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ASYMPTOTICS FOR RANDOM WALKS WITH DEPENDENT HEAVY-TAILED INCREMENTS

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Abstract: We consider a random walk $\{S_n\}$ with dependent heavy-tailed increments and negative drift. We study the asymptotics for the tail probability $\mathbf{P}\{\sup_n S_n > x\}$ as $x \to \infty$. If the increments of $\{S_n\}$ are independent then the exact asymptotic behavior of $\mathbf{P}\{\sup_n S_n > x\}$ is well known. We investigate the case in which the increments are given as a one-sided asymptotically stationary linear process. The tail behavior of $\sup_n S_n$ turns out to depend heavily on the coefficients of this linear process.

Keywords: random walk, dependent increment, heavy tails, subexponential distribution, tail asymptotics

§ 1. Introduction

In a series of recent papers, see e.g. [1–6], the tail behavior is studied for the supremum of negatively drifted random walks with dependent heavy-tailed increments. In the present article, we continue these studies and consider a stochastic model which can be justified as follows. Suppose that the nominal return of some manufacturing or financial system per unit time is equal to some constant $a > 0$. In a variety of practical situations, this nominal return is, however, not exactly achieved by the actual returns in the individual unit time period. We therefore assume that the actual return in the nth period is subject to some random perturbations $\eta_1, \dots, \eta_n$ with zero mean which arise in the first n periods due to unexpected claim costs or extra income. For example, the perturbation η_n incurred in the nth period may fail to be fully reported during that period and may also affect the actual returns of later periods. More precisely, the fraction $c_0\eta_n$ of η_n is reported in the nth period, the fraction $c_1\eta_n$ in the period $n+1$, the fraction $c_2\eta_n$ in the period $n+2$, and so on, where $c_0, c_1, \dots \in [0,1]$ with $\sum_{i=0}^{\infty} c_i = 1$. Thus, on supposing that the system begins to work at time zero, the actual return in the kth period is given by the expression $a - \sum_{j=1}^{k} c_{k-j}\eta_j$. Furthermore, the sum $S_n = \xi_1 + \dots + \xi_n$, where $\xi_k = \sum_{j=1}^{k} c_{k-j}\eta_j - a$, can be seen as the total (cumulative) claim surplus in the nth period.

The results proved in this paper are valid also under more general conditions on the coefficients $c_0, c_1, \dots$. Namely, they can be arbitrary fixed real numbers such that $\sum_{i=0}^{\infty} |c_i| < \infty$. The coefficients greater than one and negative coefficients could be interpreted, for example, as the declaration of too high costs and reimbursement in later periods, respectively.

The question of whether the claim surplus process $\{S_n,\ n \ge 1\}$ is "well-behaved" or dangerous is often answered by studying the asymptotics for the tail probability $\mathbf{P}\{\sup_n S_n > x\}$ as $x \to \infty$. In this article, we derive conditions under which the exact asymptotic behavior of $\mathbf{P}\{\sup_n S_n > x\}$ can be determined. It turns out that this asymptotic tail behavior depends heavily on the choice of the coefficients $c_0, c_1, \dots$.

1.1. The model. Let $\{\eta_n,\ n = 1, 2, \dots\}$ be a sequence of independent identically distributed random variables with zero mean, $\mathbf{E}\eta_n = 0$. The distribution of η_n will be denoted by F, i.e., $F(x) = \mathbf{P}\{\eta_n \le x\}$

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for $x \in \mathbf{R}$. The right tail of F is denoted by $\overline{F}(x) = 1 - F(x)$. We will also use the notations $(x \geq 0)$

$$G_+(x) = \int_x^\infty \overline{F}(y)\,dy \quad \text{and} \quad G_-(x) = \int_x^\infty F(-y)\,dy,$$

where the integrals are finite for each x by the existence of $\mathbf{E}\eta_n$. For real numbers $B \geq 0$ and $b \geq 0$, which are not both equal to 0, let $G_{B,b}$ be the following function (here $x/0 = \infty$ for $x \geq 0$):

$$G_{B,b}(x) = BG_+(x/B) + bG_-(x/b), \quad x \geq 0.$$

By definition, $G_+ = G_{1,0}$, $G_- = G_{0,1}$ and

$$G_{B,b}(x) = \int_x^\infty (\overline{F}(y/B) + F(-y/b))\,dy.$$

Let $a > 0$ and $c_k \in \mathbf{R}$, $k \in \mathbf{N}$, be some constants not all equal to 0; $\mathbf{N} = \{0, 1, \dots\}$. Let the random variable ξ_k be given by

$$\xi_k = \sum_{j=1}^k c_{k-j}\eta_j - a.$$

Consider the partial sums

$$S_0 = 0, \quad S_n = \xi_1 + \dots + \xi_n, \quad n \geq 1.$$

Then the sequence $\{S_n,\, n \in \mathbf{N}\}$ is called a *random walk with asymptotically stationary dependent increments and negative drift*. The following representation of the partial sums S_n is useful. With the notation

$$\bar{c}_k = \sum_{i=0}^k c_i, \quad k \in \mathbf{N}, \tag{1}$$

we have the representation in terms of sums of *weighted summands*:

$$S_n = \sum_{j=1}^n \bar{c}_{n-j}\eta_j - na. \tag{2}$$

We assume everywhere that

$$\sum_{k=0}^\infty |c_k| < \infty. \tag{3}$$

Under this condition, $\{S_n\}$ satisfies the strong law of large numbers, i.e., with probability 1, $S_n/n \to -a < 0$ as $n \to \infty$; see the corresponding elementary proof in Lemma 1 below. Hence, the supremum $\sup_{n\in\mathbf{N}} S_n$ of the random walk $\{S_n\}$ is a well-defined random variable finite with probability 1.

