THEORY PROBAB. APPL. Translated from Russian Journal
Vol. 41, No. 1

LARGE-DEVIATION PROBABILITIES FOR ONE-DIMENSIONAL MARKOV CHAINS PART 1: STATIONARY DISTRIBUTIONS*

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(*Translated by K. A. Borovkov*)

Abstract. In this paper, we consider time-homogeneous and asymptotically space-homogeneous Markov chains that take values on the real line and have an invariant measure. Such a measure always exists if the chain is ergodic. In this paper, we continue the study of the asymptotic properties of $\pi([x,\infty))$ as $x\to\infty$ for the invariant measure π, which was started in [A. A. Borovkov, *Stochastic Processes in Queueing Theory*, Springer-Verlag, New York, 1976], [A. A. Borovkov, *Ergodicity and Stability of Stochastic Processes*, TVP Science Publishers, Moscow, to appear], and [A. A. Brovkov and D. Korshunov, *"Ergodicity in a sense of weak convergence, equilibrium-type identities and large deviations for Markov chains,"* in Probability Theory and Mathematical Statistics, Coronet Books, Philadelphia, 1984, pp. 89–98]. In those papers, we studied basically situations that lead to a purely exponential decrease of $\pi([x,\infty))$. Now we consider two remaining alternative variants: the case of "power" decreasing of $\pi([x,\infty))$ and the "mixed" case when $\pi([x,\infty))$ is asymptotically $l(x)e^{-\beta x}$, where $l(x)$ is an integrable function regularly varying at infinity and $\beta>0$.

Key words. Markov chain, invariant measure, rough and exact asymptotic behavior of large-deviation probabilities

1. Introduction. Let $X(n)=X(y,n)\in\mathbf{R}$, $n=0,1,\dots$, be a Markov chain homogeneous in time with initial value $X(y,0)=y\in\mathbf{R}$ and transition probabilities $P(y,B)=P(y,1,B)$, $P(y,n,B)=\mathbf{P}\{X(y,n)\in B\}$, $n\geqslant 1$, $B\in\mathcal{B}(\mathbf{R})$, where $\mathcal{B}(\mathbf{R})$ is the σ-algebra of Borel sets on the line. In the first and last sections of the paper (sections 1, 2, 8, and 9), we shall assume that the chains under consideration possess the property of asymptotic homogeneity in space, i.e., that the distributions $P(y,y+B)$ converge weakly as $y\to\infty$ to a distribution $F(B)$. If we denote by $\xi(y)=X(y,1)-y$ the increment of the chain in one step and by ξ the random variable having the distribution $F(B)$, then the above property means weak convergence of the distribution $\xi(y)$ as $y\to\infty$ to the distribution of ξ.

Furthermore, we shall assume in the first part of the paper that there exists a probability invariant measure $\pi(\cdot)$ (generally speaking, not unique), i.e., a measure satisfying the equations

$$\pi(B)=\int_{\mathbf{R}}\pi(du)\,P(u,B),\quad \pi(\mathbf{R})=1. \tag{1.1}$$

The main object under study in the first part of the paper will be the asymptotic behavior of $\pi(x)=\pi([x,\infty))$ as $x\to\infty$. In the second part, we shall study the asymptotic behavior of $P(x_0,n,[x,\infty))$ as $x\to\infty$.

To simplify the exposition, we shall restrict ourselves, as a rule, to considering chains assuming values on the positive half-line.

As we shall see (cf. also [3, section 23]), studying large deviations for π in the general case can be reduced to the case $X(n)\geqslant 0$.

*Received by the editors February 10, 1995. This research was partially supported by Russian Foundation for Fundamental Research project 93-01-01416 and International Science Foundation grant RAC000, 1994.

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Since an invariant measure π always exists if the chain is ergodic, it will exist if $\mathbf{E}\xi < 0$ and $X(n)$ is a nonnegative Harris chain (see, e.g., [13] and [3]). In the first part of the paper, condition $\mathbf{E}\xi < 0$ will always be assumed to be satisfied. We shall not exclude the case $\mathbf{E}\xi = -\infty$ unless otherwise specified.

To characterize (rather roughly) the possible asymptotic behavior of $\pi([x,\infty))$, one could distinguish the following three basic cases.

Set $\pi(x) = \pi([x,\infty))$,

$$\varphi(\mu) = \mathbf{E}e^{\mu\xi}, \quad \mu_+ = \sup\{\mu \geqslant 0\colon\ \varphi(\mu) < \infty\}, \tag{1.2}$$

$$F(t) \equiv F\big([t,\infty)\big) = \mathbf{P}\{\xi \geqslant t\}, \qquad G(t) = \int_t^\infty F(u)\,du; \tag{1.3}$$

we shall write $f(y) \sim g(y)$ as $y \to \infty$ if

$$\lim_{y\to\infty} \frac{f(y)}{g(y)} = 1$$

exists.

The functions $F(t)$ and $G(t)$ are sometimes called a "tail" and a "double-tail" of the distribution of ξ.

1) $\mu_+ > 0$, $\varphi(\mu_+) > 1$. (The possibility $\varphi(\mu_+) = 1$, $\varphi'(\mu_+) < \infty$ is also included; recall that the assumption $\mathbf{E}\xi < 0$ implies that $\varphi(\mu) - 1$ is negative for small μ.) In that case, under certain additional rather broad assumptions on the distributions of $\xi(y)$ for finite y, one will have a purely exponential decreasing $\pi(x) \sim ce^{-\beta x}$, where $\beta > 0$ is the solution of the equation $\varphi(\beta) = 1$, $c = \text{const}$

2) In the other cases ($\mu_+ = 0$ or $\mu_+ > 0$, $\varphi(\mu_+) < 1$),

$$\pi(x) \sim c\,G(x). \tag{1.4}$$

(Moreover, under certain additional rather broad assumptions, for $\mu_+ > 0$, instead of (1.4), one can also write, $\pi(x) \sim c\mathbf{P}\{\xi \geqslant x\}$).

3) The above-mentioned additional assumptions require basically that, for finite y, $\mathbf{P}\{\xi(y) \geqslant t\}$ decreases fast enough as t increases (not slower than a certain given function, which sometimes coincides with $\mathbf{P}\{\xi \geqslant t\}$). If this assumption is contradicted, then the asymptotic behavior of $\pi(x)$ can be determined by the distributions of $\xi(y)$ for finite y rather than by the distribution of ξ (see, e.g., Theorem 2 and corollaries to it). In this case the nature of the asymptotic behavior of $\pi(x)$ can be very complicated.

In [3] and [5] we obtained, for a special class of Markov chains possessing the so-called partial homogeneity property (see below), an explicit representation for $\pi(x)$. This representation allows one to analyze the asymptotic behavior of $\pi(x)$ in all three above-mentioned cases. In the same papers, the asymptotic behavior of $\pi(x)$ was extensively studied in the first case.

In what follows, we shall principally study the asymptotic behavior of $\pi(x)$ in the second and, to some extent, third cases. To give an exhaustive picture, we shall list all the known results concerning the asymptotics of $\pi(x)$.

From the point of view of applications, one could mention at least two areas where the obtained results could be applied.

(a) In many applications, $X(n)$ describes the "load" of a certain physical system, and one needs to know what is the probability that this load, in the stationary regime, will exceed a given high level x. And this is just the probability $\pi(x)$, the approximation of which is studied here.

(b) The results obtained enable one to establish the existence of the "moments" $\mathbf{E}f(X(n))$ for a given class of increasing functions f, to use these moments in optimisation problems of different kind, and to estimate these moments with the help of the Monte Carlo method. Moreover, if one has already established that $\pi(x) \sim cx^{-\alpha}e^{-\beta x}$, where some of the parameters c, α, or β are unknown in explicit form, then one could use the Monte Carlo method to estimate these parameters as well (see, e.g., [4, Chapter 5, section 5], [6], [9], and [14]). To obtain more information about these estimates, it is useful to know the next term of the asymptotics of $\pi(x)$ as well, i.e., the asymptotic behavior $\pi(x) - cx^{-\alpha}e^{-\beta x}$ as $x \to \infty$ (see section 9).

From the mathematical point of view, the results that we give below are related to the poorly studied problem of asymptotic properties of the solution of equation (1.1) under rather broad assumptions on the kernel $P(y, \cdot)$. The methods we use are basically probabilistic. They have required developing new approaches which will be very useful for studying also large-deviation probabilities for multidimensional Markov chains.

2. Statement of main theorems on large-deviation probabilities.

2.1. Homogeneous chains. As we have already noted, the study of the asymptotic behavior of $\pi(x)$ for a broad class of chains was started in [2], [3], and [5]. For the sake of having a complete picture, we recall the main results. We begin with the simplest *homogeneous* random walks with holding state at zero, which are well studied and can often be encountered in applications. They have the form

$$X(n+1) = (X(n) + \xi_n)^+,$$

where $\{\xi_n\}$ is a sequence of independent random variables distributed as ξ, $a = \mathbf{E}\xi < 0$, and $x^+ = \max(0, x)$. Set

$$S_0 = 0, \quad S_k = \sum_{i=1}^{k} \xi_i, \quad S = \sup_{k \geqslant 0} S_k, \quad S^* = \sup_{k \geqslant 1} S_k. \tag{2.1}$$

An invariant distribution for the homogeneous chain exists if and only if $\mathbf{E}\xi < 0$, and if coincides in this case with the distribution of S.

Before stating the theorem on the asymptotic behavior of the probability $\pi(x) = \mathbf{P}\{S \geqslant x\}$, we introduce (following [2, pp. 122 and 132]) the notions of local power and upper power functions.

DEFINITION. *A function $f(y)$ is said to behave like a* local power (*be an* l.p. function) *if for any t,*

$$\frac{f(y+t)}{f(y)} \to 1 \quad \textit{as } y \to \infty. \tag{2.2}$$

An l.p. function $f(y)$ is called an upper power (u.p.) *function if for some $c_0 < \infty$ and all y and p, $\frac{1}{2} \leqslant p \leqslant 1$,*

$$\frac{f(py)}{f(y)} \leqslant c_0. \tag{2.3}$$

If an l.p. function $f(y)$ is nonincreasing, then it is u.p. if and only if for any $y \geqslant 0$,

$$\frac{f(y/2)}{f(y)} \leqslant c_0 < \infty. \tag{2.4}$$

When considering lattice distributions, we shall need the same definitions, but in which y and t run over the integer values only and p is substituted by its integral part $[py]$.

