LARGE-DEVIATION PROBABILITIES FOR ONE-DIMENSIONAL MARKOV CHAINS. PART 2: PRESTATIONARY DISTRIBUTIONS IN THE EXPONENTIAL CASE*

A. A. BOROVKOV† AND D. A. KORSHUNOV†

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Abstract. This paper continues investigations of [A. A. Borovkov and A. D. Korshunov, *Theory Probab. Appl.*, 41 (1996), pp. 1–24]. We consider a time-homogeneous and asymptotically space-homogeneous Markov chain $\{X(n)\}$ that takes values on the real line and has increments possessing a finite exponential moment. The asymptotic behavior of the probability $\mathbf{P}\{X(n) \geqq x\}$ is studied as $x \to \infty$ for fixed or growing values of time n. In particular, we extract the ranges of n within which this probability is asymptotically equivalent to the tail of a stationary distribution $\pi(x)$ (the latter is studied in [A. A. Borovkov and A. D. Korshunov, *Theory Probab. Appl.*, 41 (1996), pp. 1–24] and is detailed in section 27 of [A. A. Borovkov, *Ergodicity and Stability of Stochastic Processes*, Wiley, New York, 1998]).

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1. Introduction.

Let

$$X(n) = X(y,n) \in \mathbf{R}, \quad n = 0,\ 1, \dots ,$$

be a time-homogeneous Markov chain which takes values on the real line $\mathbf{R}$ and has initial value $y \equiv X(y,0)$. We denote $P(y,B) = \mathbf{P}\{X(y,1) \in B\}$, where B is a Borel set in $\mathbf{R}$, the transition probabilities of the chain.

Let $\xi(y)$ be the increment of X in one step at point $y \in \mathbf{R}$, that is, $\xi(y) = X(y,1) - y$. In the present paper we study an *asymptotically space-homogeneous* chain that is a chain for which the distribution of $\xi(y)$ converges weakly as $y \to \infty$ to the distribution F of a random variable ξ. We assume everywhere that $m = \mathbf{E}\xi < 0$ (the case $m = -\infty$ is not excluded) and $\mathbf{P}\{\xi > 0\} > 0$.

The Laplace transform $\varphi(\lambda) \equiv \mathbf{E}e^{\lambda\xi}$ of ξ is a convex function and, therefore, the set $\{\lambda\colon \varphi(\lambda) \leqq 1\}$ is an interval of the form $[0,\beta]$, where $\beta = \sup\{\lambda\colon \varphi(\lambda) \leqq 1\}$. Since $\mathbf{P}\{\xi > 0\} > 0$, it follows that β is a finite number. The following three cases are possible:

(a) $\beta > 0$, $\varphi(\beta) = 1$, called the *Cramér* case;

(b) $\beta > 0$, $\varphi(\beta) < 1$, called the *intermediate* case;

(c) $\beta = 0$. This case includes, in particular, the distribution of ξ with regularly varying tails at infinity.

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†Sobolev Institute of Mathematics SB RAS, Academician Koptyug pr., 4, 630090 Novosibirsk, Russia (borovkov@math.nsc.ru, korschuno@math.nsc.ru).

In Part 1 (see [5]) we assumed that X was a chain possessing an invariant measure π, that is, a measure solving

$$\pi(\cdot) = \int_{\mathbf{R}} \pi(dy)\, P(y, \cdot), \quad \pi(\mathbf{R}) = 1. \tag{1.1}$$

We have studied in detail the asymptotic behavior of $\pi(x) = \pi([x,\infty))$ as $x \to \infty$ in all cases (a)–(c). The main attention was paid to the case of asymptotically homogeneous chains.

In this second part we investigate the asymptotic behavior of $\pi_n(x) = \mathbf{P}\{X(n) \geqq x\}$ as $x \to \infty$ in the Cramér and intermediate cases. We are interested in fixed values of the time parameter n and unboundedly growing n as well. The case $\beta = 0$ will be considered in the future part 3 of the work.

Let $\{\xi_n\}$ be a tuple of independent copies of ξ. Following [5], we call a Markov chain X in $\mathbf{R}^+ = [0,\infty)$, specified by the relations $X(n+1) = (X(n)+\xi_n)^+$, *space homogeneous*.

Put

$$S_0 = 0, \qquad S_k = \xi_1 + \cdots + \xi_k, \quad \text{and} \quad M_n = \max_{0 \leqq k \leqq n} S_k.$$

It is well known (see, for example, [10, Chap. VI, section 9]) that the distribution of the homogeneous chain $X(0,n)$ coincides with the distribution of M_n, that is, for any x

$$\mathbf{P}\bigl\{X(0,\, n) \geqq 0\bigr\} = \mathbf{P}\{M_n \geqq x\}. \tag{1.2}$$

Following [5] we say that a chain X is *N-partially homogeneous in space* (or simply *partially homogeneous*) if for any Borel set $B \subseteq (N,\infty)$ the transition probability $P(y,B)$ coincides with the probability $\mathbf{P}\{y+\xi \in B\}$ when y runs through the set (N,∞). In other words, the behavior of X in the domain (N,∞) coincides with the process of summation of independent random variables with common distribution F. Clearly, a homogeneous chain is 0-partially homogeneous.

In section 2 we give the considerations needed for further refinements of a number of known theorems on the large-deviation probabilities for sums of independent identically distributed random variables, provided their exponential moments exist.

In section 3 we formulate and prove a rough theorem on the large-deviation probabilities in the Cramér case for asymptotically homogeneous chains.

In section 4 we still deal with the Cramér case chains but restrict the class of chains under consideration (in comparison with section 3 and [5]) to the class of partially homogeneous in space chains meeting the Harris condition. For such chains it is possible to describe the asymptotic behavior of the probability $\pi_n(x)$ for a large spectrum of growing values of n. In particular, a domain of values of n is found in which $\pi_n(x)$ is equivalent to the tail $\pi(x)$ of the invariant distribution π.

In section 5 asymptotically homogeneous chains are investigated in the intermediate case. We show, under broad conditions on the initial distribution π_0, that $\pi_n(x) \sim \pi(x)$[1] for any rate of convergence of n and x to infinity. In addition, an essential generalization of Theorem 5 from [5] is given about the large-deviation probabilities for the stationary distribution π.

[1] We write $a_n(x) \sim b_n(x)$ as $n,\, x \to \infty$, if $\lim_{n,x\to\infty} a_n(x)/b_n(x) = 1$.