1.2. Main results. The purpose of this paper is to derive conditions under which the asymptotic behavior of the tail $\mathbf{P}\{\sup_{n\in\mathbf{N}} S_n > x\}$ can be related easily to the asymptotic behavior of the functions $G_+(x)$ and $G_-(x)$ as $x \to \infty$.

In Section 2, we derive an asymptotic lower bound for the probability $\mathbf{P}\{\sup_n S_n > x\}$. We prove in Theorem 2 that, if we take arbitrary different natural numbers $m_1, m_2 \in \mathbf{N}$ and put $C = \max\{0, \bar{c}_{m_1}\} \geq 0$ and $c = \min\{0, \bar{c}_{m_2}\} \leq 0$ then

$$\liminf_{x\to\infty} \frac{\mathbf{P}\{\sup_n S_n > x\}}{G_{C,|c|}(x)} \geq \frac{1}{a} \tag{4}$$

provided that $C + |c| > 0$ and $G_{C,|c|}$ is a *long-tailed function* (see Section 2 for the definition).

In Section 3, we obtain an asymptotic upper bound (Theorem 3). Let $\overline{C} = \sup\{0, \bar{c}_k, k \in \mathbf{N}\} \ge 0$ and $\bar{c} = \inf\{0, \bar{c}_k, k \in \mathbf{N}\} \le 0$, where $\overline{C} + |\bar{c}| > 0$ since not all c_k vanish. We prove that, if $G_{\overline{C},|\bar{c}|}$ belongs to the class $\mathscr{S}$ of *subexponential* distributions (see Section 3 for the definition) then

$$\limsup_{x\to\infty} \frac{\mathbf{P}\{\sup_n S_n > x\}}{G_{\overline{C},|\bar{c}|}(x)} \le \frac{1}{a}. \tag{5}$$

Combining (4) and (5), we immediately obtain the following asymptotic tail behavior of $\sup_n S_n$.

Theorem 1. *Let one of the following conditions hold:*
(i) *for some* m_1 *and* m_2,

$$\bar{c}_{m_1} = \overline{C} \equiv \sup\{0, \bar{c}_k, k \in \mathbf{N}\} > 0, \quad \bar{c}_{m_2} = \bar{c} \equiv \inf\{0, \bar{c}_k, k \in \mathbf{N}\} < 0;$$

(ii) $\overline{C} = \bar{c}_{m_1} > 0$ *for some* m_1, *and* $\bar{c} = 0$;
(iii) $\overline{C} = 0$ *and* $\bar{c} = \bar{c}_{m_2} < 0$ *for some* m_2.
If $G_{\overline{C},|\bar{c}|} \in \mathscr{S}$ *then*

$$\mathbf{P}\{\sup_n S_n > x\} \sim a^{-1} G_{\overline{C},|\bar{c}|}(x) \quad \text{as } x \to \infty.$$

In Section 4 we consider the rest of the possible cases unsettled by Theorem 1, namely, $\overline{C} > 0$ and $\overline{C} > \bar{c}_m$ for all m or $\bar{c} < 0$ and $\bar{c} < \bar{c}_m$ for all m. We show that then Theorem 1 remains valid under the additional condition of (intermediate) regular variation of tails at infinity.

Note that our results generalize the well-known theorem on the asymptotic tail behavior of the supremum of negatively drifted random walks with independent subexponential increments which concerns the case $c_0 = 1$, $c_1 = c_2 = \cdots = 0$ (see [7] and also [8–10]). Recently, some relevant extensions of this theorem to the case of random walks with dependent increments have been proved in [1–5]. An extension similar to our results was derived in [6], where F is assumed to have regularly varying left and right tails. This assumption of [6] is essential for the application of Karamata-type arguments. Our technique in Sections 2 and 3 is different and therefore we need not assume in Theorem 1 that F is regularly varying.

1.3. The strong law of large numbers. Let us formulate and prove the elementary law of large numbers for the sequence S_n defined by (2).

Lemma 1. $S_n/n \to -a$ *with probability* 1 *as* $n \to \infty$.

Proof. Condition (3) implies that the sequence $\bar{c}_n$ has a limit as $n \to \infty$, say $c \in \mathbf{R}$. Then, for every n and $N \in \mathbf{N}$ with $n \ge N$, we have

$$\frac{S_n + na}{n} = \frac{c}{n} \sum_{j=1}^{n} \eta_j + \frac{1}{n} \sum_{j=1}^{n-N} (\bar{c}_{n-j} - c)\eta_j + \sum_{j=0}^{N-1} (\bar{c}_j - c)\frac{\eta_{n-j}}{n}.$$

Since $\mathbf{E}|\eta_1|$ is finite, by the standard strong law of large numbers we have $\frac{c}{n}\sum_{j=1}^{n} \eta_j \to 0$ as $n \to \infty$ with probability 1. Furthermore, $|\eta_{n-j}|/n \to 0$ as $n \to \infty$ with probability 1 for each fixed $j \ge 0$. Hence, for each fixed $N \in \mathbf{N}$, we have

$$\limsup_{n\to\infty} \left| \frac{S_n + na}{n} \right| \le \limsup_{n\to\infty} \left| \frac{1}{n} \sum_{j=1}^{n-N} (\bar{c}_{n-j} - c)\eta_j \right|.$$

For every $\varepsilon > 0$ there exists N such that $|\bar{c}_n - c| \le \varepsilon$ for all $n \ge N$. Thus,

$$\limsup_{n\to\infty} \left| \frac{1}{n} \sum_{j=1}^{n-N} (\bar{c}_{n-j} - c)\eta_j \right| \le \varepsilon \limsup_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} |\eta_j| = \varepsilon \mathbf{E}|\eta_j|.$$

Since $\varepsilon > 0$ can be chosen arbitrarily small, the lemma is proved.