In the notation in (1.2), there are three possible cases: (a) $\mu_+ > 0$, $\varphi(\mu_+) \geqslant 1$; then there exists a unique root $\beta > 0$ of the equation $\varphi(\mu) = 1$, so that $\varphi(\mu) < 1$ for $\mu \in (0, \beta)$; (b) $\mu_+ > 0$, $\varphi(\mu_+) < 1$; then we put $\beta = \mu_+$; (c) $\mu_+ = 0$; in this case, we set $\beta = 0$. The parameter β can be defined in a unified way as

$$\beta = \sup\bigl\{\mu \geqslant 0\colon\ \varphi(\mu) \leqslant 1\bigr\}. \tag{2.5}$$

Theorem 1 (see [2, pp. 129 and 132]). (a) *If* $\varphi(\beta) \geqslant 1$ *and* $\varphi'(\beta) = \mathbf{E}\xi e^{\beta\xi} < \infty$, *then*

$$\pi(x) = e^{-\beta x}\bigl(c_1 + o(1)\bigr), \quad x \to \infty,$$

where $c_1 > 0$ *depends on the distribution of* ξ *and is known in explicit form* (see [2]). *If* $\varphi(\mu_+) > 1$ *and the distribution of* ξ *is lattice or has a nontrivial absolutely continuous component*, *then* $o(1)$ *can be replaced by* $o(e^{-\varepsilon x})$ *for some* $\varepsilon > 0$.

(b) *If* $\beta > 0$, $\varphi(\beta) < 1$ *and* $e^{\beta y}G(y)$ *is a u.p. function* (*i.e., satisfies conditions* (2.2) *and* (2.3)), *then* $\mathbf{E}e^{\beta S} < \infty$ *and*

$$\pi(x) \sim c_2 G(x), \quad x \to \infty;$$

$c_2 > 0$ *is also known in explicit form* (see below).

(c) *If* $\beta = 0$ *and* $G(y)$ *is a u.p. function* (*i.e., satisfies conditions* (2.2) *and* (2.4)), *then*

$$\pi(x) = \Bigl(-\frac{1}{\mathbf{E}\,\xi} + o(1)\Bigr)\, G(x), \quad x \to \infty.$$

It will follow from Theorem 5 that $c_2 = \beta(1 - \varphi(\beta))^{-1}\mathbf{E}e^{\beta S}$.

Remark 1. If $X(n)$ assumes only integer values, then the variables x and y in assertions (a) and (b) must also take on (as in Theorems 4 and 5 below as well) only integer values.

Remark 2. If $\beta > 0$ and $\widetilde{F}(t) \equiv e^{\beta t}F(t)$ is an l.p. function (i.e., satisfies condition (2.2)), then

$$G(t) \equiv \int_t^\infty F(u)\,du = \left(\frac{1}{\beta} + o(1)\right) F(t) \quad \text{as } t \to \infty.$$

In particular, in assertion (b) of the theorem, one can write

$$\pi(x) \sim \bigl(1 - \varphi(\beta)\bigr)^{-1}\mathbf{E}e^{\beta S}F(x).$$

This will be proved in section 8.

Assertions (b) and (c) of the theorem can be extended to the so-called subexponential distributions F.

Definition (see [7]). *We say that the distribution* $\widehat{F}(\cdot)$ *on* $[0,\infty)$ belongs to the class $\mathcal{S}(\beta)$, $\beta \geqslant 0$ *if* $e^{\beta t}\widehat{F}([t,\infty))$ *is an l.p. function*,

$$\widehat{\varphi}(\beta) = \int_0^\infty e^{\beta t}\,\widehat{F}(dt) < \infty,$$

and

$$\widehat{F}^{*(2)}\bigl([t,\,\infty)\bigr) \sim \widehat{c}\,\widehat{F}\bigl([t,\,\infty)\bigr),$$

as $t \to \infty$ *for some* $\widehat{c} > 0$ *or, in other words,*

$$\mathbf{P}\{\widehat{\xi}_1 + \widehat{\xi}_2 \geqslant t\} \sim \mathbf{P}\{\widehat{\xi}_1 \geqslant t\},$$

where $\widehat{\xi}_i$ *are independent and have distribution* $\widehat{F}$.

Functions from the class $\mathcal{S}(0)$ are called *subexponential* functions.

It is known (see, e.g., [7]) that necessarily $\widehat{c} = 2\widehat{\varphi}(\beta)$. It follows from Lemma 5 given in section 7 that, if $e^{\beta t}\widehat{F}([t,\infty))$ is a u.p. function and $\widehat{\varphi}(\beta) < \infty$, then the distribution $\widehat{F}$ belongs to the class $\mathcal{S}(\beta)$. The following modification of assertions (b) and (c) of Theorem 1 is true.

THEOREM 2A. (b) *If* $\beta > 0$, $\varphi(\beta) < 1$ *and the distribution of the random variable* $\xi I\{\xi \geqslant 0\}$ *belongs to the class* $\mathcal{S}(\beta)$, *then* $\mathbf{E}e^{\beta S} < \infty$ *and*

$$\pi(x) \sim c_2 G(x), \quad x \to \infty.$$

(c) *If* $\beta = 0$ *and the distribution of the random variable* $\xi I\{\xi \geqslant 0\}$ *is subexponential, then*

$$\pi(x) = \Big(-\frac{1}{\mathbf{E}}\,\xi + o(1)\Big) G(x), \quad x \to \infty.$$

Assertion (c) of the theorem can be found in [18], [7], and [8]; assertion (b) can be found in [12]. Note that in all subsequent theorems and lemmas in which we use the condition that some function is u.p., this condition can be relaxed to the condition that the corresponding distribution belongs to the class $\mathcal{S}(\beta)$.

2.2. Partially homogeneous chains. Now consider a broader class of chains called *almost homogeneous* (in space). They assume values in $\mathbf{R}^+$ and are defined by the relations

$$X(n+1) = \begin{cases} (X(n) + \xi_n)^+ & \text{for } X(n) > 0,\\ \eta_n & \text{for } X(n) = 0, \end{cases}$$

where $\{\xi_n\}$ and $\{\eta_n\}$ are two mutually independent sequences of independent random values distributed, respectively, like ξ and η, $a = \mathbf{E}\xi < 0$, $\mathbf{E}\eta < \infty$. Put

$$G^{(H)}(t) = \int_0^\infty \mathbf{P}\{\eta > t + u\}\, dH(u), \quad t > 0,$$

where $H(u)$ is the renewal function of the random variable χ, which is equal to the first positive value in the sequence $-S_1, -S_2, \dots$. Now consider the random variable γ which does not depend of S and has distribution $\mathbf{P}\{\gamma > t\} = G^{(H)}(t)/G^{(H)}(0)$.

THEOREM 2 (see [3] and [5]). *If* $-\infty < \mathbf{E}\xi < 0$, $\mathbf{E}\eta < \infty$, *then the chain* $X(n)$ *is ergodic and*

$$\pi(x) = c_3 \mathbf{P}\{S + \gamma \geqslant x\}, \quad x > 0, \quad \pi\big(\{0\}\big) = 1 - c_3,$$

where

$$c_3 = \frac{G^{(H)}(0)}{\mathbf{P}\{S^* < 0\} + G^{(H)}(0)} < 1.$$

Theorem 2 gives an explicit expression for $\pi(x)$ and a possibility to obtain a rather complete description of the asymptotic behavior of $\pi(x)$ depending on the asymptotic properties of $\mathbf{P}\{\gamma \geqslant t\}$ and $\mathbf{P}\{S \geqslant t\}$. Thus the following corollary holds (we use the notations in (2.1) and (2.5)).

Corollary 1 (see [3] and [5]). (a) *Let* $\beta > 0$. *If* $\varphi(\beta) \geqslant 1$, $\varphi'(\beta) < \infty$ (*this holds automatically if* $\varphi(\mu_+) > 1$) *and* $\mathbf{E}e^{\eta\beta} < \infty$, *then*

$$\mathbf{P}\{S \geqslant x\} \sim c_1 e^{-\beta x}, \qquad \pi(x) \sim c_3 c_1 e^{-\beta x}.$$

(b) *If* $\beta > 0$, $\varphi(\beta) < 1$, $e^{\beta t}G(t)$ *is a u.p. function, and* $\mathbf{P}\{\eta \geqslant t\} = o(\mathbf{P}\{\xi \geqslant t\})$, *then* $\mathbf{E}e^{\gamma\beta} < \infty$ *and*

$$\pi(x) \sim c_3 c_2 \mathbf{E}e^{\gamma\beta} G(x).$$

(c) *If* $\beta = 0$, $G(t)$ *is a u.p. function* (*i.e., it satisfies conditions* (2.2) *and* (2.4)), *and*

$$\int_t^\infty \mathbf{P}\{\eta \geqslant u\}\, du = o\big(G(t)\big) \quad \text{as } t \to \infty,$$

then

$$\pi(x) \sim \frac{G(x)c_3}{-\mathbf{E}\xi}.$$

The constants c_1 *and* c_2 *are taken from Theorem* 1, *and* c_3 *from Theorem* 2.

In this assertion, we have restrictions on the rate of the decrease of $\mathbf{P}\{\eta \geqslant t\}$ which reduce the role of η just to its influence on the constant factor. Now we give an alternative assertion which follows from Theorem 2 in which η plays the main role. (More precisely, assertion (a)–(b) follows from Theorems 2 and 3 and from Lemma 5 in section 7; assertion (c) follows from Theorem 10 of section 6.)

Corollary 2 [3, section 23]. (a)–(b) *Let* $\beta > 0$. *If* $e^{\mu t}\mathbf{P}\{\eta \geqslant t\}$ *is a u.p. function for some* $\mu \in [0, \beta)$, *and* $\mathbf{E}e^{\mu\eta} < \infty$, *then*

$$\pi(x) \sim c_3 \mathbf{E}e^{\mu S}\mathbf{P}\{\gamma \geqslant x\}.$$

(c) *Let* $\beta = 0$. *If* $G_\eta(t) = \int_t^\infty \mathbf{P}\{\eta > u\}\, du$ *is a u.p. function and* $G(t) = o(G_\eta(t))$, *then*

$$\pi(x) \sim (1 - c_3)\, \frac{G_\eta(x)}{-\mathbf{E}\xi}.$$

In a similar way, one could consider also other types of relations between the distributions of S and η. The basic rule can be expressed, roughly speaking, as follows. If $\mathbf{P}\{\gamma \geqslant x\} = o(\mathbf{P}\{S \geqslant x\})$, then the asymptotic behavior of $\pi(x)$ repeats, up to a constant factor, that of $\mathbf{P}\{S \geqslant x\}$. If $\mathbf{P}\{S \geqslant x\} = o(\mathbf{P}\{\gamma \geqslant x\})$, then the asymptotic behavior of $\pi(x)$ repeats, up to a constant factor, that of $\mathbf{P}\{\gamma \geqslant x\}$. The rule can be simplified even more by observing that, for "regular" distributions of η, the functions $G^{(H)}(t)$ and $G_\eta(t) = \int_t^\infty \mathbf{P}\{\eta > u\}\, du$ behave asymptotically in the same way (up to a constant factor). Hence, in the above rule, the function $\mathbf{P}\{\gamma \geqslant t\}$ (or $G^{(H)}(t)$) can be replaced by the double tail $G_\eta(t)$ of the distribution of η. In cases (b) and (c) of Theorem 1, $\mathbf{P}\{S \geqslant t\} \sim c'G(t)$, $c' = \mathrm{const}$, and determining the asymptotic behavior of $\pi(x)$ reduces to the comparison of the double tails of ξ and η. In case (a), everything is determined by the comparison of $G_\eta(t)$ and $e^{-\beta t}$.