2. Large-deviation probabilities for sums of random variables.

2.1. Deviation (or rate) function. In this subsection we recall the notion of deviation function and some of its properties (see, for example, [2, Chap. 8, section 8] or [6, section 1, subsection 7]). We need the following notation: $m = \mathbf{E}\xi$, $\varphi(\lambda) = \mathbf{E}e^{\lambda\xi}$, $\lambda_+ = \sup\{\lambda\colon\ \varphi(\lambda) < \infty\}$ (here and throughout we assume that $\lambda_+ > 0$), and

$$\alpha_+ = \lim_{\lambda\to\lambda_+} \frac{\varphi'(\lambda)}{\varphi(\lambda)} = \lim_{\lambda\to\lambda_+} \frac{d\log\varphi(\lambda)}{d\lambda}.$$

If ξ is a random variable bounded from above, then $\lambda_+ = \infty$, and α_+ coincides with the essential supremum of the values of ξ, that is, with $\sup\{x\colon\ F(x) < 1\}$. If ξ is not bounded from below and $\lambda_+ = \infty$, then $\alpha_+ = \infty$. If ξ is not bounded from above and λ_+ is finite, then α_+ may take a priori any value within the interval $(m, \infty]$, including a negative value.

It is known (see, for example, [10, Chap. XVI, section 7]) that $\log\varphi(\lambda)$ is a strictly convex function. Therefore, for $\alpha \in [m, \alpha_+)$ the maximum of the difference $\alpha\lambda - \log\varphi(\lambda)$ with respect to λ is attained at that unique point $\lambda(\alpha)$, at which the tangent line to the function $\log\varphi(\lambda)$ is parallel to the line $\alpha\lambda$, that is, at which $d\log\varphi(\lambda)/d\lambda = \alpha$. The function $\lambda(\alpha)$ is increasing and $\lambda(m) = 0$. The function

$$\Lambda(\alpha) \equiv \sup_{\lambda}\ \{\alpha\lambda - \log\varphi(\lambda)\} = \alpha\lambda(\alpha) - \log\varphi(\lambda(\alpha)), \quad \alpha \in [m, \alpha_+), \tag{2.1}$$

is called a *deviation* (*or rate*) *function*. Differentiating the previous equality we get the relation $\Lambda'(\alpha) = \lambda(\alpha)$; therefore, Λ is a strictly convex function. We have $\Lambda(m) = \Lambda'(m) = 0$.

In the case when $\alpha_+ < \infty$ and $\lambda_+ < \infty$, we define $\Lambda(\alpha)$ for $\alpha \geqq \alpha_+$ in the linear way: $\Lambda(\alpha) = \Lambda(\alpha_+ - 0) + (\alpha - \alpha_+)\lambda_+$, setting additionally $\lambda(\alpha) = \lambda(\alpha_+ - 0)$ for $\alpha \geqq \alpha_+$. If $\alpha_+ < \infty$ and $\lambda_+ = \infty$, that is, if ξ is a random variable bounded from above (with necessity by α_+), we set $\Lambda(\alpha_+) = -\log\mathbf{P}\{\xi = \alpha_+\}$ and $\Lambda(\alpha) = \infty$ for $\alpha > \alpha_+$ if $\mathbf{P}\{\xi = \alpha_+\} > 0$; $\Lambda(\alpha) = \infty$ for $\alpha \geqq \alpha_+$ if $\mathbf{P}\{\xi = \alpha_+\} = 0$. Note that a formal search for the supremum in (2.1) leads just to these values of the deviation function for $\alpha > \alpha_+$.

If ξ is not bounded from above, $\Lambda(\alpha)$ is finite and continuously differentiable for all $\alpha \geqq 0$. If ξ is bounded from above (that is, if $\lambda_+ = \infty$ and $\alpha_+ < \infty$), $\Lambda(\alpha)$ has a unique discontinuity at the point $\alpha = \alpha_+$.

2.2. Rough asymptotics for large-deviation probabilities. According to the large-deviation principle for sums of independent random variables (see, for example, [6, section 1, Lemma 7]) for any fixed $\alpha \geqq m$ as $n \to \infty$ we have

$$n^{-1}\log\mathbf{P}\left\{\frac{S_n}{n} \geqq \alpha\right\} \longrightarrow -\Lambda(\alpha). \tag{2.2}$$

This principle is still in force in the following stronger form. Let $\alpha_1 > m$ be an arbitrary number such that $\Lambda(\alpha_1) < \infty$. Then, as $n \to \infty$,

$$n^{-1}\log\mathbf{P}\left\{\frac{S_n}{n} \geqq \alpha\right\} = -\Lambda(\alpha) + o(1) \tag{2.3}$$

uniformly in $\alpha \in [m, \alpha_1]$. The left-hand side in (2.3) equals $-\infty$ if $\Lambda(\alpha) = \infty$. The statement in the uniform form follows from (2.2) in view of the uniform continuity of $\Lambda(\alpha)$ on the compact $[m, \alpha_1]$ and monotonicity of the left-hand side of (2.3) in α.

In the next two auxiliary subsections we give two refinements of the central limit theorem for lattice and nonlattice summands which are uniform with respect to a parameter.

2.3. Local central limit theorem uniform with respect to a parameter (the lattice case). In this subsection ξ_k are supposed to be lattice with $\sigma^2 = \mathbf{D}\xi < \infty$. Let b and h be numbers such that $\mathbf{P}\{\xi = b + kh,\ k \in \mathbf{Z}\} = 1$, and the lattice with step h is the minimal one possessing this property. According to a local limit theorem for the distribution of sums S_n (see [8, section 43])

$$\mathbf{P}\left\{S_n = nb + kh\right\} - \frac{h}{\sqrt{2\pi n}\sigma} e^{-(nb+kh-nm_1)^2/2n\sigma^2} = o\left(\frac{1}{\sqrt{n}}\right)$$

as $n \to \infty$ uniformly for all $k \in \mathbf{Z}$. In the lemma below we list conditions under which this statement is valid uniformly in a parameter on which the distribution of ξ depends. Namely, we consider lattice random variables $\xi_k^{[r]}$, the distribution of which depends on a parameter r from an arbitrary parametric set and for each r is lattice with minimal lattice $\{b + kh,\ k \in \mathbf{Z}\}$.