§ 2. The Lower Bounds

We first state some asymptotic properties of long-tailed distributions. They will be used in Subsection 2.2 in order to derive an asymptotic lower bound for the tail of the supremum of sums.

2.1. Properties of long-tailed distributions. Let $\mathscr{L}$ be the collection of all nonincreasing functions $f : \mathbf{R} \to (0, \infty)$ such that, for each $y \in \mathbf{R}$,

$$\lim_{x\to\infty} f(x+y)/f(x) = 1.$$

The distribution F is called *right long-tailed* if $\overline{F}(x) \in \mathscr{L}$. To simplify notation, we will write $F \in \mathscr{L}$ if the distribution F is right long-tailed. Note that $G_+ \in \mathscr{L}$ if $F \in \mathscr{L}$.

The distribution F is called *left long-tailed* if $F(-x) \in \mathscr{L}$. We denote by $\mathscr{L}^-$ the family of all distributions on $\mathbf{R}$ with this property. Notice that the distribution F of a random variable η belongs to $\mathscr{L}^-$ if and only if the distribution of $-\eta$ belongs to $\mathscr{L}$.

Lemma 2. *Let $f \in \mathscr{L}$. Then there exists an increasing function $g : \mathbf{R} \to \mathbf{R}_+ = [0, \infty)$ such that $g(x) \to \infty$ as $x \to \infty$ and*

$$\lim_{x\to\infty} f(x+g(x))/f(x) = 1.$$

PROOF. From the definition of the class $\mathscr{L}$ we see that there exists an increasing sequence of real numbers $\{x_n,\ n \ge 1\}$ such that $x_n \ge n$ and

$$f(x+n)/f(x) \ge 1 - 1/n \quad \text{for all} \quad x \ge x_n.$$

Define

$$g(x) = \begin{cases} 0 & \text{if } x < x_1, \\ n & \text{if } x_n \le x < x_{n+1}. \end{cases}$$

Since $x_n \to \infty$, we have $g(x) \to \infty$ as $x \to \infty$ and, for $x_n \le x < x_{n+1}$,

$$f(x+g(x))/f(x) \ge 1 - 1/n,$$

which implies

$$\liminf_{x\to\infty} f(x+g(x))/f(x) \ge 1.$$

On the other hand, for every nonnegative function g we have $f(x+g(x)) \le f(x)$. This completes the proof.

Corollary 1. *Assume that $f \in \mathscr{L}$. Then, for every function $g(x)$ provided by Lemma 2, we have*

$$\lim_{x\to\infty} \inf_{y\ge x} f(y+g(x))/f(y) = 1.$$

PROOF. Fix $\varepsilon > 0$. Then, by the result of Lemma 2, there exists an x_0 such that $f(x+g(x))/f(x) \ge 1-\varepsilon$ for all $x \ge x_0$. By the monotonicity of g, for each $y \ge x \ge x_0$, we thus obtain

$$f(y+g(x))/f(y) \ge f(y+g(y))/f(y) \ge 1 - \varepsilon.$$

Lemma 3. *Let the sequence $T_1, T_2, \ldots$ of random variables be such that $T_n/n \to 0$ as $n \to \infty$ with probability 1. Then there exists a nondecreasing function $h : \mathbf{N} \to \mathbf{R}_+$ such that $h(n) = o(n)$ as $n \to \infty$ and*

$$\lim_{z\to\infty} \mathbf{P}\Big\{ \bigcap_{n\ge1} \{|T_n| \le z + h(n)\} \Big\} = 1.$$

PROOF. Since $T_n/n \to 0$ with probability 1, there exists a sequence of integers $\{N_k,\ k \ge 1\}$ such that $N_k \to \infty$ and

$$\mathbf{P}\Big\{\bigcup_{n\ge N_k}\{|T_n| > n/k\}\Big\} \le 2^{-k} \tag{6}$$

for all $k = 1, 2, \ldots$, where without loss of generality we can assume that $N_{k+1} \ge N_k + 1$. Define

$$h^*(n) = \begin{cases} n & \text{if } n < N_1, \\ n/k & \text{if } N_k \le n < N_{k+1}. \end{cases} \tag{7}$$

Since $N_k \to \infty$, it follows that $h^*(n) = o(n)$. For each fixed $M \in \mathbf{N}$, we have

$$\mathbf{P}\Big\{\bigcup_{n\ge 1}\{|T_n| > z + h^*(n)\}\Big\} \le \sum_{n=1}^{N_M-1}\mathbf{P}\{|T_n| > z\} + \mathbf{P}\Big\{\bigcup_{n\ge N_M}\{|T_n| > h^*(n)\}\Big\}.$$

Therefore,

$$\limsup_{z\to\infty}\mathbf{P}\Big\{\bigcup_{n\ge 1}\{|T_n| > z + h^*(n)\}\Big\} \le \mathbf{P}\Big\{\bigcup_{n\ge N_M}\{|T_n| > h^*(n)\}\Big\}.$$

Using (6) and (7), we obtain the following estimates:

$$\begin{aligned}\mathbf{P}\Big\{\bigcup_{n\ge N_M}\{|T_n| > h^*(n)\}\Big\} &\le \sum_{k=M}^{\infty}\mathbf{P}\Big\{\bigcup_{N_k\le n<N_{k+1}}\{|T_n| > h^*(n)\}\Big\} \\ &\le \sum_{k=M}^{\infty}\mathbf{P}\Big\{\bigcup_{n\ge N_k}\{|T_n| > n/k\}\Big\} \le \sum_{k=M}^{\infty}2^{-k} = 2^{-M+1}.\end{aligned}$$

Since M is arbitrary, letting $M \to \infty$ yields

$$\limsup_{z\to\infty}\mathbf{P}\Big\{\bigcup_{n\ge 1}\{|T_n| > z + h^*(n)\}\Big\} = 0,$$

which is equivalent to

$$\lim_{z\to\infty}\mathbf{P}\Big\{\bigcap_{n\ge 1}\{|T_n| \le z + h^*(n)\}\Big\} = 1.$$

Putting now $h(n) \equiv \max\{h^*(k),\ k \le n\}$, we obtain a nondecreasing function $h(n) = o(n)$ which satisfies the assertion of the lemma.