The reason for giving a complete description of the asymptotic behavior of $\pi(x)$ for almost homogeneous Markov chains is that the picture is basically preserved for a much more general class of the so-called *partially homogeneous* chains.

Denote by $\xi(y) = X(y, 1) - y$ the one-step increment of the chain starting at the point y. A chain X that takes values in $\mathbf{R}$ is called N-*partially homogeneous* (or just *partially homogeneous*) if for $y \in (N, \infty)$ and $y + B \subset (N, \infty)$, the distribution $P(y, y + B) = F(B)$ does not depend on y. A random variable with the distribution

$F(\cdot)$ will be denoted by ξ. An almost homogeneous walk is 0-partially homogeneous. For $y \leqslant N$, the distribution of $\xi(y)$ can be arbitrary.

Consider a "merged" (or "averaged") chain $X^{(N)}$ with values in $[N, \infty)$, for which the state $\{N\}$ "corresponds" to the domain $(-\infty, N]$ for the chain X. Define the transition probabilities $P^{(N)}(y, B)$ of the chain $X^{(N)}$ by the relations

$$\begin{aligned} P^{(N)}(y,\, B) &= P(y,\, B) \quad \text{for } y > N,\ B \subset (N,\, \infty), \\ P^{(N)}\big(y,\, \{N\}\big) &= P\big(y,\, (-\infty,\, N]\big) \quad \text{for } y > N, \\ P^{(N)}(N,\, B) &= \int_{-\infty}^{N} \frac{\pi(dy)}{\pi((-\infty,\, N])}\, P(y,\, B) \quad \text{for } B \subset (N,\, \infty), \\ P^{(N)}\big(N,\, \{N\}\big) &= 1 - P^{(N)}\big(N,\, (N,\, \infty)\big). \end{aligned} \tag{2.6}$$

It is not hard to note (see also [3, section 23] and [11, section 7]) that the new chain $X^{(N)}$ possesses the property that there exists an invariant measure $\pi^{(N)}$ for this chain coinciding with π in the domain (N, ∞).

Therefore, from the point of view of the asymptotic properties of $\pi(x)$, the chains X and $X^{(N)}$ are equivalent as $x \to \infty$.

If a chain X is N-partially homogeneous, then the chain $X^{(N)} - N$ is almost homogeneous (0-partially homogeneous), and one can apply to it Theorem 2 and Corollaries 1 and 2, where one must think of $\eta \geqslant 0$ as a random variable with the distribution

$$\mathbf{P}\{\eta \geqslant t\} = \int_{-\infty}^{N} \frac{\pi(du)}{\pi((-\infty, N])}\, \mathbf{P}\big\{u + \xi(u) \geqslant N + t\big\}, \quad t > 0.$$

Therefore, if, say, the conditions of Corollary 1(a) concerning the variable ξ are satisfied and

$$\sup_{y \leqslant N} \mathbf{E} e^{(y+\xi(y))\beta} < \infty,$$

then the conditions of this corollary will also be satisfied for η (which applies to the chain $X^{(N)} - N$), and hence

$$\pi(x) \sim c_3 c_1 e^{-\beta(x-N)}.$$

The above facts allow one to claim that the problem of the asymptotic analyzis of $\pi(x)$ for partially homogeneous chains is rather well studied.

Note also that for the oscillating random walk (which is 0-partially homogeneous) the distribution of η, law π and constant c for the "enlarged" chain $X^{(0)}$ can be found in explicit form (see [4]).

2.3. Asymptotically homogeneous chains. The study of *asymptotically homogeneous* chains, that is, chains for which we know only that the distribution of $\xi(y)$ converges weakly as $y \to \infty$ to that of a random variable ξ is more complicated. We shall write this as $\xi(y) \Rightarrow \xi$ as $y \to \infty$. Here the variety of the asymptotic behavior of $\pi(x)$ can be very rich. However, after imposing several natural restrictions, the picture on the whole will be similar to that obtained in Corollaries 1 and 2.

First, we turn to rough asymptotics. As above, set $\beta = \sup\{\mu \geqslant 0\colon\ \varphi(\mu) \leqslant 1\}$.

THEOREM 3 (see [3] and [5]). *Let $\xi(y) \Rightarrow \xi$ as $y \to \infty$, and $\sup_y \mathbf{E} e^{\beta\xi(y)} < \infty$. Then*

$$\lim_{x \to \infty} \frac{\log \pi(x)}{x} = -\beta.$$

The theorem implies that the large-deviation principle holds (see, e.g., [17, p. 3]) with the deviation function $I(t) = -\beta t$.

Studying exact asymptotics of $\pi(x)$ requires stricter conditions. Only the Cramér case has been considered until now, that is, the case when there exists a $\beta > 0$ such that $\varphi(\beta) = 1$.

THEOREM 4 (see [3] and [5]). *Let* $\varphi(\beta) = 1$, $\varphi'(\beta) = \mathbf{E}\xi e^{\beta\xi} < \infty$, *and*

$$\int_{-\infty}^{\infty} e^{\beta t} \Big| \mathbf{P}\big\{\xi(y) < t\big\} - \mathbf{P}\{\xi < t\} \Big| \, dt \leqslant l(y), \tag{2.7}$$

where $l(y)$ *is regularly varying at infinity with the exponent* $-\alpha$, *i.e.*, $l(uy) \sim u^{-\alpha} l(y)$ *as* $y \to \infty$ *for any fixed* $u > 0$. *Further*, *let*

$$\int_0^{\infty} l(y)\, dy < \infty \tag{2.8}$$

(*so that* $\alpha \geqslant 1$). *Then*

$$\pi(x) = e^{-\beta x}\big(c_4 + o(1)\big), \quad 0 \leqslant c_4 < \infty, \quad x \to \infty. \tag{2.9}$$

Moreover, *if for all* y,

$$\pi(y) > 0 \quad \textit{and} \quad \mathbf{E}\, e^{\beta\xi(y)} \geqslant 1 - \gamma(y), \tag{2.10}$$

where

$$\gamma(y) \geqslant 0 \quad \textit{and} \quad \int_1^{\infty} \gamma(y)\, y(\log y)\, dy < \infty,$$

then $c_4 > 0$.

Remark 3. When (2.7) does not hold, then as Corollary 2 shows, the asymptotic behavior of $\pi(x)$ can be essentially different. The violation of (2.7) consisting, for instance, of the form that $\mathbf{E}e^{\beta\xi(y)} = \infty$, $y \leqslant y_0$, means that $\mathbf{P}\{\xi(y) \geqslant t\}$ vanishes, as $t \to \infty$ and $y \leqslant y_0$, much slower than $\mathbf{P}\{\xi \geqslant t\}$. In that case, the main contribution to $\pi(x)$ will asymptotically be $\mathbf{P}\{\xi(y) \geqslant t\}$, $y \leqslant y_0$, and the asymptotic behavior itself can be estimated by constructing majorants $\xi_+ \geqslant_{\mathrm{st}} \xi(y)$ and minorants $\xi_- \leqslant_{\mathrm{st}} \xi(y)$, which will be close to each other for large y, $\mathbf{E}\xi_+ < 0$, and then by constructing partially homogeneous chains X_+ and X_- which will, respectively, majorize and minorize the original chain X.

Remark 4. Condition (2.8) is close to necessary, as the following example indicates. Let the chain X take values in $\mathbf{Z}^+ = \{0, 1, 2, \dots\}$, and let $\mathbf{P}\{\xi(x) = -1\} = (1 + \delta + l(x))/2$, $\mathbf{P}\{\xi(x) = 1\} = (1 - \delta - l(x))/2$, $x \in \mathbf{Z}^+$, $\delta > 0$, $l(x) \downarrow 0$. Then $\beta = \log(1+\delta)/(1-\delta)$ and, as computations show (see [10] and [11, p. 98]),

$$\pi\big(\{x\}\big) \sim c_5 e^{-\beta x} \exp\left\{-c_6 \sum_{k=1}^{x} l(k)\right\}$$

for some $c_5, c_6 > 0$. In the case where condition (2.8) is violated, that is, when $\sum l(x) = \infty$, the asymptotic behavior of $\pi(\{x\})$ is different from $e^{-\beta x}$.

Remark 5. The problem of whether condition (2.10) is essential remains open.

There are still two possibilities that we have not yet discussed: $\varphi(\beta) < 1$ and $\beta = 0$.

First, consider the case where $\varphi(\beta) < 1$. Then one has $\beta = \mu_+ > 0$. Set $\widetilde{F}(y) = e^{\beta y}\mathbf{P}\{\xi \geqslant y\}$.

THEOREM 5. *Let* $\beta > 0$, $\varphi(\beta) < 1$, *the function* $\widetilde{F}$ *be u.p.* (*i.e., satisfy conditions* (2.2) *and* (2.3)), *and*

$$\Bigl|\mathbf{P}\bigl\{\xi(y) \geqslant t\bigr\} - \mathbf{P}\{\xi \geqslant t\}\Bigr| \leqslant \delta(y)\,\mathbf{P}\{\xi \geqslant t\}, \tag{2.11}$$

where $\delta(y) \downarrow 0$ *as* $y \to \infty$. *Then if for each* y *there exists the limit*

$$\lim_{t\to\infty} \frac{\mathbf{P}\{\xi(y) \geqslant t\}}{\mathbf{P}\{\xi \geqslant t\}} = c(y) < \infty, \tag{2.12}$$

we have that

$$\int_0^\infty e^{\beta y}\,\pi(dy) < \infty, \tag{2.13}$$

$$\pi(x) \sim G(x)\,\frac{\beta}{1-\varphi(\beta)} \int_0^\infty c(y) e^{\beta y}\,\pi(dy). \tag{2.14}$$

Remark 6. Condition (2.12) means that all of the "tails" $\mathbf{P}\{\xi(y) \geqslant t\}$ decrease "not slower" than $\mathbf{P}\{\xi \geqslant t\}$. If this property is violated, we can recommend the same method of estimation of $\pi(x)$ as in Remark 2.

Theorem 5 will be proved in section 8.

Now let $\beta = 0$. This happens, for example, when $\mathbf{P}\{\xi \geqslant t\}$ decreases as a power of t.