LEMMA 1. *Let* $\{(\xi^{[r]})^2\}$ *be a family* (*in* r) *of random variables uniformly integrable and such that*, *for any* $\varepsilon > 0$

$$\sup_r \sup_{\mu \in [\varepsilon, 2\pi/h-\varepsilon]} |\mathbf{E}e^{i\mu\xi^{[r]}}| < 1 \tag{2.4}$$

(*here* i *is the imaginary unit*). *Then uniformly in* k *and* r *the following relation holds*:

$$\mathbf{P}\left\{S_n^{[r]} = nb + kh\right\} - \frac{h}{\sqrt{2\pi n}\sigma^{[r]}}\, e^{-(nb+kh-nm_1^{[r]})^2/2n(\sigma^{[r]})^2} = o\left(\frac{1}{\sqrt{n}}\right). \tag{2.5}$$

Remark 1. Since for any r the distribution of $\xi^{[r]}$ is lattice with the minimal step of the lattice equal to h, for any $\varepsilon > 0$ the estimate $|\mathbf{E}e^{i\mu\xi^{[r]}}| < 1$ is valid uniformly in $\mu \in [\varepsilon, 2\pi/h - \varepsilon]$ and condition (2.4) is an analogue of this relation being uniform in r.

Proof of Lemma 1. The condition of the uniform integrability of the squares of the random variables implies that in a vicinity of point $\mu = 0$ the expansion

$$\mathbf{E}e^{i\mu(\xi^{[r]}-m_1^{[r]})} = 1 - \frac{(\sigma^{[r]})^2\mu^2}{2} + o(\mu^2)$$

holds uniformly in r. In particular, the characteristic functions $\mathbf{E}e^{i\mu(S_n^{[r]}-nm_1^{[r]})/\sqrt{n}\sigma^{[r]}}$ converge to the characteristic function $e^{-\mu^2/2}$ of the standard normal law uniformly in μ from any compact and uniformly in r. These facts and condition (2.4) further allow us to use, literally, the proof of the local theorem in [8, section 43].

Lemma 1 implies the following statement.

COROLLARY 1. *Let* $\{(\xi^{[r]})^2\}$ *be a family* (*in* r) *of random variables that is uniformly integrable. Let*, *in addition, a finite set* $K \subset \mathbf{Z}$ *be such that the greatest common divisor of the numbers from* K *is equal to one and for any* $k \in K$ *and* r *the inequality* $\mathbf{P}\{\xi^{[r]} = b + kh\} \geqq \varepsilon$ *is valid for some* $\varepsilon > 0$. *Then relation* (2.5) *holds true uniformly in* k *and* r.

2.4. Expansion in the central limit theorem uniform with respect to a parameter (the nonlattice case). Now let ξ_k be nonlattice with zero mean and finite third moment $m_3 = \mathbf{E}\xi^3$. The following expansion for the distribution of $S_n = \xi_1 + \cdots + \xi_n$ is known which refines the central limit theorem (see, for example, Theorem 1 in [10, Chap. XVI, section 4]): As $n \to \infty$

$$\mathbf{P}\left\{\frac{S_n}{\sqrt{n\mathbf{D}\xi_1}} < u\right\} - \Phi(u) - \frac{m_3}{6\,\sigma^3\sqrt{n}}(1-u^2)\,\Phi'(u) = o\left(\frac{1}{\sqrt{n}}\right)$$

uniformly with respect to all u, where $\Phi(u)$ is the distribution function of the standard normal law. In this auxiliary subsection we formulate a generalization of this expansion to the case when the distribution of ξ depends on a parameter r from an arbitrary parametric set. Namely, we consider random variables $\xi_k^{[r]}$ each of which has nonlattice distribution with zero mean, unit variance, and finite third moment $m_3^{[r]} = \mathbf{E}(\xi^{[r]})^3$.

LEMMA 2. *Let, for any compact $K \subset \mathbf{R}$ containing no zero,*

$$\sup_r \sup_{\mu \in K} \left|\mathbf{E}e^{i\mu\xi^{[r]}}\right| < 1 \tag{2.6}$$

(*here i is the imaginary unit*). *Let, in addition, the family* (*in r*) *of random variables $\{(\xi^{[r]})^3\}$ be uniformly integrable. Then the distribution function $F_n^{[r]}(u)$ of the random variable $S_n^{[r]}/\sqrt{n}$ satisfies, as $n \to \infty$, the relation*

$$F_n^{[r]}(u) - \Phi(u) - \frac{m_3^{[r]}}{6\sqrt{n}}\,(1-u^2)\,\Phi'(u) = o\left(\frac{1}{\sqrt{n}}\right)$$

uniformly in $u \in \mathbf{R}$ and r.

Remark 2. Since for any r the distribution of the random variables $\xi^{[r]}$ is a nonlattice, $\sup_{\mu \in K} |\mathbf{E}e^{i\mu\xi^{[r]}}| < 1$ for any compact $K \subset \mathbf{R}$ containing no zero, and condition (2.6) is simply the uniform analogue of this relation.

Proof. The proof follows from the arguments of [10, Chap. XVI, section 4] and is based on the following estimate (4.4) given there: For any $\varepsilon > 0$

$$\left|F_n^{[r]}(u) - \Psi(u)\right| \leqq \int_{-a\sqrt{n}}^{a\sqrt{n}} \left|\frac{(\mathbf{E}e^{i\mu\xi^{[r]}/\sqrt{n}})^n - \gamma(\mu)}{\mu}\right| d\mu + \frac{\varepsilon}{\sqrt{n}}, \tag{2.7}$$

where

$$\Psi(u) = \Phi(u) + \frac{m_3^{[r]}}{6\sqrt{n}}\,(1-u^2)\,\Phi'(u)$$

is a function of bounded variation with Fourier transform

$$\psi(\mu) \equiv \int_{-\infty}^{\infty} e^{i\mu u}\,d\Psi(u) = e^{-\mu^2/2}\left[1 + \frac{m_3^{[r]}}{6\sqrt{n}}\,(i\mu)^3\right],$$

and a is a constant such that $24|\Psi'(u)| < \varepsilon a$ for all u and r. One can find such an a since the third moments are uniformly bounded.