Lemma 4. *Let $a > 0$ and $n_1 \ge 1$. Let $h : \mathbf{N} \to \mathbf{R}_+$ be a function such that $h(n) = o(n)$ as $n \to \infty$. If $G_+ \in \mathscr{L}$ then, as $z \to \infty$,*

$$\sum_{n=n_1}^{\infty}\overline{F}(z+na) \sim a^{-1}G_+(z); \qquad \sum_{n=n_1}^{\infty}\overline{F}(z+na+h(n)) \sim a^{-1}G_+(z).$$

PROOF. Given a distribution F, we have

$$\sum_{n=n_1}^{\infty}\overline{F}(z+na) \le \sum_{n=1}^{\infty}\int_{n-1}^{n}\overline{F}(z+ay)\,dy = \int_0^{\infty}\overline{F}(z+ay)\,dy = a^{-1}G_+(z). \tag{8}$$

On the other hand,

$$\sum_{n=n_1}^{\infty} \overline{F}(z+na) \geq \sum_{n=n_1}^{\infty} \int_{n}^{n+1} \overline{F}(z+ay)\,dy$$

$$= \int_{n_1}^{\infty} \overline{F}(z+ay)\,dy = a^{-1}G_+(z+an_1) \sim a^{-1}G_+(z)$$

as $z \to \infty$, since $G_+ \in \mathscr{L}$. The first equivalence of the lemma is thus proved. To prove the second equivalence, fix $\varepsilon > 0$. First, we have

$$\sum_{n=n_1}^{\infty} \overline{F}(z+na+h(n)) \leq \sum_{n=n_1}^{\infty} \overline{F}(z+na) \leq a^{-1}G_+(z).$$

Next, since $h(n) = o(n)$, there exists $N \geq n_1$ such that $h(n) \leq \varepsilon n$ for all $n \geq N$. Therefore,

$$\sum_{n=n_1}^{\infty} \overline{F}(z+na+h(n)) \geq \sum_{n=N}^{\infty} \overline{F}(z+n(a+\varepsilon)) \sim (a+\varepsilon)^{-1}G_+(z)$$

as $z \to \infty$, in view of the first equivalence of the lemma. Owing to the arbitrariness of $\varepsilon > 0$, this implies the second equivalence of the lemma.

Let $b_k \in \mathbf{R}$, $k \in \mathbf{N}$, be a bounded convergent sequence. Thus, the supremum $b = \sup_k |b_k|$ is finite. Put

$$T_n = \sum_{k=1}^{n} b_{n-k}\eta_k$$

and, for arbitrary natural numbers $n \geq 1$, $m \geq 0$, $n > m$, define

$$T_n^{(m)} = \sum_{k=1}^{n-m-1} b_{n-k}\eta_k.$$

By definition, for $n > m$ we have

$$T_n = T_n^{(m)} + \sum_{k=n-m}^{n} b_{n-k}\eta_k.$$

The sequences $\{T_n\}$ and $\{T_n^{(m)}\}$ fulfill the condition of Lemma 3. Indeed, since $\mathbf{E}\eta_i = 0$, we have $\lim_{n\to\infty} T_n/n = 0$ and $\lim_{n\to\infty} T_n^{(m_1)}/n = 0$ with probability 1 by the strong law of large numbers (see Lemma 1). Hence, for every function $g(x)$ with $g(x) \to \infty$, there exists a function $h(n)$ such that $h(n) = o(n)$ and

$$\lim_{x\to\infty} \mathbf{P}\Big\{\bigcap_{n\geq 1}\{|T_n| \leq g(x) + h(n)\}\Big\} = 1, \tag{9}$$

$$\lim_{x\to\infty} \mathbf{P}\Big\{\bigcap_{n\geq m_1}\{|T_n^{(m)}| \leq g(x) + h(n)\}\Big\} = 1. \tag{10}$$

Further, for $n > m_1 \in \mathbf{N}$ we define the event

$$B_n = \bigcap_{j=1}^{n-m_1-1}\{|T_j| \leq g(x)+h(j)\} \cap \{|T_n^{(m_1)}| \leq g(x)+h(n)\}$$

$$\cap\{b_{m_1}\eta_{n-m_1} > x+(2+m_1b)g(x)+na+2h(n)\} \cap \bigcap_{j=n-m_1+1}^{n}\{|\eta_j| \leq g(x)\},$$

and for $n > m_2 \in \mathbf{N}$, the event

$$B_n^- = \bigcap_{j=1}^{n-m_2-2} \{|T_j| \le g(x){+}h(j)\} \cap \bigl\{\bigl|T_n^{(m_2)}\bigr| \le g(x){+}h(n)\bigr\}$$

$$\cap \{b_{m_2}\eta_{n-m_2} > x{+}(2{+}m_2 b)g(x){+}na{+}2h(n)\} \cap \bigcap_{j=n-m_2+1}^{n} \{|\eta_j| \le g(x)\}.$$

Lemma 5. *Let m_1, $m_2 \in \mathbf{N}$ be natural numbers such that $b_{m_1} \ge 0$ and $b_{m_2} \le 0$. Then the events B_n, $n > m_1$, and B_n^-, $n > m_2$, are pairwise disjoint.*