THEOREM 6. *Let the chain* $\{X(n)\}$ *be asymptotically homogeneous*, *i.e.*, $\xi(y) \Rightarrow \xi$ *as* $y \to \infty$, $a \equiv \mathbf{E}\xi < 0$. *Assume that the jumps* $\{\xi(y)\}$ *of the chain* $\{X(n)\}$ *are uniformly* (*in* y) *integrable. Let the function* $G(t)$ *be u.p.* (*i.e., satisfy conditions* (2.2) *and* (2.4)) *and*, *for some* $c(y)$,

$$\frac{\mathbf{P}\{\xi(y) \geqslant t\}}{\mathbf{P}\{\xi \geqslant t\}} \longrightarrow c(y)$$

as $t \to \infty$ *uniformly in* y. *Then the asymptotic equivalence*

$$\pi(x) \sim \frac{G(x)}{-a} \int_0^\infty c(y)\,\pi(dy) \tag{2.15}$$

takes place as $x \to \infty$.

Thus relation (2.14) progresses "by continuity" into (2.15).

Theorem 6 follows from Theorem 10, which will be stated and proved in section 6.

Sections 3, 4, and 6 also contain a certain generalization of Theorem 6 (see Theorem 10) and several estimates for $\pi(x)$, which are of independent interest. Theorems 8 and 9 imply also the following rough theorem on large deviations in the "power" case.

THEOREM 7. *Let* $\xi(y) \Rightarrow \xi$, $a \equiv \mathbf{E}\xi < 0$ *and assume the random variables* $|\xi(y)|$ *are uniformly integrable in* y. *Let* $\alpha > 1$ *and, for any* $\varepsilon > 0$, *let there exist* $c' > 0$, $c'' < \infty$, *and* t_0 *such that, for* $t > t_0$ *and all* y,

$$c' t^{-\alpha-\varepsilon} \leqslant \mathbf{P}\bigl\{\xi(y) \geqslant t\bigr\} \leqslant c'' t^{-\alpha+\varepsilon}.$$

Then

$$\log \pi(x) \sim -(\alpha - 1)\log x.$$

Convergence rate estimates in Theorem 4 (the Cramér case), which are needed for constructing statistical estimators, are given in section 9.

3. The lower bound for the invariant distribution tail. Let the chain $\{X(n)\}$, $n = 0, 1, 2, \ldots$, assume, as before, values in $\mathbf{R}^+$, and let $\xi(y)$ be a random variable corresponding to the jump of the chain $\{X(n)\}$ from the state y; $F_y(\cdot)$ is its distribution, i.e.,

$$F_y(B) = \mathbf{P}\bigl\{\xi(y) \in B\bigr\} = P(y,\, y + B), \quad B \in \mathcal{B}(\mathbf{R}^+).$$

Put $F_y(t) = F_y([t, \infty))$ and

$$G_y(t) \equiv \int_t^\infty F_y(u)\, du, \tag{3.1}$$

$$\underline{a}(y) \equiv \inf_{u>y} \mathbf{E}\,\xi(u). \tag{3.2}$$

The following lower bound holds for the large-deviation probabilities. Here and in what follows, we do not assume asymptotic homogeneity of the chain X unless otherwise stipulated.

Lemma 1. *Let*

$$\sup_y \mathbf{E}\,\bigl|\xi(y)\bigr| < \infty \tag{3.3}$$

and let N be such that $\pi((N,\infty)) > 0$. Then $\underline{a}(N) \leqslant 0$ and

$$\pi\bigl((N,\infty)\bigr) \geqslant \frac{1}{-\underline{a}(N)} \int_0^N G_u(N-u)\,\pi(du) \geqslant \frac{1}{-\underline{a}(N)} \int_0^N G_u(N)\,\pi(du)$$

(*we assume here that* $\frac{0}{0} = 0$).

This inequality estimates the value of the invariant measure on (N,∞) in terms of the values of the same measure, but in the complement domain $[0, N]$. We shall show later that, in the "regular case," the measure π in the right-hand side affects only the constant factor in the lower bound for $\pi(N)$.

Proof. Consider the enlarged chain $\{X^{(N)}(n)\}$ taking values in the state space $[N,\infty)$ and jumps $\xi^{(N)}(y)$ (see (2.6)). Since, by construction, we have $\xi^{(N)}(y) \geqslant_{\mathrm{st}} \xi(y)$ for $y > N$, then by the definition in (3.2) of the function $\underline{a}(y)$, for $y > N$, we have the inequality

$$\mathbf{E}\,\xi^{(N)}(y) \geqslant \underline{a}(N). \tag{3.4}$$

By condition (3.3),

$$\sup_{y\geqslant N} \mathbf{E}|\xi^{(N)}(y)| < \infty.$$

Therefore, one can apply the equilibrium identity (see, e.g., [3, section 8] or [11, section 2]) to obtain that

$$0 = \int_N^\infty \mathbf{E}\,\xi^{(N)}(u)\,\pi^{(N)}(du). \tag{3.5}$$

By virtue of (3.4), this means that

$$\begin{aligned}\mathbf{E}\,\xi^{(N)}(N)\,\bigl(\pi((-\infty,N]\bigr) = \mathbf{E}\,\xi^{(N)}(N)\pi^{(N)}\bigl(\{N\}\bigr) &= -\int_{N+0}^\infty \mathbf{E}\,\xi^{(N)}(u)\,\pi(du)\\ &\leqslant -\underline{a}(N)\int_{N+0}^\infty \pi(du).\end{aligned} \tag{3.6}$$

From this and the inequality $\pi((N,\infty)) > 0$, we derive that, in particular, $\underline{a}(N) \leqslant 0$.

It also follows from (3.6) that

$$\pi\big((N,\,\infty)\big) \geqslant \frac{1}{-\underline{a}(N)}\,\pi\big((-\infty,\,N]\big)\,\mathbf{E}\,\xi^{(N)}(N). \tag{3.7}$$

Further,

$$\begin{aligned}\pi\big((-\infty,\,N]\big)\mathbf{E}\,\xi^{(N)}(N) &= \pi\big((-\infty,\,N]\big)\frac{\int_{-\infty}^{N}\mathbf{E}\{u+\xi(u)-N;\,u+\xi(u)>N\}\,\pi(du)}{\pi((-\infty,N])}\\ &= \int_{-\infty}^{N}\int_{N-u}^{\infty}F_u(t)\,dt\,\pi(du) = \int_{-\infty}^{N}G_u(N-u)\,\pi(du). \qquad (3.8)\end{aligned}$$

The assertion of the lemma follows from this and (3.7).

4. The upper bound for the invariant distribution tail. Let $q(y)$ be a nonnegative nondecreasing integrable function; set

$$Q(y) \equiv \int_{y}^{\infty} q(u)\,du.$$

Let $\beta \in [0,\infty)$, let the function $Q_1(y) \equiv e^{\beta y}Q(y)$ be u.p. (that is, satisfy condition (2.2) and (2.3)), and let

$$\int_{0}^{\infty} e^{\beta y}q(y)\,dy < \infty. \tag{4.1}$$

LEMMA 2. *For any* $y,t \geqslant 0$, *let*

$$\mathbf{P}\big\{\xi(y) \geqslant t\big\} \leqslant q(t) \tag{4.2}$$

and for some number N *and random variable* $\bar{\xi}$, $\mathbf{E}\bar{\xi} < 0$, *let the following relation hold*:

$$\xi(y) \leqslant_{\mathrm{st}} \bar{\xi} \quad \textit{for } y \geqslant N. \tag{4.3}$$

Then if (a) $\beta > 0$ *and* $\mathbf{E}e^{\beta\bar{\xi}} < 1$ *or* (b) $\beta = 0$, *then there exists a* c *such that*, *for all* x,

$$\pi(x) \leqslant c\,Q(x).$$

Proof. We begin with the case $\beta > 0$, $\mathbf{E}e^{\beta\bar{\xi}} < 1$. By conditions (4.1), (4.2), and (4.3), there exist a random variable ζ and a number T such that $\zeta \geqslant -T$ with probability 1,

$$\zeta \geqslant_{\mathrm{st}} \xi(y) \quad \text{for } y \geqslant N, \tag{4.4}$$

$$\mathbf{P}\{\zeta \geqslant t\} = q(t) \quad \text{for } t \geqslant T, \tag{4.5}$$

$$\mathbf{E}\zeta < 0 \quad \text{and} \quad \mathbf{E}e^{\beta\zeta} < 1. \tag{4.6}$$

Set $M = N + 2T$. We shall now prove that for all y and z such that $0 \leqslant y \leqslant z$ and $z \geqslant M$, we have the inequality

$$y + \xi(y) \leqslant_{\mathrm{st}} z + \zeta. \tag{4.7}$$

If $N \leqslant y \leqslant z$ and $z \geqslant M$, then this inequality holds by (4.4). If $y < N$ and $z \geqslant M$, then $z - y + u \geqslant T$ for $u \geqslant -T$ and hence, by virtue of (4.2) and (4.5), one has the inequalities

$$\begin{aligned}\mathbf{P}\bigl\{y + \xi(y) \geqslant z + u\bigr\} \leqslant q(z - y + u) = \mathbf{P}\{\zeta \geqslant z - y + u\}\\ \leqslant \mathbf{P}\{z + \zeta \geqslant z + u\}.\end{aligned}$$

For $u < -T$, the last inequality holds since $\zeta \geqslant -T$. Thus inequality (4.7) also holds in the case where $y < N$ and $z \geqslant M$. Therefore, (4.7) indeed takes place.

Consider the nonnegative chain $\{Y(n)\}$ defined by the equality

$$Y(n+1) = \bigl(Y(n) + \zeta_n\bigr)^+,$$

where the random variables $\{\zeta_n\}_{n=0}^\infty$ are independent copies of ζ.

Let $X(0)$ have the distribution π. Then for any n, the distribution of $X(n)$ is also π. If $X(n-1) \leqslant_{\mathrm{st}} M + Y(n-1)$, then we have by (4.7) that

$$X(n) \leqslant_{\mathrm{st}} M + Y(n-1) + \zeta_n \leqslant M + \bigl(Y(n-1) + \zeta_n\bigr)^+ = M + Y(n).$$

Therefore, if $Y(0) \equiv_{\mathrm{st}} X(0)$, then $X(n) \leqslant_{\mathrm{st}} M + Y(n)$ for any n. In particular, for any x,

$$\pi(x) \leqslant \lim_{n\to\infty} \mathbf{P}\bigl\{M + Y(n) \geqslant x\bigr\} = \pi_Y\bigl([x - M, \infty)\bigr), \tag{4.8}$$

where π_Y denotes the stationary distribution of the chain Y. The chain $\{Y(n)\}$ is homogeneous assuming values in $\mathbf{R}^+$, the "generating" random variable ζ satisfying the conditions of Theorem 1. Therefore, for some $c < \infty$, one has the inequality $\pi_Y([x - M, \infty)) \leqslant cQ(x - M)$. In combination with (4.8) and the convergence $Q(x - M)/Q(x) \to e^{\beta M}$, the last inequality proves the assertion of the lemma in the case $\beta > 0$.