We divide the interval of integration in (2.7) into two parts. By (2.6) the maximum of $|\mathbf{E}e^{i\mu\xi^{[r]}}|$ over the domain $0 < \delta \leqq |\mu| \leqq a < \infty$ is strictly less than one. As mentioned in [10, Chap. XVI, section 4], it follows that the integral over the domain $|\mu| \in [\delta\sqrt{n}, a\sqrt{n}\,]$ tends to zero more rapidly than any power of $1/n$. Further, as the third moments of the random variables in question are uniformly integrable, the third derivatives of the functions $\log \mathbf{E}e^{i\mu\xi^{[r]}}$ are continuous in a vicinity of zero uniformly in r. Therefore, according to [10, Chap. XVI, section 2] there exists $\delta > 0$ such that the integrand in (2.7) admits the following estimate for $|\mu| \leqq \delta\sqrt{n}$:

$$\left| \frac{(\mathbf{E}e^{i\mu\xi^{[r]}/\sqrt{n}})^n - \psi(\mu)}{\mu} \right| \leqq e^{-\mu^2/4} \left(\frac{\varepsilon}{\sqrt{n}}|\mu| + \frac{(m_3^{[r]})^2}{72n}\,|\mu|^5 \right),$$

and consequently the right-hand side in (2.7) is less than $1000\,\varepsilon/\sqrt{n}$ for large n. Recalling that $\varepsilon > 0$ may be selected arbitrarily small completes the proof of the lemma.

2.5. Exact asymptotics for the large-deviation probabilities of sums.

Definition 1. *A distribution F_λ is called the* Cramér transform *of a distribution F at point λ if $F_\lambda(du) = e^{\lambda u}F(du)/\varphi(\lambda)$, provided the Laplace transform of F exists at point λ.*

The distribution F_λ with $\lambda = \lambda(\alpha)$ is called the *Cramér transform $F^{(\alpha)}$ with parameter α over the distribution F*. A random variable with distribution $F^{(\alpha)}$ is denoted by $\xi^{(\alpha)}$. Observe that $\xi^{(\alpha)} = \xi$ if $\alpha = m$. According to this definition, $\mathbf{E}\xi^{(\alpha)} = \varphi'(\lambda)/\varphi(\lambda)\,|_{\lambda=\lambda(\alpha)} = \alpha$,

$$\big(\sigma^{(\alpha)}\big)^2 = \mathbf{D}\xi^{(\alpha)} = \frac{\varphi''(\lambda)}{\varphi(\lambda)}\bigg|_{\lambda=\lambda(\alpha)} - \alpha^2 = \big(\log\varphi(\lambda)\big)''\,\big|_{\lambda=\lambda(\alpha)}.$$

Differentiating the identity $\varphi'(\lambda)/\varphi(\lambda)\,|_{\lambda=\lambda(\alpha)} = \alpha$ with respect to α and taking into account the equality $\Lambda'(\alpha) = \lambda(\alpha)$ we arrive at the relation

$$\big(\sigma^{(\alpha)}\big)^2 = \frac{1}{\Lambda''(\alpha)}. \tag{2.8}$$

In particular, $\Lambda''(m) = 1/\mathbf{D}\xi$.

The following two lemmas generalize Theorem A in [14] and some statements in [9]. Set $\gamma = \sigma^{(\alpha)}\lambda(\alpha)\,\sqrt{n}$.

Lemma 3. *Let F be a nonlattice distribution, and let α_1 and α_2 be numbers such that $m \leqq \alpha_1 < \alpha_2 < \alpha_+$; if $\alpha_1 = m$, then we additionally assume the finiteness of $\mathbf{E}|\xi|^3$. Then, as $n \to \infty$, the following representation is true uniformly in $\alpha \in [\alpha_1, \alpha_2]$:*

$$\mathbf{P}\left\{\frac{S_n}{n} \geqq \alpha\right\} = e^{-n\Lambda(\alpha)}e^{\gamma^2/2}\big(1 - \Phi(\gamma)\big)\,\big(1 + o(1)\big).$$

Remark 3. Below, by proving the statement we establish a stronger relation which refines the order of the remainder $o(1)$ (see (2.10)).

The asymptotic equality described in the lemma is universal in that it contains an approximation of the distribution of sums both in the domain of normal deviations and in the domain of large deviations. For example, we know that $1-\Phi(\gamma) \sim e^{-\gamma^2/2}/\sqrt{2\pi}\gamma$ as $\gamma \to \infty$. Hence we deduce the following statement.

Corollary 2. *If $\mathbf{E}\xi < \alpha_1$, then as $n \to \infty$ we have uniformly in $\alpha \in [\alpha_1, \alpha_2]$*

$$\mathbf{P}\left\{\frac{S_n}{n} \geqq \alpha\right\} \sim \frac{1}{\sqrt{2\pi n}\sigma^{(\alpha)}\lambda(\alpha)}\,e^{-n\Lambda(\alpha)}.$$

In addition, if $\mathbf{E}|\xi|^3 < \infty$ *and* $y(n) \to \infty$, $y(n) = o(\sqrt{n})$, *then the previous asymptotic representation is valid as* $n \to \infty$ *uniformly in* $\alpha \in [\mathbf{E}\xi + y(n)/\sqrt{n}, \alpha_2]$.

On the other hand, the expansion $\Lambda(\alpha) = \gamma^2/2n + O(|\alpha - m|^3)$ is valid in a vicinity of point $\alpha = m$ which implies the following statement.

COROLLARY 3. *If* $\mathbf{E}|\xi|^3 < \infty$, *then, as* $n \to \infty$, *the following relation is valid uniformly in the domain* $\alpha \in [m, m + o(n^{-1/3})]$:

$$\mathbf{P}\left\{\frac{S_n}{n} \geqq \alpha\right\} \sim 1 - \Phi\Big(\sigma^{(\alpha)}\,\lambda(\alpha)\,\sqrt{n}\Big) \sim 1 - \Phi\big((\alpha - \mathbf{E}\xi)\,\sqrt{n}\big).$$

Proof of Lemma 3. Let $\{\xi_k^{(\alpha)}\}$ be a tuple of independent copies of the random variable $\xi^{(\alpha)}$; $\zeta_k^{(\alpha)} = (\xi_k^{(\alpha)} - \alpha)/\sigma^{(\alpha)}$. Put

$$S_n^{(\alpha)} = \xi_1^{(\alpha)} + \cdots + \xi_n^{(\alpha)}, \qquad F_n^{(\alpha)}(u) = \mathbf{P}\left\{\frac{\zeta_1^{(\alpha)} + \cdots + \zeta_n^{(\alpha)}}{\sqrt{n}} < u\right\}. \tag{2.9}$$