Proof. Consider, for example, any two events among $\{B_n, n > m_1\}$, say, B_k and B_n, $m_1 < k < n$. If $n \le k + m_1$ then for $\omega \in B_n$ we have

$$b_{m_1}\eta_{n-m_1}(\omega) > x + (2{+}m_1 b)g(x) + na + 2h(n) \ge b_{m_1} g(x),$$

whereas for $\omega \in B_k$,

$$b_{m_1}\eta_{n-m_1}(\omega) \le b_{m_1}|\eta_{n-m_1}(\omega)| \le b_{m_1} g(x).$$

If $n > k + m_1$ then for $\omega \in B_k$ we have

$$T_k(\omega) = T_k^{(m_1)}(\omega) + b_{m_1}\eta_{k-m_1}(\omega) + \sum_{j=k-m_1+1}^{k} b_{k-j}\eta_j(\omega)$$

$$> -g(x) - h(k) + x + (2{+}m_1 b)g(x) + ka + 2h(k) - m_1 b g(x) \ge g(x) + h(k),$$

whereas for $\omega \in B_n$ we have $|T_k(\omega)| \le g(x) + h(k)$. The rest of the proof follows by similar arguments.

Lemma 6. *Let $b_{m_1} > 0$ and $G_+ \in \mathscr{L}$. Let $g(x) \to \infty$ be a function such that*

$$G_+((x + g(x))/b_{m_1}) \sim G_+(x/b_{m_1}) \quad \text{as} \ \ x \to \infty.$$

Then

$$\lim_{x\to\infty} \frac{\mathbf{P}\{\bigcup_{n>m_1} B_n\}}{b_{m_1} G_+(x/b_{m_1})} = \frac{1}{a}.$$

Proof. The function $g(x)$ exists due to Lemma 2. By Lemma 5,

$$\mathbf{P}\Bigl\{\bigcup_{n>m_1} B_n\Bigr\} = \sum_{n>m_1} \mathbf{P}\{B_n\}$$

$$= \sum_{n>m_1} \mathbf{P}\Biggl\{\bigcap_{j=1}^{n-m_1-1} \{|T_j| \le g(x){+}h(j)\} \cap \bigl\{\bigl|T_n^{(m_1)}\bigr| \le g(x){+}h(n)\bigr\}\Biggr\}$$

$$\times \mathbf{P}^{m_1}\{|\eta_1| \le g(x)\} \mathbf{P}\{b_{m_1}\eta_{n-m_1} > x{+}(2{+}m_1 b)g(x){+}na{+}2h(n)\}.$$

This gives the upper bound

$$\mathbf{P}\Bigl\{\bigcup_{n>m_1} B_n\Bigr\} \le \sum_{n>m_1} \mathbf{P}\{b_{m_1}\eta_1 > x + na\} \tag{11}$$

and the lower bound

$$\mathbf{P}\Bigl\{\bigcup_{n>m_1} B_n\Bigr\}$$

$$\ge \mathbf{P}\Bigl\{\bigcap_{n\ge 1} \{|T_n| \le g(x){+}h(n)\} \cap \bigcap_{n\ge m_1} \bigl\{\bigl|T_n^{(m_1)}\bigr| \le g(x){+}h(n)\bigr\}\Bigr\}$$

$$\times \mathbf{P}^{m_1}\{|\eta_1| \le g(x)\} \sum_{n>m_1} \mathbf{P}\{b_{m_1}\eta_1 > x{+}(2{+}m_1 b)g(x){+}na{+}2h(n)\}. \tag{12}$$

We have the convergence $\mathbf{P}\{|\eta_1| \le g(x)\} \to 1$ as $x \to \infty$, as well those of (9) and (10). Hence, (11), (12), and Lemma 4 with $z = (x + (2{+}m_1 b)g(x))/b_{m_1}$ lead to the assertion of the lemma.

Lemma 7. *Let $b_{m_1} \ge 0$, $b_{m_2} \le 0$, and $b_{m_1} + |b_{m_2}| > 0$. Let $G_{b_{m_1},|b_{m_2}|} \in \mathscr{L}$ and let $g(x) \to \infty$ be a function such that, with $m = \max\{m_1, m_2\}$,*

$$G_{b_{m_1},|b_{m_2}|}(x + (2+mb)g(x)) \sim G_{b_{m_1},|b_{m_2}|}(x) \quad \text{as } x \to \infty.$$

Then

$$\liminf_{x\to\infty} \frac{\mathbf{P}\{\bigcup_{n>m} (B_n \cup B_n^-)\}}{G_{b_{m_1},|b_{m_2}|}(x)} \ge \frac{1}{a}.$$

Remark. Notice that $G_{B,b} \in \mathscr{L}$ if both $G_+ \in \mathscr{L}$ and $G_- \in \mathscr{L}^-$. Another sufficient condition for $G_{B,b} \in \mathscr{L}$ is $G_+ \in \mathscr{L}$ and $G_-(x/b) = o(G_+(x/B))$ as $x \to \infty$. The function $g(x)$ in Lemma 7 exists, since $(2+mb)g(x)$ can be taken as the function $g(x)$ in Lemma 2.