Let $\beta = 0$. Then the argument above continue to hold when one excludes the inequality $\mathbf{E}e^{\beta\zeta} < 1$ from formula (4.6). Lemma 2 is proved.

5. Two lemmas on the "local property." Let a positive nonincreasing function $f(x)$ be l.p., that is, satisfy condition (2.2). Since the function f is nonincreasing, it is l.p. if and only if there exists a sequence of points $T_0 < T_1 < \cdots < T_n < \cdots$ such that

$$T_n - T_{n-1} \uparrow \infty \quad \text{as } n \to \infty, \tag{5.1}$$

$$\frac{f(T_n)}{f(T_{n-1})} \to 1 \quad \text{as } n \to \infty. \tag{5.2}$$

Let $h(x)$ be a nonnegative nonincreasing function.

LEMMA 3. *Let $h(x) \leqslant f(x)$. Then there exists a sequence of segments $[t_n, s_n] \subseteq [T_{n-1}, T_n]$ such that $s_n - t_n \to \infty$ and $h(t_n) - h(s_n) = o(f(t_n))$ as $n \to \infty$.*

Proof. By virtue of (5.1), there exists a sequence u_n, $u_n > 0$ such that $u_n \to \infty$ and $u_n = o(T_n - T_{n-1})$ as $n \to \infty$. Denote by l_n the greatest integer not exceeding $(T_n - T_{n-1})/u_n$; by the choice of u_n, we have $l_n \to \infty$ as $n \to \infty$.

To prove the lemma, it suffices to show that, for any n, there exists a point $t_n \in [T_{n-1}, T_n - u_n]$, for which

$$h(t_n) - h(t_n + u_n) = o\bigl(f(t_n)\bigr), \quad n \to \infty.$$

Assume that, on the contrary, the last relation does not hold. Then there exists a number $\varepsilon > 0$ and a sequence of indices $n(k)$, $n(k) \uparrow \infty$ as $k \to \infty$, such that, for any $t \in [T_{n(k)-1}, T_{n(k)} - u_{n(k)}]$

$$h(t) - h(t + u_{n(k)}) \geqslant \varepsilon f(t). \tag{5.3}$$

In particular, by the inequality $h \leqslant f$,

$$h(t + u_{n(k)}) \leqslant h(t) - \varepsilon f(t) \leqslant (1-\varepsilon)\, h(t).$$

Consequently,

$$\begin{aligned} h(T_{n(k)} - u_{n(k)}) &\leqslant (1-\varepsilon)^{l_{n(k)}-1} h(T_{n(k)-1}) \leqslant (1-\varepsilon)^{l_{n(k)}-1} f(T_{n(k)-1}) \\ &= o\big(f(T_{n(k)} - u_{n(k)})\big), \quad k \to \infty, \end{aligned}$$

in view of the convergence $l_{n(k)} \to \infty$ and (5.2). This contradicts (5.3) when $t = T_{n(k)} - u_{n(k)}$. The lemma is proved.

LEMMA 4. *Let $h(x) \leqslant f(x)$. Then there exist functions $t(x)$ and $s(x)$ such that $[t(x), s(x)] \subseteq [0, x]$,*

$$s(x) - t(x) \longrightarrow \infty, \qquad \frac{f(N(x))}{f(x)} \longrightarrow 1,$$
$$h\big(t(x)\big) - h\big(s(x)\big) = o\big(f(x)\big) \quad \textit{as } x \to \infty,$$

where $N(x)$ is the midpoint of the segment $[t(x), s(x)]$.

Proof. If $x \in [T_n, T_{n+1})$, we set $t(x) \equiv t_n$, $s(x) \equiv s_n$. Then

$$\big[t(x),\, s(x)\big] = [t_n,\, s_n] \subseteq [T_{n-1},\, T_n] \subseteq [0,\, x],$$

$s(x) - t(x) \to \infty$ and, by Lemma 3, $h(t(x)) - h(s(x)) = o(f(t_n))$ as $x \to \infty$. Since the function f does not increase, $f(T_{n-1}) \geqslant f(t_n) \geqslant f(N(x)) \geqslant f(x) \geqslant f(T_{n+1})$. The required properties of the functions $N(x)$, $t(x)$, and $s(x)$ follow from here by condition (5.2). The lemma is proved.

6. Theorems on large deviations in the case $\beta = 0$**.** Let $\underline{G}(y)$ be a positive nonincreasing function, $\underline{G}(y) \downarrow 0$; the function $G_y(t)$ is defined in (3.1). Set

$$\underline{c}(y) \equiv \liminf_{t\to\infty} \frac{G_y(t)}{\underline{G}(t)}, \qquad \underline{c} \equiv \int_0^\infty \underline{c}(y)\,\pi(dy), \tag{6.1}$$

$$\underline{a}(y) \equiv \inf_{z>y} \mathbf{E}\xi(z), \qquad \underline{a} \equiv \lim_{y\to\infty} \underline{a}(y). \tag{6.2}$$

The following lower bound holds for large-deviation probabilities for the invariant distribution.

THEOREM 8. *Let $\sup_y \mathbf{E}|\xi(y)| < \infty$. Then*

$$\liminf_{x\to\infty} \frac{\pi(x)}{\underline{G}(x)} \geqslant \frac{\underline{c}}{-\underline{a}}$$

(*we assume here that $\frac{0}{0} = 0$ and $\frac{\infty}{\infty} = 0$*).

As examples show (see Theorem 1 (c), Corollary 1 (c)), the constant in the right-hand side of the inequality is exact in the case of "power" distribution tails. To make the inequality as informative as possible, one should take $\underline{G}$ to be the "heaviest tail" among G_y.

Proof of Theorem 8. Choose arbitrary c_0, $c_0 < \underline{c}$, and a_0, $a_0 < \underline{a}$. By virtue of the definitions in (6.1) and (6.2) as well as Fatou's lemma, there exists an $x_0 < \infty$, for which $\underline{a}(x_0) > a_0$ and

$$\liminf_{t\to\infty} \int_0^{x_0} \frac{G_y(t)}{\underline{G}(t)}\,\pi(dy) \geqslant c_0.$$

By Lemma 1, for $x > x_0$ one has the bound

$$\frac{\pi(x)}{\underline{G}(x)} \geqslant \frac{1}{-\underline{a}(x_0)} \int_0^{x_0} \frac{G_y(t)}{\underline{G}(t)}\,\pi(dy).$$

Consequently,

$$\liminf_{x\to\infty} \frac{\pi(x)}{\underline{G}(x)} \geqslant \frac{c_0}{-a_0}.$$

Since the numbers $c_0 < \underline{c}$ and $a_0 < \underline{a}$ were arbitrary, the theorem is proved.

In the following theorem, an upper bound for the large-deviation probabilities for the invariant measure is given, which is more exact than that in Lemma 2. We shall assume that the function $\overline{G}(t)$ is sufficiently regular. Namely, it is assumed in what follows that $\overline{G}(t) = \int_t^\infty g(u)\,du$, where the function g is positive and nonincreasing. Assume also that the function $\overline{G}(t)$ is u.p. (i.e., satisfies conditions (2.2) and (2.4)). Set

$$\overline{c}(y) \equiv \limsup_{t\to\infty} \frac{G_y(t)}{\overline{G}(t)}, \qquad \overline{c} \equiv \int_0^\infty \overline{c}(y)\,\pi(dy), \tag{6.3}$$

$$\overline{a}(y) \equiv \sup_{z>y} \mathbf{E}\xi(z), \qquad \overline{a} \equiv \lim_{y\to\infty} \overline{a}(y). \tag{6.4}$$

THEOREM 9. *Let the jumps $\{\xi(y)\}$ of the chain $\{X(n)\}$ be uniformly integrable in y. Further, let there exist numbers N, $\widehat{c} < \infty$ and a random variable $\overline{\xi}$, $\mathbf{E}\overline{\xi} < 0$, such that*

$$\mathbf{P}\{\xi(y) \geqslant t\} \leqslant \widehat{c}\,g(t) \quad \textit{for } y,t \geqslant 0, \tag{6.5}$$

$$\xi(y) \leqslant_{\mathrm{st}} \overline{\xi} \qquad \textit{for } y \geqslant N. \tag{6.6}$$

Then if the function $\overline{G}$ satisfies conditions (2.2) *and* (2.4), *then the relation*

$$\limsup_{x\to\infty} \frac{\pi(x)}{\overline{G}(x)} \leqslant \frac{\overline{c}}{-\overline{a}}$$

holds.

Remark 7. By condition (6.5), $\overline{c}(y) \leqslant \widehat{c}$ for any $y \geqslant 0$; hence $\overline{c} \leqslant \widehat{c}$. By condition (6.6), $\overline{a} < 0$.

Proof of Theorem 9. From Lemma 2, it follows that for $q(y) = \widehat{c}\,g(y)$, $Q(y) = \widehat{c}\,\overline{G}(y)$, there exists a c_* such that

$$\pi(x) \leqslant c_*\overline{G}(x). \tag{6.7}$$

Therefore, since $\overline{G}(x)$ is l.p. and does not increase, the conditions of Lemma 4 for $f(x) = c_*\overline{G}(x)$ and $h(x) = \pi(x)$ are satisfied. Hence there exist functions $t(x)$ and $s(x)$ such that $[t(x), s(x)] \subseteq [0, x]$,

$$s(x) - t(x) \longrightarrow \infty, \tag{6.8}$$

$$\frac{\overline{G}(N(x))}{\overline{G}(x)} \longrightarrow 1, \tag{6.9}$$

$$\pi\Big(\big[t(x),\, s(x)\big]\Big) = o\big(\overline{G}(x)\big) \tag{6.10}$$

as $x \to \infty$, where $N(x)$ is the center of the segment $[t(x), s(x)]$.

Consider the enlarged Markov chain $\{X^{(N(x))}(n)\}$ assuming values in the state space $[N(x), \infty)$ (see (2.6)). From the equilibrium identity (3.5), we get

$$0 = \mathbf{E}\xi^{(N(x))}\big(N(x)\big)\,\pi\Big(\big[0, N(x)\big]\Big) + \int_{N(x)+0}^{\infty} \mathbf{E}\xi^{(N(x))}(y)\,\pi(dy),$$

where, according to (3.8),

$$\mathbf{E}\xi^{(N(x))}\big(N(x)\big)\pi\Big(\big[0, N(x)\big]\Big) = \int_0^{N(x)} G_y\big(N(x) - y\big)\,\pi(dy).$$

Hence

$$\begin{aligned} I_1 \equiv -\int_{s(x)+0}^{\infty} \mathbf{E}\xi^{(N(x))}(y)\,\pi(dy) &= \int_{N(x)+0}^{s(x)} \mathbf{E}\xi^{(N(x))}(y)\,\pi(dy) \\ &+ \bigg(\int_0^{N(x)/2} + \int_{N(x)/2+0}^{t(x)} + \int_{t(x)+0}^{N(x)}\bigg) G_y(N(x)-y)\,\pi(dy) \\ &\equiv I_2 + I_3 + I_4 + I_5. \end{aligned} \tag{6.11}$$

(We assume without loss of generality that $N(x)/2 < t(x)$.)