The following "inversion formula" is valid (see, for example, [2, Chap. 8, section 8])

$$\mathbf{P}\{S_n \in du\} = \varphi^n\big(\lambda(\alpha)\big)\, e^{-\lambda(\alpha)u}\mathbf{P}\{S_n^{(\alpha)} \in du\}$$

and, respectively,

$$\begin{aligned}\mathbf{P}\left\{\frac{S_n}{n} \geqq \alpha\right\} &= \varphi^n\big(\lambda(\alpha)\big)\int_x^\infty e^{-\lambda(\alpha)u}\,\mathbf{P}\big\{S_n^{(\alpha)} \in du\big\}\\ &= \varphi^n\big(\lambda(\alpha)\big)\, e^{-\lambda(\alpha)\alpha n}\int_0^\infty e^{-\lambda(\alpha)\sigma^{(\alpha)}\sqrt{n}u}\, dF_n^{(\alpha)}(u).\end{aligned}$$

Recalling the definition of Λ and integrating by parts we obtain the equality

$$\mathbf{P}\left\{\frac{S_n}{n} \geqq \alpha\right\} = e^{-n\Lambda(\alpha)}\gamma\int_0^\infty e^{-\gamma u}\big(F_n^{(\alpha)}(u) - F_n^{(\alpha)}(0)\big)\, du.$$

The set of random variables $\{\zeta_1^{(\alpha)}, \alpha_1 \leqq \alpha \leqq \alpha_2\}$ meets the conditions of Lemma 2. Applying it to estimate the difference $F_n^{(\alpha)}(u) - F_n^{(\alpha)}(0)$ in the last equality we arrive at the relation

$$\begin{aligned}\mathbf{P}\left\{\frac{S_n}{n} \geqq \alpha\right\} = e^{-n\Lambda(\alpha)}\gamma\Bigg[&\int_0^\infty e^{-\gamma u}\big(\Phi(u) - \Phi(0)\big)\, du\\ &+ \frac{m_3^{[r]}}{6\sigma^{(\alpha)3}\sqrt{n}}\int_0^\infty e^{-\gamma u}\,\big((1-u^2)\,\Phi'(u) - \Phi'(0)\big)\, du\\ &+ o\Big(\frac{1}{\sqrt{n}}\Big)\int_0^\infty e^{-\gamma u}\, du\Bigg].\end{aligned}$$

Note now that the identity $\int_0^\infty e^{-\gamma u}e^{-u^2/2}du = e^{\gamma^2/2}(1-\Phi(\gamma))\sqrt{2\pi}$ is valid (one can check it by extracting a perfect trinomial square in the exponent of the exponent). By differentiating this identity with respect to γ one can find the values of the integrals $\int_0^\infty u^k e^{-\gamma u}e^{-u^2/2}\, du$, $k = 1, 2, 3$. Proceeding in this way we arrive at the equalities

$$\begin{aligned}\int_0^\infty e^{-\gamma u}\big(\Phi(u) - \Phi(0)\big)\, du &= \frac{e^{\gamma^2/2}(1-\Phi(\gamma))}{\gamma},\\ \int_0^\infty e^{-\gamma u}\big((1-u^2)\,\Phi'(u) - \Phi'(0)\big)\, du &= \frac{\gamma}{\sqrt{2\pi}} - \gamma^2 e^{\gamma^2/2}\big(1-\Phi(\gamma)\big) - \frac{1}{\gamma\sqrt{2\pi}}.\end{aligned}$$

Consequently,

$$\text{(2.10)}\quad \mathbf{P}\left\{\frac{S_n}{n} \geqq \alpha\right\} = e^{-n\Lambda(\alpha)} e^{\gamma^2/2}\big(1-\Phi(\gamma)\big) + \frac{e^{-n\Lambda(\alpha)}}{\sqrt{n}} \left(O\left(\frac{1}{1+\gamma^2}\right) + o(1)\right).$$

The last relation implies the statement of the lemma.

LEMMA 4. *Let F be a lattice distribution with minimal lattice $\{b+kh,\, k \in \mathbf{Z}\}$ and let α_1 and α_2 be numbers such that $m \leqq \alpha_1 < \alpha_2 < \alpha_+$; if $\alpha_1 = m$, we additionally assume the finiteness of $\mathbf{E}\xi^2$. Then for $x = nb + kh$, $n \to \infty$,*

$$\mathbf{P}\{S_n = x\} \sim \frac{h}{\sqrt{2\pi n}\sigma^{(\alpha)}}\, e^{-n\Lambda(x/n)}$$

uniformly in x such that $\alpha = x/n \in [\alpha_1, \alpha_2]$.

Proof. The inversion formula in the lattice case has the form

$$\mathbf{P}\{S_n = x\} = \varphi^n\big(\lambda(\alpha)\big)\, e^{-\lambda(\alpha)\alpha n}\, \mathbf{P}\{S_n^{(\alpha)} = x\}.$$

The family of random variables $\{\xi_1^{(\alpha)}, \alpha \in [\alpha_1, \alpha_2]\}$ meets the conditions of Corollary 1. Therefore, $\mathbf{P}\{S_n^{(\alpha)} = x\} \sim h/\sqrt{2\pi n}\sigma^{(\alpha)}$; hence the needed asymptotic representation follows.

COROLLARY 4. *If $\alpha_1 > m$, then, as $n \to \infty$, we have, uniformly in $\alpha \in [\alpha_1, \alpha_2]$ such that $(\alpha - b)n/h$ is an integer,*

$$\mathbf{P}\left\{\frac{S_n}{n} \geqq \alpha\right\} \sim \frac{h}{\sqrt{2\pi n}\sigma^{(\alpha)}(1 - e^{\lambda(\alpha)h})}\, e^{-n\Lambda(\alpha)}.$$

2.6. Asymptotic behavior of taboo probabilities of large deviations of sums. Set $M_-^{(\alpha)} = \min_{k>0} S_k^{(\alpha)}$, $b(v,\alpha) = \mathbf{P}\{M_-^{(\alpha)} \geqq v\}$. In what follows we need the following statement about the asymptotic behavior of the probability:

$$q_n(v,\, x) \equiv \mathbf{P}\{S_k \geqq v \text{ for all } k < n,\ S_n \geqq x\}.$$