Proof of Lemma 7. By Lemma 5,

$$\mathbf{P}\Big\{\bigcup_{n>m} (B_n \cup B_n^-)\Big\} = \mathbf{P}\Big\{\bigcup_{n>m} B_n\Big\} + \mathbf{P}\Big\{\bigcup_{n>m} B_n^-\Big\}.$$

Following now the guidelines of the proof of Lemma 6, we deduce that for every $\varepsilon > 0$ there exists x_0 such that, for $x \ge x_0$,

$$\mathbf{P}\Big\{\bigcup_{n>m_1} B_n\Big\} \ge (1-\varepsilon) \sum_{n>m_1} \mathbf{P}\{b_{m_1}\eta_1 > x + (2+mb)g(x) + na + 2h(n)\};$$

$$\mathbf{P}\Big\{\bigcup_{n>m_2} B_n^-\Big\} \ge (1-\varepsilon) \sum_{n>m_2} \mathbf{P}\{|b_{m_2}|\eta_1 < -x - (2+mb)g(x) - na - 2h(n)\}.$$

Therefore, according to Lemma 4,

$$\begin{aligned}\mathbf{P}\Big\{\bigcup_{n>m} (B_n \cup B_n^-)\Big\} &\ge (1-\varepsilon) \sum_{n>m} \bigg(\overline{F}\bigg(\frac{x+(2+mb)g(x)+na+2h(n)}{b_{m_1}}\bigg) \\ &\qquad + F\bigg(-\frac{x+(2+mb)g(x)+na+2h(n)}{|b_{m_2}|}\bigg)\bigg) \\ &\sim (1-\varepsilon)a^{-1}G_{b_{m_1},|b_{m_2}|}(x + (2+mb)g(x)) \sim (1-\varepsilon)a^{-1}G_{b_{m_1},|b_{m_2}|}(x).\end{aligned}$$

2.2. Asymptotic lower bounds for the tail of the supremum. We are now in a position to derive an asymptotic lower bound for the tail $\mathbf{P}\{\sup_n S_n > x\}$ as $x \to \infty$.

Theorem 2. *Let m_1, $m_2 \in \mathbf{N}$ be arbitrary different natural numbers. Put $C = \max\{0, \bar{c}_{m_1}\} \ge 0$ and $c = \min\{0, \bar{c}_{m_2}\} \le 0$. If $C + |c| > 0$ and $G_{C,|c|} \in \mathscr{L}$ then*

$$\liminf_{x\to\infty} \frac{\mathbf{P}\{\sup_n S_n > x\}}{G_{C,|c|}(x)} \ge \frac{1}{a}. \tag{13}$$

Proof. Put $b_k = \bar{c}_k$ and $T_n = S_n + na$ in Lemma 7. Let $m = \max\{m_1, m_2\}$ and let $g(x) \to \infty$ be a function such that

$$G_{C,|c|}(x + (2+mb)g(x)) \sim G_{C,|c|}(x)$$

as $x \to \infty$; it exists due to Lemma 2 (see the remark after Lemma 7). For $n > m$, consider the events

$$\widetilde{B}_n = \big\{\big|T_n^{(m_1)}\big| \le g(x) + h(n)\big\} \cap \{C\eta_{n-m_1} > x + (2+m_1b)g(x) + na + 2h(n)\} \\ \cap \bigcap_{j=n-m_1+1}^{n} \{|\eta_j| \le g(x)\},$$

$$\widetilde{B}_n^- = \{|T_n^{(m_2)}| \le g(x) + h(n)\} \cap \{|c|\eta_{n-m_2} > x + (2{+}m_2 b)g(x) + na + 2h(n)\}$$
$$\cap \bigcap_{j=n-m_2+1}^{n} \{|\eta_j| \le g(x)\},$$

where $h(n)$ is the function considered in (9) and (10). By definition, $B_n \subseteq \widetilde{B}_n \subseteq \{S_n > x\}$ and $B_n^- \subseteq \widetilde{B}_n^- \subseteq \{S_n > x\}$. Thus,

$$\mathbf{P}\{\sup_n S_n > x\} \ge \mathbf{P}\Big\{\bigcup_n (B_n \cup B_n^-)\Big\}.$$

Now the assertion follows from Lemma 7.

The following statements are immediate consequences of the proof of Theorem 2.

Corollary 2. *Let $\bar{c}_m > 0$ for some $m \ge 0$ and $G_+ \in \mathscr{L}$. Then*

$$\liminf_{x\to\infty} \frac{\mathbf{P}\{\sup_n S_n > x\}}{\bar{c}_m G_+(x/\bar{c}_m)} \ge \frac{1}{a}.$$

Corollary 3. *Let $\bar{c}_m < 0$ for some $m \ge 0$ and $G_- \in \mathscr{L}^-$. Then*

$$\liminf_{x\to\infty} \frac{\mathbf{P}\{\sup_n S_n > x\}}{|\bar{c}_m| G_-(x/|\bar{c}_m|)} \ge \frac{1}{a}.$$

§ 3. The Upper Bounds

We first introduce the class of subexponential distributions. They will be used in this section in order to derive asymptotic upper bounds for the tail $\mathbf{P}\{\sup_n S_n > x\}$ as $x \to \infty$.

The distribution G on $\mathbf{R}_+$ is called *subexponential* if $G(x) < 1$ for all $x \ge 0$ and

$$\lim_{x\to\infty} \frac{\overline{G*G}(x)}{\overline{G}(x)} = 2, \tag{14}$$

where $\overline{G*G}(x)$ denotes the tail of the convolution

$$G*G(x) = \int_0^x G(x-y)\, G(dy).$$

We denote the family of all subexponential distributions by $\mathscr{S}$. It is well known that $\mathscr{S} \subset \mathscr{L}$.

To simplify notation, we will write $G_{B,b} \in \mathscr{S}$ if $G_{B,b}(x)/G_{B,b}(0)$, $x \ge 0$, is the tail of a subexponential distribution. In particular, $G_+ \in \mathscr{S}$ if the *integrated tail* $G_+(x)/G_+(0)$, $x \ge 0$, of the distribution function $F(x)$ is the tail of a subexponential distribution.