Since the jumps $\xi(y)$ of the chain $\{X(n)\}$ are uniformly integrable in y and $s(x) - N(x) \to \infty$,

$$\sup_{y > s(x)} \mathbf{E}\xi^{(N(x))}(y) \longrightarrow \overline{a} \quad \text{as } x \to \infty.$$

Therefore, $I_1 \geqslant (-\overline{a} + o(1))\pi((s(x), \infty))$ as $x \to \infty$. Consequently, in view of the inequality $s(x) \leqslant x$ and (6.7),

$$I_1 \geqslant -\overline{a}\pi(x) + o\big(\overline{G}(x)\big), \quad x \to \infty. \tag{6.12}$$

Now we estimate I_2. Since

$$\sup_{y > N(x)} \mathbf{E}\xi^{(N(x))}(y) \leqslant \sup_{y > N(x)} \mathbf{E}\big|\xi(y)\big| \leqslant \sup_{y \geqslant 0} \mathbf{E}\big|\xi(y)\big| < \infty,$$

we have by (6.10) that

$$I_2 = O\Big(\pi\big(\big(N(x),\, s(x)\big]\big)\Big) = o\big(\overline{G}(x)\big), \quad x \to \infty. \tag{6.13}$$

Let us estimate the value $I_3/\overline{G}(x)$. If $y \leqslant N(x)/2$, then by condition (2.4),

$$\frac{G_y(N(x) - y)}{\overline{G}(N(x))} \leqslant c_0.$$

Moreover, by the definition in (6.3), for any y

$$\limsup_{x \to \infty} \frac{G_y(N(x) - y)}{\overline{G}(N(x))} = \overline{c}(y).$$

Therefore, by (6.9) and Fatou's lemma,

$$\limsup_{x\to\infty}\frac{I_3}{\overline{G}(N(x))}=\limsup_{x\to\infty}\frac{I_3}{\overline{G}(x)}\leqslant\int_0^\infty \bar{c}(y)\,\pi(dy)\equiv\bar{c}. \tag{6.14}$$

Consecutively using condition (6.5), inequality (6.7), relation (6.9), and the fact that the function $\overline{G}$ satisfies condition (2.4), we estimate $I_4/\overline{G}(x)$:

$$\begin{aligned}\frac{I_4}{\overline{G}(x)}&\leqslant\int_{N(x)/2}^{t(x)}\frac{G_y(N(x)-t(x))}{\overline{G}(x)}\,\pi(dy)\leqslant\widehat{c}\,\frac{\overline{G}(N(x)-t(x))}{\overline{G}(x)}\,\pi\Big(\Big[\frac{N(x)}{2},\infty\Big)\Big)\\ &\leqslant\widehat{c}\,c_*\frac{\overline{G}(N(x)-t(x))\overline{G}(N(x)/2)}{\overline{G}(x)}=O\Big(\overline{G}\big(N(x)-t(x)\big)\Big)\longrightarrow 0\end{aligned} \tag{6.15}$$

as $x\to\infty$ since $N(x)-t(x)\to\infty$. Estimate $I_5/\overline{G}(x)$: by condition (6.5),

$$\frac{I_5}{\overline{G}(x)}\leqslant\frac{\widehat{c}\,\overline{G}(0)}{\overline{G}(x)}\,\pi\Big(\big[t(x),\,N(x)\big]\Big)\longrightarrow 0 \tag{6.16}$$

as $x\to\infty$ by virtue of (6.10). Substituting relations (6.12)–(6.16) into (6.11), we obtain the assertion of the theorem.

Theorems 8 and 9 entail the following result.

THEOREM 10. *Let the jumps $\{\xi(y)\}$ of the chain $\{X(n)\}$ be uniformly integrable in y and let*

$$\mathbf{E}\xi(y)\to a<0\quad\text{as } y\to\infty. \tag{6.17}$$

For some number N and random variable $\bar{\xi}$, $\mathbf{E}\bar{\xi}<0$, let

$$\xi(y)\leqslant_{\mathrm{st}}\bar{\xi}\quad\text{for } y\geqslant N, \tag{6.18}$$

and let a nonincreasing function $g(t)$ be such that

$$\mathbf{P}\big\{\xi(y)\geqslant t\big\}\leqslant g(t)\quad\text{for any } y,\ t\geqslant 0. \tag{6.19}$$

Let there exist $G_*(t)=\int_t^\infty g(u)\,du$ and let the function G_* be u.p. (*i.e., satisfy conditions* (2.2) *and* (2.4)). *Then if for any y the limit*

$$\lim_{t\to\infty}\frac{G_y(t)}{G_*(t)}\equiv c(y),$$

exists, then the equality

$$\lim_{x\to\infty}\frac{\pi(x)}{G_*(x)}=\frac{1}{-a}\int_0^\infty c(y)\,\pi(dy)$$

holds.

Theorem 10 implies Theorem 6 on the exact asymptotics of the large-deviation probabilities for asymptotically homogeneous chains in the case where $\beta=0$. Theorems 8 and 9 also entail Theorem 7 on the rough asymptotics in the case where $\beta=0$.

7. Asymptotics of the distributions of the convolutions of measures. Let μ be a probability measure on $\mathbf{R}$, and let μ_1 and μ_2 be two arbitrary (generally speaking, signed) measures on $\mathbf{R}$, the variance of which admits the bound

$$\operatorname*{Var}_{[x,\infty)}\mu_1\leqslant ce^{\beta|x|/2},\qquad \operatorname*{Var}_{[x,\infty)}\mu_2\leqslant ce^{\beta|x|/2} \tag{7.1}$$

for some $c < \infty$ and $\beta \geqslant 0$. Set $\mu(x) \equiv \mu([x, \infty))$ and $\mu_k(x) \equiv \mu_k([x, \infty))$, $k = 1, 2$. Assume that

$$\mu_k(x) = \big(\rho_k + o(1)\big)\,\mu(x), \quad k = 1,\ 2, \tag{7.2}$$

as $x \to \infty$, where $\rho_k \in \mathbf{R}$, and also assume that

$$b \equiv \int_{\mathbf{R}} e^{\beta y}\,\mu(dy) < \infty. \tag{7.3}$$

Then the integrals $b_1 \equiv \int_{\mathbf{R}} e^{\beta y}\,\mu_1(dy)$ and $b_2 \equiv \int_{\mathbf{R}} e^{\beta y}\,\mu_2(dy)$ exist.

LEMMA 5. *If $e^{\beta x}\mu(x)$ is a u.p. function (i.e., satisfies conditions (2.2) and (2.3)), then the relation*

$$(\mu_2 * \mu_1)\big([x, \infty)\big) = \mu(x)\,\big(\rho_2 b_1 + \rho_1 b_2 + o(1)\big)$$

holds as $x \to \infty$.

Proof. We have

$$\begin{aligned}(\mu_2 * \mu_1)\big([x, \infty)\big) &= -\bigg(\int_{-\infty}^{x/2} + \int_{x/2}^{\infty}\bigg)\mu_1(x-t)\,d\mu_2(t) \\ &= -\int_{-\infty}^{x/2} \mu_1(x-t)\,d\mu_2(t) - \mu_1(x-t)\mu_2(t)\big|_{x/2}^{\infty} \\ &\quad + \int_{x/2}^{\infty} \mu_2(t)\,d\mu_1(x-t).\end{aligned}$$

From condition (7.1), we derive that $\mu_1(x-t) = O(e^{\beta t/2})$ as $t \to \infty$ for any fixed x, and conditions (7.2) and (7.3) imply that $\mu_2(t) = o(e^{-\beta t})$. Therefore,

$$\begin{aligned}\frac{(\mu_2 * \mu_1)([x, \infty))}{\mu(x)} &= -\int_{-\infty}^{x/2} \frac{\mu_1(x-t)}{\mu(x)}\,d\mu_2(t) + \frac{\mu_1(x/2)\mu_2(x/2)}{\mu(x)} \\ &\quad - \int_{-\infty}^{x/2} \frac{\mu_2(x-t)}{\mu(x)}\,d\mu_1(t) \equiv I_1 + I_2 + I_3.\end{aligned} \tag{7.4}$$

By virtue of conditions (7.2) and (7.3) and the relation $\mu(x/2) = o(e^{-\beta x/2})$, we have

$$\begin{aligned}\mu_1\Big(\frac{x}{2}\Big)\mu_2\Big(\frac{x}{2}\Big) &= O\bigg(\mu\Big(\frac{x}{2}\Big)\,\mu\Big(\frac{x}{2}\Big)\bigg) = \mu(x)\,O(e^{\beta x/2})\,\mu\Big(\frac{x}{2}\Big) \\ &= o\big(\mu(x)\big), \quad x \to \infty.\end{aligned} \tag{7.5}$$

Next, we evaluate the limit I_1. Since $\mu_1(x-t)/\mu(x-t) \longrightarrow \rho_1$ as $x \to \infty$ uniformly in $t \leqslant x/2$,

$$I_1 = \big(-\rho_1 + o(1)\big)\int_{-\infty}^{x/2} \frac{\mu(x-t)}{\mu(x)}\,d\mu_2(t) \quad \text{as } x \to \infty. \tag{7.6}$$

By condition (2.2), for each fixed t,

$$\frac{\mu(x-t)}{\mu(x)} \longrightarrow e^{\beta t} \quad \text{as } x \to \infty.$$

Moreover, for $0 \leqslant t \leqslant x/2$, by condition (2.3), we have $\mu(x-t)/\mu(x) \leqslant e^{\beta t} c_0$ and, for $t<0$, $\mu(x-t)/\mu(x) \leqslant 1$. Thus in view of condition (7.3), the integrand in (7.6) admits an integrable majorant. Therefore, by Lebesgue dominated convergence,

$$I_1 \longrightarrow -\rho_1 \int_{-\infty}^{\infty} e^{\beta t}\, d\mu_2(t) = \rho_1 b_2 \quad \text{as } x \to \infty. \tag{7.7}$$

Since the measures μ_1 and μ_2 participate in (7.4) in a symmetric way, $I_3 \to \rho_2 b_1$ as $x \to \infty$. Substituting the values of the limits I_1 and I_3 in (7.4) and taking into account (7.5), we obtain the assertion of the lemma.

Now we shall prove the following lemma for the kth convolution of the measure μ.