LEMMA 5. *Let α_1 and α_2 be arbitrary numbers such that $0 < \alpha_1 < \alpha_2$ and $\Lambda(\alpha_2) < \infty$; $v_0 \in \mathbf{R}$. Then, as $x \to \infty$, the following relation is valid uniformly in v and n such that $v \leqq v_0$ and $x/n \in [\alpha_1, \alpha_2]$:*

$$n^{-1} \log q_n(v,\, x) \geqq -\Lambda\left(\frac{x}{n}\right) + o(1).$$

If, in addition, $\alpha_2 < \alpha_+$, then the following equivalence takes place:

$$q_n(v,x) \sim b\left(v, \frac{x}{n}\right) \mathbf{P}\{S_n \geqq x\}.$$

Proof. The proof of the second statement is contained, in fact, in [3] and [4]. The only problem is that in [3] and [4] an additional Cramér condition is imposed on the characteristic function of distribution F. However, this additional condition is used to provide correct references to the previous results containing uniform asymptotic relations for large-deviation probabilities. By our arguments we have obtained the needed uniform asymptotic representations (Corollary 2 and Lemma 4) with no use of the Cramér condition on the characteristic function.

The first statement of the lemma can be deduced from the second one by applying the truncation method in a way that is similar to those used in the proof of Lemma 7 in [6, section 1]. The lemma is proved.

2.7. Some results related to the sums of probabilities $\mathbf{P}\{S_k \geqq x\}$. Let $\beta > 0$, $\varphi(\beta) = 1$, and $\varphi'(\beta) < \infty$; in particular, $\alpha_+ \geqq \varphi'(\beta) > 0$. Set $\alpha_0 \equiv \varphi'(\beta)/\varphi(\beta) = \varphi'(\beta)$. We have $\alpha_0 \in (0, \alpha_+]$, $\lambda(\alpha_0) = \beta$, and $\Lambda(\alpha_0) = \beta\alpha_0$.

As the renewal function is constructed by the sums S_k, the following representation is valid as $x \to \infty$:

$$H(x) \equiv \sum_{k \geqq 1} \mathbf{P}\{S_k \geqq x\} = \bigl(c_H + o(1)\bigr)\, e^{-\beta x}. \tag{2.11}$$

If ξ is a nonlattice random variable, then $c_H = 1/\beta\alpha_0$. If ξ is lattice with minimal lattice $\{kh,\ k \in \mathbf{Z}\}$, then $c_H = h/(1 - e^{-\beta h})\alpha_0$, and it is necessary to take x as a multiple of h.

To check (2.11), we consider the renewal function $H^{(\alpha_0)}$ constructed by the sums $S_k^{(\alpha_0)}$ with a positive mean drift. Then the inversion formula

$$H(x) = e^{-\beta x} \int_x^\infty e^{-\beta(u-x)}\, H^{(\alpha_0)}(du)$$

is valid. Applying the local renewal theorem for $H^{(\alpha_0)}$ leads to (2.11) (compare also with [7] where the asymptotic behavior of $H(x)$ is studied in more detail).

Throughout what follows $n_0 = n_0(x) = x/\alpha_0$. The asymptotic behavior of the partial sums of the probabilities $\mathbf{P}\{S_k \geqq x\}$ is described by the following lemma.

LEMMA 6. *Let $\varphi(\lambda) < \infty$ for a $\lambda > \beta$, that is, $\lambda_+ > \beta$, and let $y(x) \to \infty$ be a function such that $y(x) = o(x^{1/6})$ as $x \to \infty$. Then the following relation takes place uniformly in $y \geqq -y(x)$:*

$$\sum_{k \leqq y\sqrt{x}} \mathbf{P}\{S_{n_0+k} \geqq x\} = \bigl(c_H + o(1)\bigr)\, e^{-\beta x}\, \Phi\biggl(\frac{y\alpha_0^{3/2}}{\sigma^{(\alpha_0)}}\biggr) \quad \text{as } x \to \infty;$$

in the lattice case it is necessary to take x to be a multiple of the lattice step.

Preparatory to proving the lemma we introduce one more function. Denote

$$V(\alpha) = \frac{\Lambda(\alpha)}{\alpha}.$$

Since

$$V'(\alpha) = \alpha^{-2}\bigl(\lambda(\alpha)\,\alpha - \Lambda(\alpha)\bigr) = \alpha^{-2} \log \varphi\bigl(\lambda(\alpha)\bigr),$$

it follows that $V'(\alpha) < 0$ if $\alpha \in (0, \alpha_0)$; $V'(\alpha) > 0$ if $\alpha > \alpha_0$, and $V'(\alpha_0) = 0$. Thus, $V(\alpha)$ attains its minimal value on $(0, \infty)$ at point α_0 and this value is $\Lambda(\alpha_0)/\alpha_0 = \beta$. We have $V''(\alpha_0) = 1/\alpha_0(\sigma^{(\alpha_0)})^2$.

Proof of Lemma 6. By Corollaries 2 and 4 the following asymptotic representation is valid uniformly in $|k| \leqq 2y(x)\sqrt{x} = o(x^{2/3})$:

$$\begin{aligned} \mathbf{P}\{S_{n_0+k} \geqq x\} &\sim \frac{\alpha_0 c_H}{\sqrt{2\pi n_0}\,\sigma^{(\alpha_0)}}\, e^{-(n_0+k)\Lambda(x/(n_0+k))} \\ &= \frac{\alpha_0^{3/2} c_H}{\sqrt{2\pi x}\,\sigma^{(\alpha_0)}} e^{-xV(x/(n_0+k))}. \end{aligned} \tag{2.12}$$

In view of the equalities $V(\alpha_0) = \beta$ and $V'(\alpha_0) = 0$ we have the expansion

$$V\left(\frac{x}{n_0+k}\right) = \beta + \frac{V''(\theta)}{2}\left(\frac{x}{n_0+k} - \alpha_0\right)^2,$$

where θ lies between α_0 and $x/(n_0+k)$; in particular, $\theta \to \alpha_0$. Since

$$\frac{x}{n_0+k} - \alpha_0 = \frac{\alpha_0}{1+k/n_0} - \alpha_0 = -\frac{\alpha_0^2 k}{x}\left(1 + O\left(\frac{k}{x}\right)\right)$$

and $V''(\theta) = V''(\alpha_0) + O(k/x) = 1/\alpha_0\,(\sigma^{(\alpha_0)})^2(1+O(k/x))$ uniformly in $|k| \leqq 2y(x)\sqrt{x}$, we have for such values of k

$$e^{-xV\left(\frac{x}{n_0+k}\right)} = e^{-\beta x} e^{-\alpha_0^3 (k/\sqrt{x})^2/2(\sigma^{(\alpha_0)})^2 + o(1)}. \tag{2.13}$$