It is well known that the tail behavior of the supremum of partial sums of independent identically distributed random variables is given by

$$\mathbf{P}\bigg\{\sup_{n\ge1}\bigg\{\sum_{k=1}^{n} \eta_k - na\bigg\} > x\bigg\} \sim a^{-1} G_+(x) \quad \text{as } x \to \infty, \tag{15}$$

provided that $G_+ \in \mathscr{S}$; see [7] and also [8–10]. It turns out that $G_+ \in \mathscr{S}$ is not only sufficient, but also necessary for (15); see [11].

Lemma 8. *Let $b_k \in \mathbf{R}$, $k \in \mathbf{N}$, $B \ge \sup\{0, b_k, k \in \mathbf{N}\}$, and $b \le \inf\{0, b_k, k \in \mathbf{N}\}$. Suppose that the limit*

$$\lim_{k\to\infty} b_k = \tilde{b} \tag{16}$$

exists. If $B + |b| > 0$ and $G_{B,|b|} \in \mathscr{S}$ then

$$\limsup_{x\to\infty} \frac{\mathbf{P}\Bigl\{\sup\limits_n \Bigl\{\sum\limits_{k=1}^n b_{n-k}\eta_k - na\Bigr\} > x\Bigr\}}{G_{B,|b|}(x)} \le \frac{1}{a}.$$

Remark. The condition $G_{B,|b|} \in \mathscr{S}$ is fulfilled, e.g., if $G_+ \in \mathscr{S}$ and $G_-(x/b) = (\gamma + o(1))G^+(x/B)$ as $x \to \infty$ for some $\gamma \ge 0$.

Proof of Lemma 8. Our argument is based on a truncation technique. For a real $z > 0$ and for a random variable η with distribution F, put

$$\eta^{[z]}(\omega) \equiv \begin{cases} B\eta(\omega) & \text{if } \eta(\omega) > z, \\ \tilde{b}\eta(\omega) & \text{if } -z \le \eta(\omega) \le z, \\ b\eta(\omega) & \text{if } \eta(\omega) < -z. \end{cases}$$

For $x > \max\{B, -b, |\tilde{b}|\}z$,

$$\mathbf{P}(\eta^{[z]} > x) = \mathbf{P}(B\eta > x) + \mathbf{P}(b\eta > x) = \overline{F}(x/B) + F(-x/|b|). \tag{17}$$

Since $G_{B,|b|} \in \mathscr{S}$, the integrated tail distribution of $\eta_1^{[z]}$ is subexponential. Furthermore, for $\omega \in \Omega$ and $b' \in [b, B]$, we have

$$\begin{aligned} b'\eta(\omega) &\le \begin{cases} B\eta(\omega) & \text{if } \eta(\omega) > z, \\ b'\eta(\omega) & \text{if } -z < \eta(\omega) \le z, \\ b\eta(\omega) & \text{if } \eta(\omega) < -z \end{cases} \\ &= \begin{cases} \eta^{[z]}(\omega) & \text{if } \eta(\omega) > z, \\ \eta^{[z]}(\omega) + (b' - \tilde{b})\eta(\omega) & \text{if } -z < \eta(\omega) \le z, \\ \eta^{[z]}(\omega) & \text{if } \eta(\omega) < -z \end{cases} \\ &\le \eta^{[z]}(\omega) + |\tilde{b} - b'|z. \end{aligned}$$

Therefore,

$$\sum_{k=1}^n b_{n-k}\eta_k \le \sum_{k=1}^n \eta_k^{[z]} + z\sum_{k=0}^{n-1} |\tilde{b} - b_k|.$$

Fix $\varepsilon \in (0, a/2)$. Since $b_k \to \tilde{b}$, there exists K such that $|b_k - \tilde{b}| \le \varepsilon$ for all $k \ge K$. Hence,

$$\sum_{k=1}^n b_{n-k}\eta_k \le \sum_{k=1}^n \eta_k^{[z]} + z\sum_{k=0}^{K} |\tilde{b} - b_k| + n\varepsilon \equiv \sum_{k=1}^n \eta_k^{[z]} + \hat{b}z + n\varepsilon,$$

where

$$\hat{b} \equiv \sum_{k=0}^{K} |\tilde{b} - b_k|.$$

Since $\mathbf{E}\eta_1 = 0$, there exists a sufficiently large $z > 0$ such that $\mathbf{E}\eta_1^{[z]} \le \varepsilon$. In view of (15) and (17), we have

$$\mathbf{P}\biggl(\sup_{n\ge1}\biggl\{\sum_{k=1}^n \eta_k^{[z]} - na\biggr\} > x\biggr) \sim \frac{1}{a - \mathbf{E}\eta_1^{[z]}} G_{B,|b|}(x) \quad \text{as} \quad x \to \infty.$$

Hence,

$$\mathbf{P}\left(\sup_n\left\{\sum_{k=1}^n b_{n-k}\eta_k - na\right\} > x\right) \le \mathbf{P}\left(\sup_n\left\{\sum_{k=1}^n \eta_k^{[z]} - n(a-\varepsilon)\right\} > x - \hat{b}z\right)$$
$$\le \frac{1+o(1)}{a-2\varepsilon} G_{B,|b|}(x-\hat{b}z) \sim \frac{1}{a-2\varepsilon} G_{B,|b|}(x).$$

Since $\varepsilon > 0$ was chosen arbitrarily, the proof is complete.

The last lemma implies the following asymptotic upper bound for the tail $\mathbf{P}\{\sup_n S_n > x\}$.