Lemma 6. *If $e^{\beta x}\mu(x)$ is a u.p. function, then, for any $k = 1, 2, \ldots$,*

$$\mu^{*(k)}\big([x,\infty)\big) = \mu(x)\left(kb^{k-1} + o(1)\right) \quad \textit{as } x \to \infty.$$

Moreover, the following estimate is true: for any $\delta > 0$ there exist $c < \infty$ and $x_0 < \infty$ such that, for $x \geqslant x_0$ and any $k = 1, 2, \ldots$,

$$\mu^{*(k)}\big([x,\infty)\big) \leqslant c\mu(x)\,(b+\delta)^k.$$

Proof. The first assertion follows from Lemma 5. We shall prove now the second by carefully estimating each of the three summands in (7.4) for $\mu_1 = \mu$ and $\mu_2 = \mu^{*(k-1)}$. As was noted above, one has, as $x \to \infty$,

$$-\int_{-\infty}^{x/2} \frac{\mu(x-t)}{\mu(x)}\, d\mu(t) \longrightarrow -\int e^{t\beta}\, d\mu(t) \equiv b.$$

Therefore, there exists an $x_0 = x_0(\delta)$ such that, for all $x \geqslant x_0$,

$$-\int_{-\infty}^{x/2} \frac{\mu(x-t)}{\mu(x)}\, d\mu(t) \leqslant (b+\delta). \tag{7.8}$$

Set

$$A_k \equiv \sup_{x \geqslant x_0} \frac{\mu^{*(k)}([x,\infty))}{\mu([x,\infty))}.$$

Now we find an upper bound for A_k in terms of A_{k-1}. By condition (2.3) and the Chebyshev inequality, one has

$$\frac{\mu(x/2)\mu_2(x/2)}{\mu(x)} \leqslant \frac{c_0 e^{x\beta/2}\mu(x)\, b^{k-1} e^{-x\beta/2}}{\mu(x)} = c_0 b^{k-1}. \tag{7.9}$$

By the definition of A_{k-1} and $x_0(\delta)$, for $x \geqslant 2x_0(\delta)$, one has the inequalities

$$-\int_{-\infty}^{x/2} \frac{\mu_2(x-t)}{\mu(x)}\, d\mu(t) \leqslant -A_{k-1} \int_{-\infty}^{x/2} \frac{\mu(x-t)}{\mu(x)}\, d\mu(t) \leqslant A_{k-1}(b+\delta)$$

in view of (7.8). For $x_0 \leqslant x \leqslant 2x_0$, we obtain

$$-\int_{-\infty}^{x/2} \frac{\mu_2(x-t)}{\mu(x)}\, d\mu(t) \leqslant -\int_{-\infty}^{x/2} \frac{b^{k-1} e^{-\beta(x-t)}}{\mu(x)}\, d\mu(t) \leqslant \frac{b^{k-1}}{\mu(x)} \leqslant c_1 b^{k-1},$$

where $c_1 \equiv 1/\mu(2x_0) < \infty$. It follows from the last two estimates that for any $x \geqslant x_0(\delta)$,

$$-\int_{-\infty}^{x/2} \frac{\mu_2(x-t)}{\mu(x)}\, d\mu(t) \leqslant A_{k-1}(b+\delta) + c_1 b^{k-1}. \tag{7.10}$$

In view of condition (2.3), for any $x \geqslant 0$,

$$-\int_{-\infty}^{x/2} \frac{\mu(x-t)}{\mu(x)}\, d\mu_2(t) \leqslant -c_0 \int_{-\infty}^{x/2} e^{\beta t}\, d\mu_2(t) \leqslant c_0 b^{k-1}. \tag{7.11}$$

Substituting estimates (7.9), (7.10), and (7.11) in equality (7.4) (in which we take the distribution μ_1 to be μ and that of μ_2 to be $\mu^{*(k-1)}$), we derive the inequality

$$A_k \leqslant c_2 b^{k-1} + A_{k-1}(b+\delta).$$

This entails that $A_k \leqslant c_2 k(b+\delta)^{k-1}$. The last inequality is equivalent to the estimate stated in the lemma.

8. Exact asymptotics of the large-deviation probabilities for asymptotically homogeneous chains in the case $\varphi(\beta) < 1$. In this section, we shall prove Theorem 5. First we show that Remark 2 is correct. As observed in [2, p. 129], for an l.p. function $\widetilde{F}(t)$, for any $\varepsilon > 0$ and all large enough t,

$$\frac{\widetilde{F}(t+n)}{\widetilde{F}(t)} = \frac{\widetilde{F}(t+1)}{\widetilde{F}(t)} \frac{\widetilde{F}(t+2)}{\widetilde{F}(t+1)} \cdots \frac{\widetilde{F}(t+n)}{\widetilde{F}(t+n-1)} < e^{\varepsilon n}.$$

Therefore, in the ratio

$$\frac{G(t)}{\mathbf{P}\{\xi \geqslant t\}} = e^{\beta t} \frac{G(t)}{\widetilde{F}(t)} = \int_t^\infty e^{-\beta(y-t)} \frac{\widetilde{F}(y)}{\widetilde{F}(t)}\, dy = \int_0^\infty e^{-\beta u} \frac{\widetilde{F}(t+u)}{\widetilde{F}(t)}\, du,$$

the integrand has the limit $e^{-\beta u}$ as $t \to \infty$ and admits the upper bound $\widehat{c}\, e^{(-\beta+\varepsilon)u}$. Remark 2 follows from here by the Lebesgue dominated convergence theorem.

Proof of Theorem 5. We shall use the notation $F_y(t) = \mathbf{P}\{\xi(y) \geqslant t\}$, $F(t) = \mathbf{P}\{\xi \geqslant t\}$, $\varphi_y(\lambda) = \mathbf{E}e^{\lambda \xi(y)}$, and $\varphi(\lambda) = \mathbf{E}e^{\lambda\xi}$. The Kolmogorov–Chapman equation (1.1) for the stationary measure π is equivalent to the following equation for Laplace transforms:

$$\int_0^\infty e^{\lambda y}\, \pi(dy) = \int_0^\infty \varphi_y(\lambda) e^{\lambda y}\, \pi(dy) = \int_0^\infty (\varphi(\lambda) + \varphi_y(\lambda) - \varphi(\lambda)) e^{\lambda y}\, \pi(dy).$$

Therefore, for $0 \leqslant \lambda \leqslant \beta$,

$$\int_0^\infty e^{\lambda y}\, \pi(dy) = \frac{\int_0^\infty (\varphi_y(\lambda) - \varphi(\lambda)) e^{\lambda y}\, \pi(dy)}{1 - \varphi(\lambda)} \equiv \frac{1}{1-\varphi(\lambda)}\, \psi(\lambda). \tag{8.1}$$

Let $H(t) \equiv \sum_{k=0}^\infty F^{*(k)}([t,\infty))$ be the renewal function for the distribution F; in particular,

$$\frac{1}{1-\varphi(\lambda)} = -\int_{\mathbf{R}} e^{\lambda u}\, dH(u).$$

By condition $\varphi(\beta) < 1$ and Lemma 6, the series

$$\sum_{k=0}^{\infty} \frac{F^{*(k)}([t,\infty))}{F(t)}$$

converges uniformly in t in the domain (x_0, ∞) (for any $x_0 \in \mathbf{R}$). Hence by virtue of the same lemma,

$$\frac{H(t)}{F(t)} = \sum_{k=0}^{\infty} \frac{F^{*(k)}([t,\infty))}{F(t)} \to \sum_{k=0}^{\infty} k\varphi^{k-1}(\beta) = \frac{1}{(1-\varphi(\beta))^2}$$

as $t \to \infty$, that is,

$$H(t) = \frac{1+o(1)}{(1-\varphi(\beta))^2}\, F(t). \tag{8.2}$$

For $0 \leqslant \lambda \leqslant \beta$, we have the equalities

$$\begin{aligned}
\psi(\lambda) &\equiv \int_0^\infty \left(\int_{\mathbf{R}} e^{\lambda(t+y)}\, d_t\big(F(t) - F_y(t)\big) \right) \pi(dy) \\
&= \int_0^\infty \left(\int_{\mathbf{R}} e^{\lambda t}\, d_t\big(F(t-y) - F_y(t-y)\big) \right) \pi(dy) \\
&= \int_{\mathbf{R}} e^{\lambda t}\, d_t\left(\int_0^\infty \big(F(t-y) - F_y(t-y)\big)\, \pi(dy) \right).
\end{aligned}$$

Therefore, $\psi(\lambda)$ is the Laplace transform of the measure

$$\mu_2(dt) = d_t \int_0^\infty \big(F(t-y) - F_y(t-y)\big)\, \pi(dy) \tag{8.3}$$

on $\mathbf{R}$. Let us prove that

$$\mu_2\big([t,\infty)\big) = \big(\rho_* + o(1)\big)\, F(t) \tag{8.4}$$

as $t \to \infty$ for some $\rho_* \in \mathbf{R}$. One has

$$\frac{\mu_2([t,\infty))}{F(t)} \equiv \int_0^\infty g(t,y)\, \pi(dy), \tag{8.5}$$

where

$$g(t,y) = \frac{\mathbf{P}\{\xi(y) \geqslant t-y\} - \mathbf{P}\{\xi \geqslant t-y\}}{\mathbf{P}\{\xi \geqslant t\}}.$$

In view of conditions (2.2) and (2.12), for any fixed $y \geqslant 0$, the integrand has a limit as $t \to \infty$:

$$\lim_{t\to\infty} g(t,\, y) = c(y)\, e^{\beta y} - e^{\beta y}. \tag{8.6}$$

Moreover, by condition (2.11), we have the bound

$$\big|g(t,\, y)\big| \leqslant \delta(y) \frac{\mathbf{P}\{\xi \geqslant t-y\}}{\mathbf{P}\{\xi \geqslant t\}}. \tag{8.7}$$