Let $\varepsilon > 0$. Substituting (2.13) in (2.12), we obtain the following equivalence which is valid uniformly over all y_1 and y_2 such that $y_2 - y_1 \geqq \varepsilon$, $|y_1|$, $|y_2| \leqq 2y(x)$:

$$\begin{aligned}\sum_{k=y_1\sqrt{x}}^{y_2\sqrt{x}} \mathbf{P}\{S_{n_0+k} \geqq x\} &\sim e^{-\beta x} \frac{\alpha_0^{3/2} c_H}{\sqrt{2\pi x}\,\sigma^{(\alpha_0)}} \sum_{k=y_1\sqrt{x}}^{y_2\sqrt{x}} e^{-\alpha_0^3 (k/\sqrt{x})^2/2(\sigma^{(\alpha_0)})^2} \\ &\sim e^{-\beta x} \frac{\alpha_0^{3/2} c_H}{\sqrt{2\pi}\,\sigma^{(\alpha_0)}} \int_{y_1}^{y_2} e^{-\alpha_0^3 t^2/2(\sigma^{(\alpha_0)})^2}\,dt \\ &= c_H e^{-\beta x}\left(\Phi\left(y_2 \frac{\alpha_0^{3/2}}{\sigma^{(\alpha_0)}}\right) - \Phi\left(y_1 \frac{\alpha_0^{3/2}}{\sigma^{(\alpha_0)}}\right)\right).\end{aligned} \tag{2.14}$$

By the exponential Chebyshev inequality we have

$$\mathbf{P}\{S_{n_0+k} \geqq x\} \leqq e^{-\lambda x} \varphi^{n_0+k}(\lambda).$$

Summing this inequality over $k \leqq -2y(x)\sqrt{x}$ and letting $\lambda = \lambda(x/(n_0 - 2y(x)\sqrt{x}))$ we obtain the estimate

$$\sum_{k\leqq -2y(x)\sqrt{x}} \mathbf{P}\{S_{n_0+k} \geqq x\} \leqq n_0 e^{-\lambda x} \varphi^{n_0 - 2y(x)\sqrt{x}}(\lambda) = \frac{x}{\alpha_0}\, e^{-xV(x/(n_0-2y(x)\sqrt{x}))}.$$

Hence, in view of (2.13),

$$\begin{aligned}\sum_{k\leqq -2y(x)\sqrt{x}} \mathbf{P}\{S_{n_0+k} \geqq x\} &\leqq cxe^{-\beta x} e^{-2\alpha_0^3 y^2(x)/(\sigma^{(\alpha_0)})^2} \\ &= o\left(e^{-\beta x} e^{-\alpha_0^3 y^2(x)/2(\sigma^{(\alpha_0)})^2}\right).\end{aligned} \tag{2.15}$$

Relations (2.14) and (2.15) imply the statement of the lemma.

COROLLARY 5. *Let $\lambda_+ > \beta$, $v_0 \in \mathbf{R}$, and let $y(x) \to \infty$ be a function such that $y(x) = o(x^{1/6})$ as $x \to \infty$. Then, uniformly in $y \geqq -y(x)$ and $v \leqq v_0$ we have as $x \to \infty$ (the notation $q_n(v,x)$ is taken from Lemma* 5)

$$\sum_{k\leqq y\sqrt{x}} q_{n_0+k}(v,x) = b(v,\alpha_0)\left(c_H + o(1)\right) e^{-\beta x}\, \Phi\left(\frac{y\alpha_0^{3/2}}{\sigma^{(\alpha_0)}}\right);$$

in the lattice case one should take x to be a multiple of the lattice step.

In particular, if $k_1(x)$ and $k_2(x)$ are sequences of integers such that $k_1(x) < k_2(x)$ and $k_2(x) - k_1(x) = o(\sqrt{x})$, then the following estimate is valid as $x \to \infty$:

$$\sum_{k\in[k_1(x),k_2(x)]} q_k(v,x) = o(e^{-\beta x}).$$

Proof. The principle contribution into the sum in Lemma 6 is made by the summands corresponding to the values of k of the order $o(x)$. For such values of k, $b(v, x/(n_0+k))$ tends to $b(v, \alpha_0)$. For this reason the corollary follows from Lemmas 5 and 6.

3. The Cramér case: Large-deviation principle for an asymptotically homogeneous chain. In this and the succeeding sections we investigate the Cramér case when $\beta > 0$, $\varphi(\beta) = 1$, and $\alpha_0 = \varphi'(\beta) < \infty$.

3.1. A homogeneous chain with zero initial condition. In this subsection we describe some properties of the distribution of $M_n = \max\{0, S_1, \dots, S_n\}$. Note that (2.3) implies

$$x^{-1} \log \mathbf{P}\{S_n \geqq x\} = -V\left(\frac{x}{n}\right) + o(1) \tag{3.1}$$

as $x \to \infty$ uniformly in n such that $x/n \in (0, \alpha_1]$ where, as before,

$$V(\alpha) = \frac{\Lambda(\alpha)}{\alpha}.$$

Here $\alpha_1 > 0$ is an arbitrary number such that $\Lambda(\alpha_1) < \infty$. Clearly, both parts of (3.1) are equal to $-\infty$ if $\Lambda(x/n) = \infty$.

Introduce continuous functions $\widetilde{V}(\alpha)$ and $\widetilde{\lambda}(\alpha)$:

$$\widetilde{V}(\alpha) \equiv \inf_{\alpha' \geqq \alpha} V(\alpha') = \begin{cases} \beta & \text{for } \alpha \leqq \alpha_0, \\ V(\alpha) & \text{for } \alpha \geqq \alpha_0, \end{cases} \qquad \widetilde{\lambda}(\alpha) \equiv \begin{cases} \beta & \text{for } \alpha \leqq \alpha_0, \\ \lambda(\alpha) & \text{for } \alpha \geqq \alpha_0. \end{cases}$$

The following lemma contains a large-deviation principle for M_n.