Theorem 3. *Let*

$$\overline{C} = \sup\{0, \bar{c}_k, k \in \mathbf{N}\} \ge 0, \quad \bar{c} = \inf\{0, \bar{c}_k, k \in \mathbf{N}\} \le 0.$$

If $G_{\overline{C},|\bar{c}|} \in \mathscr{S}$ *then*

$$\limsup_{x\to\infty} \frac{\mathbf{P}\{\sup_n S_n > x\}}{G_{\overline{C},|\bar{c}|}(x)} \le \frac{1}{a}. \tag{18}$$

Proof follows from Lemma 8 with $b_k = \bar{c}_k$, $B = \overline{C}$, and $b = \bar{c}$. Condition (16) holds because of (3).

In the case when the coefficients $\bar{c}_k$ are either all nonnegative or all nonpositive, we obtain the following two immediate consequences of Theorem 3.

Corollary 4. *Assume that* $\bar{c}_k \ge 0$ *for all* $k \in \mathbf{N}$. *Let* $G_+ \in \mathscr{S}$ *and* $\overline{C} = \sup_k \bar{c}_k > 0$. *Then*

$$\limsup_{x\to\infty} \frac{\mathbf{P}\{\sup_n S_n > x\}}{\overline{C} G_+(x/\overline{C})} \le \frac{1}{a}.$$

Corollary 5. *Assume that* $\bar{c}_k \le 0$ *for all* $k \in \mathbf{N}$. *Let* $G_- \in \mathscr{S}$ *and* $\bar{c} = \inf_k \bar{c}_k < 0$. *Then*

$$\limsup_{x\to\infty} \frac{\mathbf{P}\{\sup_n S_n > x\}}{|\bar{c}| G_-(x/|\bar{c}|)} \le \frac{1}{a}.$$

§ 4. Asymptotics for Regularly Varying Tails

In this section we study the rest of the possible cases unsettled by Theorem 1.

A function $f : \mathbf{R}_+ \to \mathbf{R}_+$ is said to be *intermediate regularly varying* if

$$\lim_{\delta\downarrow 0} \lim_{x\to\infty} \frac{f(x(1+\delta))}{f(x)} = 1. \tag{19}$$

We denote the family of all functions satisfying (19) by $\mathscr{IR}$. For example, the functions regularly varying at infinity belong to the class $\mathscr{IR}$. If a distribution G has an intermediate regularly varying tail then $G \in \mathscr{S}$.

Theorem 4. *Let* $G_{\overline{C},|\bar{c}|} \in \mathscr{S}$ *and assume that one of the following conditions hold:*

(i) $\overline{C} > 0$, $\overline{C} > \bar{c}_m$ *for all* m, $\bar{c} = \bar{c}_{m_2} < 0$ *for some* m_2, *and* $G_+ \in \mathscr{IR}$;

(ii) $\overline{C} = \bar{c}_{m_1} > 0$ *for some* m_1, $\bar{c} < 0$, $\bar{c} < \bar{c}_m$ *for all* m, *and* $G_- \in \mathscr{IR}$;

(iii) $\overline{C} > 0$, $\overline{C} > \bar{c}_m$ *for all* m, $\bar{c} < 0$, $\bar{c} < \bar{c}_m$ *for all* m, *and* $G_{\overline{C},|\bar{c}|} \in \mathscr{IR}$.

Then

$$\mathbf{P}\{\sup_n S_n > x\} \sim a^{-1} G_{\overline{C},|\bar{c}|}(x) \quad \text{as } x \to \infty. \tag{20}$$

Proof. Fix $\varepsilon > 0$ and suppose that condition (i) is fulfilled. By (19), there exist $\delta \in (0, \varepsilon]$ and $x_0 > 0$ such that for $x \geq x_0$

$$\frac{G_+(x/(\overline{C} - \delta))}{G_+(x/\overline{C})} \geq 1 - \varepsilon. \tag{21}$$

Since $\sup_{k \geq 0} \bar{c}_k = \overline{C}$, there exists k_0 such that $\bar{c}_{k_0} \geq \overline{C} - \delta$. Now it follows from (13) that

$$\liminf_{x \to \infty} \frac{\mathbf{P}\{\sup_n S_n > x\}}{G_{\bar{c}_{k_0}, |\bar{c}_{m_2}|}(x)} \geq \frac{1}{a}. \tag{22}$$

Using the equalities

$$\frac{G_{\bar{c}_{k_0}, |\bar{c}_{m_2}|}(x)}{G_{\overline{C}, |\bar{c}|}(x)} = \frac{G_{\bar{c}_{k_0}, |\bar{c}|}(x)}{G_{\overline{C}, |\bar{c}|}(x)} = \frac{\bar{c}_{k_0} G_+(x/\bar{c}_{k_0}) + |\bar{c}| G_-(x/|\bar{c}|)}{\overline{C} G_+(x/\overline{C}) + |\bar{c}| G_-(x/|\bar{c}|)}$$

and (21), we find that for $x \geq x_0$

$$\frac{G_{\bar{c}_{k_0}, |\bar{c}_{m_2}|}(x)}{G_{\overline{C}, |\bar{c}|}(x)} \geq \frac{(\overline{C} - \delta)(1 - \varepsilon) G_+(x/\overline{C}) + |\bar{c}| G_-(x/|\bar{c}|)}{\overline{C} G_+(x/\overline{C}) + |\bar{c}| G_-(x/|\bar{c}|)}$$
$$\geq (\overline{C} - \delta)(1 - \varepsilon)/\overline{C} \geq (\overline{C} - \varepsilon)(1 - \varepsilon)/\overline{C}.$$

Since $\varepsilon > 0$ is arbitrary, it follows from (22) that

$$\liminf_{x \to \infty} \frac{\mathbf{P}\{\sup_n S_n > x\}}{G_{\overline{C}, |\bar{c}|}(x)} \geq \frac{1}{a}.$$

Combining this inequality with the upper bound (18), we come to (20). The proof for the cases (ii) and (iii) can be carried out in the same way.

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