilon \equiv \inf_{x \in \mathbf{R}} \mathbf{E}(\xi(x) \wedge A) > 0. \quad (26)$$

In addition, let

$$\delta \equiv \inf_{x \in \mathbf{R}} \mathbf{P}\{X_n > x \text{ for all } n \geq 1 | X_0 = x\} > 0. \quad (27)$$

Then $U(x, x+h] \leq (A+h)/\varepsilon\delta$ for all $x \in \mathbf{R}$ and $h > 0$; in particular, (8) holds.

Proof. Inequality (5) implies

$$\begin{aligned} U(x, x+h] &= \int_{\mathbf{R}} Q(y, (x, x+h]) \mu_0(dy) \\ &\leq \sup_{y \in (x, x+h]} Q(y, (x, x+h]). \end{aligned}$$

Therefore, it suffices to prove that

$$Q(y, (x, x+h]) \leq (A+h)/\varepsilon\delta \quad (28)$$

for all $y \in (x, x+h]$. Given $X_0 \in (x, x+h]$, consider the stopping time

$$\tau = \min\{n \geq 1 : X_n > x+h\}.$$

Since $X_\tau \wedge (x+h+A) - X_0 \leq A+h$ with probability 1,

$$\begin{aligned} A+h &\geq \mathbf{E}(X_\tau \wedge (x+h+A) - X_0) \\ &= \sum_{n=1}^{\infty} \mathbf{E}[X_n \wedge (x+h+A) - X_{n-1} \wedge (x+h+A)] \mathbf{I}\{\tau \geq n\}. \end{aligned}$$

Hence, the definition of τ implies

$$\begin{aligned} A+h &\geq \sum_{n=1}^{\infty} \mathbf{E}\{X_n \wedge (x+h+A) - X_{n-1} \wedge (x+h+A); \tau \geq n\} \\ &= \sum_{n=1}^{\infty} \mathbf{E}\{X_n \wedge (x+h+A) - X_{n-1} | \tau \geq n\} \mathbf{P}\{\tau \geq n\}. \end{aligned}$$

The Markov property and condition (26) yield

$$\mathbf{E}\{X_n \wedge (x+h+A) - X_{n-1} | \tau \geq n\} \geq \mathbf{E}(\xi(X_{n-1}) \wedge A) \geq \varepsilon$$

for all n . Therefore,

$$A+h \geq \varepsilon \sum_{n=1}^{\infty} \mathbf{P}\{\tau \geq n\} = \varepsilon \mathbf{E}\tau.$$

So, the expected number of visits to the interval $(x, x + h]$ till the first exit from $(-\infty, x + h]$ does not exceed $(A + h)/\varepsilon$, independently of the initial state $X_0 \in (x, x + h]$. By condition (27), after exit from $(-\infty, x + h]$ the chain is above the level X_τ forever with probability at least δ ; in particular, it does not visit the interval $(x, x + h]$ any more. With probability at most $1 - \delta$ the chain visits this interval again, and so on. Concluding, we get that the expected number of visits to the interval $(x, x + h]$ cannot exceed the value

$$\frac{A + h}{\varepsilon} \sum_{n=0}^{\infty} (1 - \delta)^n = \frac{A + h}{\varepsilon \delta},$$

and (28) is proved. The proof of Theorem 2 is complete.

The latter theorem yields the following

Corollary 1. *Let the family of jumps $\{\xi(x), x \in \mathbf{R}\}$ possess an integrable minorant with a positive mean, that is, there exist a random variable ζ such that $\mathbf{E}\zeta > 0$ and $\xi(x) \geq_{\text{st}} \zeta$ for any $x \in \mathbf{R}$. Then*

$$U(x, x + h] \leq (A + h)A/\varepsilon^2$$

for any $A > 0$ such that $\varepsilon \equiv \mathbf{E}(\zeta \wedge A) > 0$; in particular, (8) holds.

Proof. Consider the partial sums $Z_n = \zeta_1 + \dots + \zeta_n$ of an independent copies of ζ . Denote the first ascending ladder epoch by $\chi = \min\{n \geq 1 : Z_n > 0\}$. It is well known (see, for example, Theorem 2.3(c) in [3, Ch. VIII]) that

$$\mathbf{P}\{Z_n > 0 \text{ for all } n \geq 1\} = 1/\mathbf{E}\chi.$$

Since

$$\mathbf{P}\{X_n > x \text{ for all } n \geq 1 | X_0 = x\} \geq \mathbf{P}\{Z_n > 0 \text{ for all } n \geq 1\}$$

by the minorisation condition, the δ in Theorem 2 is at least $1/\mathbf{E}\chi$. Taking into account the inequality $\mathbf{E}\chi \leq A/\varepsilon$, we get $\delta \geq \varepsilon/A$, which implies the corollary conclusion.

If the chain X has a non-negative jumps $\xi(x) \geq 0$, then the minorisation condition is equivalent to the existence of a positive A such that

$$\gamma \equiv \inf_{x \in \mathbf{R}} \mathbf{P}\{\xi(x) > A\} > 0. \quad (29)$$

In that case one can choose ζ taking values 0 and A with probabilities $1 - \gamma$ and γ respectively; then $\varepsilon \geq \gamma A$ and $U(x, x + h] \leq (A + h)/\gamma^2 A$.

We conclude with a counterexample demonstrating that condition (7) in Theorem 1 is essential; the existence of integrable majorant cannot be relaxed to the condition of the uniform integrability of jumps. We consider

an integer valued chain X . For any state $k \in \mathbf{Z}^+$, define the transition probabilities in the following way: given $2^n - 1 \leq k \leq 2^{n+1} - 2$,

$$\begin{aligned} p_{k,k} &= 1/2, \\ p_{k,k+1} &= 1/2 - p_{k,2^{n+1}}, \\ p_{k,2^{n+1}} &= \frac{1}{(2^{n+1} - k) \ln(n + e^2)} \leq \frac{1}{2}. \end{aligned}$$

The corresponding jumps $\xi(k)$ converge weakly as $k \rightarrow \infty$ to the Bernoulli distribution; they are uniformly integrable. But we can observe a concentration of relatively large masses at points 2^{n+1} ; the renewal measure at point 2^{n+1} is not less than, up to a positive constant,

$$\sum_{k=2^n-1}^{2^{n+1}-2} \frac{1}{(2^{n+1} - k) \ln(n + e^2)} = \frac{1}{\ln(n + e^2)} \sum_{k=2}^{2^{n+1}} \frac{1}{k} \sim \frac{n \ln 2}{\ln n}.$$

Hence, there is no convergence $U\{k\} \rightarrow 1/\mathbf{E}\xi = 2$ and the key renewal theorem does not hold for the chain constructed.

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