If $0 \leqslant y \leqslant t/2$, then by the last inequality and condition (2.3), the function g admits the estimate

$$\big|g(t,\, y)\big| \leqslant \delta(0)\, e^{\beta y} c_0.$$

Hence by convergence (8.6) and the Lebesgue dominated convergence theorem, the limit

$$\lim_{t\to\infty}\int_0^{t/2} g(t,\,y)\,\pi(dy) = \int_0^{\infty}\bigl(c(y)-1\bigr)\,e^{\beta y}\,\pi(dy) \tag{8.8}$$

exists. In view of Lemma 2 and Remark 2, $\pi(y) \leqslant F(y)$. Hence it follows from inequality (8.7) and Lemma 5 (for $\mu_1 = \mu_2 = F$) that

$$\begin{aligned}\left|\int_{t/2}^{\infty} g(t,\,y)\,\pi(dy)\right| &\leqslant \delta\Bigl(\frac{t}{2}\Bigr)\int_0^{\infty}\frac{F(t-y)}{F(t)}\,\pi(dy)\\ &\leqslant c\delta\Bigl(\frac{t}{2}\Bigr)\int_0^{\infty}\frac{F(t-y)}{F(t)}\,dF(y) = O\Bigl(\delta\Bigl(\frac{t}{2}\Bigr)\Bigr)\longrightarrow 0\end{aligned}$$

as $t\to\infty$. It follows from here and (8.8) that the relation (8.4) does take place for

$$\rho_* = \int_0^{\infty}\bigl(c(y)-1\bigr)\,e^{\beta y}\,\pi(dy).$$

Therefore, in view of (8.2) and (8.4), conditions of Lemma 5 are satisfied for measures: μ, generated by the random variable ξ; μ_1, generated by the renewal function H; μ_2, defined in (8.3); and $\rho_1 = (1-\varphi(\beta))^{-2}$, $\rho_2 = \rho_*$. Consequently, equality (8.1) implies the relation

$$\pi(x) = F(x)\Bigl(\bigl(1-\varphi(\beta)\bigr)^{-2}b_2 + \rho_* b_1 + o(1)\Bigr), \quad x\to\infty, \tag{8.9}$$

where $b_1 = (1-\varphi(\beta))^{-1}$, $b_2 = \psi(\beta)$. We evaluate b_2 making use of the definition (8.1) of the function ψ:

$$\begin{aligned}b_2 = \psi(\beta) &= \int_0^{\infty}\bigl(e^{\beta\xi(y)}-\varphi(\beta)\bigr)\,e^{\beta y}\,\pi(dy)\\ &= \int_0^{\infty}\bigl(e^{\beta(y+\xi(y))}-e^{\beta y}\bigr)\,\pi(dy) + \bigl(1-\varphi(\beta)\bigr)\int_0^{\infty} e^{\beta y}\,\pi(dy).\end{aligned}$$

In view of the theorem on the mean drift (for the function $e^{\beta y}$; see [5, equality (14)]), the first integral is equal to zero. Therefore,

$$b_2 = \bigl(1-\varphi(\beta)\bigr)\int_0^{\infty} e^{\beta y}\,\pi(dy).$$

Substituting this value into (8.9) and taking into account Remark 2, we obtain the assertion of the theorem about the asymptotics of $\pi(x)$.

9. Estimating the second term of the asymptotics of $\pi(x)$ in the Cramér case. For the sake of simplicity, we consider a chain $\{X(n)\}$ taking values in $\mathbf{Z}^+$. Everywhere in what follows $x, y \in \mathbf{Z}^+$. Assume that the chain is asymptotically homogeneous, that is, $\xi(y) \Rightarrow \xi$ as $y\to\infty$, and there exists a $\beta > 0$ such that $\varphi(\beta) \equiv \mathbf{E}e^{\beta\xi} = 1$.

THEOREM 11. (a) *Let an integer $k \geqslant 1$ be such that*

$$\varphi^{(k+1)}(\beta) = \mathbf{E}\xi^{k+1}e^{\beta\xi} < \infty, \tag{9.1}$$

$$\sup_y \mathbf{E}\xi^{k+1}(y)e^{\beta\xi(y)} < \infty, \tag{9.2}$$

$$\int_{-\infty}^{\infty} e^{\beta t}\Big|\mathbf{P}\big\{\xi(y)<t\big\}-\mathbf{P}\{\xi<t\}\Big|\,dt \leqslant \frac{c_1}{y^{k+1}}. \tag{9.3}$$

Then $\pi(x)=ce^{-\beta x}+O(x^{-k}e^{-\beta x})$ *as* $x\to\infty$, *where* $0\leqslant c<\infty$. *If* $k>1$, *then* $c>0$.

(b) *Let* $\varepsilon>0$ *be such that*

$$\sup_y \mathbf{E}e^{(\beta+\varepsilon)\xi(y)}<\infty, \tag{9.4}$$

$$\int_{-\infty}^{\infty} e^{\beta t}\Big|\mathbf{P}\big\{\xi(y)<t\big\}-\mathbf{P}\{\xi<t\}\Big|\,dt \leqslant c_2e^{-\varepsilon y}. \tag{9.5}$$

Then for some $\delta>0$, *one has* $\pi(x)=ce^{-\beta x}+o(e^{-(\beta+\delta)x})$ *as* $x\to\infty$, *where* $c>0$.

Proof. (a) It follows from condition (9.3) that

$$\mathbf{E}e^{\beta\xi(x)}\geqslant \mathbf{E}e^{\beta\xi}-\frac{c_1}{x^{k+1}}=1-\frac{c_1}{x^{k+1}},$$

and since the series $\sum x\log x/x^{k+1}$ converges for $k>1$, we have from Theorem 4 that the constant c is positive for such values of k.

Let $\tilde\xi$ be a random variable with the distribution

$$\widetilde F(dt)=\mathbf{P}\{\tilde\xi\in dt\}=e^{\beta t}\,\mathbf{P}\{\xi\in dt\},\qquad \widetilde a\equiv \mathbf{E}\tilde\xi=\mathbf{E}\xi e^{\beta\xi}<\infty,$$

and

$$\widetilde H(y)=\sum_{j=0}^{\infty}\widetilde F^{*(j)}\big((-\infty,\,y)\big)$$

be the renewal function for the random variable $\tilde\xi$.

It is shown in the proof of Theorem 5 in [3, section 23] that

$$\left|\int_{z-1}^{z} e^{\beta x}\pi(x)\,dx-\frac{1}{\widetilde a}\int_{-\infty}^{\infty}dR(y)\right|\leqslant \int_{-\infty}^{\infty} r(z-y)\,\big|dR(y)\big|, \tag{9.6}$$

where $r(z)=|\widetilde H(z)-\widetilde H(z-1)-1/\widetilde a|$ and

$$dR(y)=e^{\beta y}\sum_{x=0}^{\infty}\big(F_x(y-x)-F(y-x)\big)\,\pi_x\,dy,\quad \pi_x=\pi\big(\{x\}\big).$$

It was also shown there that the total variation of the function R is finite. Now we estimate the total variation of the function R on the set $[N,\infty)$. By Theorem 4, we have $e^{\beta x}\pi_x\leqslant c_3$ uniformly in x and, consequently,

$$\begin{aligned}
\int_N^{\infty}\big|dR(y)\big| &\leqslant \int_N^{\infty} e^{\beta y}\sum_{x=0}^{\infty}\big|F_x(y-x)-F(y-x)\big|\,\pi_x\,dy\\
&=\sum_{x=0}^{\infty}e^{\beta x}\pi_x\int_N^{\infty}e^{\beta(y-x)}\big|F_x(y-x)-F(y-x)\big|\,dy\\
&\leqslant \sum_{x=0}^{\infty}c_3\int_{N-x}^{\infty}e^{\beta t}\big|F_x(t)-F(t)\big|\,dt\\
&=\left(\sum_{x=0}^{N/2}+\sum_{x=N/2}^{\infty}\right)c_3\int_{N-x}^{\infty}e^{\beta t}\big|F_x(t)-F(t)\big|\,dt=\Sigma_1+\Sigma_2.
\end{aligned}$$

We estimate Σ_1. By a Chebyshev-type inequality and in view of conditions (9.1) and (9.2), one has

$$\int_y^\infty e^{\beta t} F_x(t)\,dt = \frac{1}{\beta} e^{\beta t} F_x(t)\Big|_y^\infty - \frac{1}{\beta}\int_y^\infty e^{\beta t}\,dF_x(t) \leqslant c_4 y^{-(k+1)},$$
$$\int_y^\infty e^{\beta t} F(t)\,dt \leqslant c_5 y^{-(k+1)}.$$

Therefore,

$$\Sigma_1 \leqslant c_3(c_4+c_5)\sum_{x=0}^{N/2}\frac{1}{(N-x)^{k+1}} = O(N^{-k}).$$

Now we estimate Σ_2 using condition (9.3):

$$\Sigma_2 \leqslant c_3 \sum_{x=N/2}^{\infty}\frac{c_1}{x^{k+1}} = O(N^{-k}).$$

Thus

$$\operatorname*{Var}_{[N,\infty)} R(y) \equiv \int_N^\infty \bigl|dR(y)\bigr| = O(N^{-k}), \quad N\to\infty. \tag{9.7}$$

By condition (9.1), the $(k+1)$st moment of the random variable $\tilde{\xi}$ is finite. Hence by Corollary 1 in [15], we have the estimate

$$r(x) \equiv \Bigl|\widetilde{H}(x) - \widetilde{H}(x-1) - \frac{1}{\tilde{a}}\Bigr| \leqslant c_6 x^{-k},$$

where $\sup_x r(x) = c_7 < \infty$. Substituting the estimates for r into (9.6), we obtain

$$\begin{aligned}
\Bigl|\pi_N e^{\beta N}\frac{1-1/e}{\beta} - \frac{1}{\tilde{a}}\int_{-\infty}^\infty dR(y)\Bigr| &\leqslant \int_{-\infty}^\infty r(N-y)\,|dR(y)| \\
&= \Bigl(\int_{-\infty}^{N/2} + \int_{N/2}^\infty\Bigr) r(N-y)\,\bigl|dR(y)\bigr| \\
&\leqslant c_6\Bigl(\frac{N}{2}\Bigr)^{-k}\int_{-\infty}^{N/2}\bigl|dR(y)\bigr| + c_7\int_{N/2}^\infty \bigl|dR(y)\bigr| = O(N^{-k}),
\end{aligned}$$

in view of the finiteness of the total variation of R and (9.7). This implies assertion (a) of the theorem.

(b) If conditions (9.4) and (9.5) of the theorem are satisfied, then

$$\operatorname*{Var}_{[N,\infty)} R(y) = O(e^{-\varepsilon N/2}), \quad N\to\infty. \tag{9.8}$$

By (9.4), we have $\mathbf{E}e^{\varepsilon_0\tilde{\xi}} < \infty$ for any $\varepsilon_0 < \varepsilon$. Moreover,

$$\mathbf{E}\tilde{\xi}^2 = \mathbf{E}\xi^2 e^{\beta\xi} < \infty.$$

Hence by the theorem from [16], for some $\varepsilon_1 > 0$,

$$r(x) \equiv \Bigl|\widetilde{H}(x) - \widetilde{H}(x-1) - \frac{1}{\tilde{a}}\Bigr| = o\bigl(e^{-\varepsilon_1 x}\bigr), \quad x\to\infty. \tag{9.9}$$

Now (9.8) and (9.9) entail assertion (b) of the theorem for $\delta = \min(\varepsilon/2, \varepsilon_1)$. Note that, as follows from [16], there exists an $\varepsilon_2 > 0$ such that

$$\mathbf{E}\, e^{z\tilde{\xi}} = \mathbf{E}\, e^{(\beta+z)\xi} \neq 1$$

for all z from the band $0 < \mathrm{Re} z \leqslant \varepsilon_2$; ε_1 can be taken to be any number from the interval $(0, \varepsilon_2)$.

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