LEMMA 7. *Let $\alpha_1 > 0$ be such that $\Lambda(\alpha_1) < \infty$. Then, as $x \to \infty$, we have*

$$x^{-1} \log \mathbf{P}\{M_n \geqq x\} = -\widetilde{V}\left(\frac{x}{n}\right) + o(1)$$

uniformly in n such that $x/n \in (0, \alpha_1]$; $x^{-1} \log \mathbf{P}\{M_n \geqq x\} = -\infty$ if $\Lambda(x/n) = \infty$.

Proof. Let $\lambda \in [\beta, \lambda_+)$. Then $\mathbf{E}e^{\lambda\xi} \geqq 1$ and the sequence $e^{\lambda S_n}$ constitute a submartingale. Hence, applying the Doob inequality for nonnegative submartingales we get for any x the estimate

$$\mathbf{P}\{M_n \geqq x\} = \mathbf{P}\left\{\sup_{k \leqq n} e^{\lambda S_k} \geqq e^{\lambda x}\right\} \leqq e^{-\lambda x}\, \mathbf{E}e^{\lambda S_n}. \tag{3.2}$$

The right-hand side of the last inequality equals

$$e^{-\lambda x} \varphi^n(\lambda) = \exp\left\{-n\left(\frac{\lambda x}{n} - \log \varphi(\lambda)\right)\right\}.$$

If $\alpha \equiv x/n \in [\alpha_0, \infty)$, then $\lambda(\alpha) \geqq \beta$ and, consequently,

$$\mathbf{P}\{M_n \geqq x\} \leqq e^{-\lambda(\alpha)x}\, \varphi^n\big(\lambda(\alpha)\big) = e^{-n\Lambda(x/n)} = e^{-xV(x/n)}. \tag{3.3}$$

In addition, (3.2) with $\lambda = \beta$ yields the Cramér estimate

$$\mathbf{P}\{M_n \geqq x\} \leqq e^{-\beta x}, \tag{3.4}$$

valid for all n and x. Combining the last two estimates we deduce for any n and x the inequality

$$\mathbf{P}\{M_n \geqq x\} \leqq e^{-x\widetilde{V}(x/n)}. \tag{3.5}$$

Using the inequality $\mathbf{P}\{M_n \geqq x\} \geqq \mathbf{P}\{S_m \geqq x\}$, valid for any $m \leqq n$, and letting, in (3.1), $m = n$ in the case $x/n \geqq \alpha_0$ and $m = x/\alpha_0$ otherwise, we obtain the following estimate from below as $x \to \infty$:

$$x^{-1} \log \mathbf{P}\{M_n \geqq x\} \geqq -\widetilde{V}\left(\frac{x}{n}\right) + o(1). \tag{3.6}$$

Relations (3.5) and (3.6) imply the statement of the lemma.

Observe that one can get estimate (3.5) from above making use of the inequality

$$\mathbf{P}\{M_n \geqq x\} \leqq \sum_{k=1}^{n} \mathbf{P}\{S_k \geqq x\}$$

and the obvious estimates $\mathbf{P}\{S_k \geqq x\} \leqq e^{-k\,\Lambda(x/k)} = e^{-x\,V(x/k)}$.

The next lemma is a slight generalization of Theorem 10 in [1] in the part dealing with the range of values of n in which the tails of the distributions M_n and M_∞ are equivalent.

LEMMA 8. *Let $y(x) \to \infty$ as $x \to \infty$. Then, as $x \to \infty$, the following equivalence takes place uniformly in $n \geqq x/\alpha_0 + y(x)\sqrt{x}$:*

$$\mathbf{P}\{M_n \geqq x\} \sim \mathbf{P}\{M_\infty \geqq x\}.$$

Proof. We know the Cramér estimate (see, for example, [10, Chap. XII, section 5]) according to which as $x \to \infty$ (if the common distribution of summands is lattice it is necessary to select x as a multiplier of the lattice step)

$$\mathbf{P}\{M_\infty \geqq x\} \sim ce^{-\beta x}, \quad c \in (0,1). \tag{3.7}$$

Let $\lambda \in [0, \beta]$. Then $\mathbf{E}e^{\lambda\xi} \leqq 1$ and the sequence $e^{\lambda S_n}$ constitutes a supermartingale. Hence by the Doob inequality for nonnegative supermartingales we have for any x

$$\mathbf{P}\left\{\sup_{k \geqq n+1} S_k \geqq x\right\} = \mathbf{P}\left\{\sup_{k \geqq n+1} e^{\lambda S_k} \geqq e^{\lambda x}\right\} \leqq e^{-\lambda x}\,\mathbf{E}e^{\lambda S_n+1}.$$

Therefore,

$$\mathbf{P}\{M_\infty \geqq x\} - \mathbf{P}\{M_n \geqq x\} \leqq e^{-\lambda x}\,\mathbf{E}e^{\lambda S_{n+1}} = e^{-\lambda x}\varphi^{n+1}(\lambda) \leqq e^{-n(\lambda x/n - \log\varphi(\lambda))}.$$

If $\alpha \equiv x/n < \alpha_0$, then $\lambda(\alpha) < \beta$ and, respectively,

$$\mathbf{P}\{M_\infty \geqq x\} - \mathbf{P}\{M_n \geqq x\} \leqq e^{-n\Lambda(x/n)}.$$

Λ is a strictly convex function and its second derivative on the interval $[0, \alpha_0]$ is separated from below by a number $\delta > 0$; consequently,

$$\Lambda(\alpha) \geqq \Lambda(\alpha_0) + \Lambda'(\alpha_0)\,(\alpha - \alpha_0) + \frac{\delta(\alpha - \alpha_0)^2}{2} = \beta\alpha + \frac{\delta(\alpha - \alpha_0)^2}{2}.$$

The range of values of time n under consideration may be characterized by the inequality $\alpha_0 - \alpha \geqq y(x)/\sqrt{n}$ (generally speaking, by changing the values of y by a finite factor). In view of the preceding two inequalities we have in this range

$$\mathbf{P}\{M_\infty \geqq x\} - \mathbf{P}\{M_n \geqq x\} \leqq e^{-\beta x}\, e^{-\delta y^2(x)/2} = o(e^{-\beta x}) \quad \text{as } x \to \infty,$$

which combined with (3.7) completes the proof.

3.2. The upper bounds for the large-deviation probabilities of an asymptotically homogeneous Markov chain. In this subsection we assume that X is a real-valued asymptotically homogeneous Markov chain.

LEMMA 9. *Let $\lambda_1 \in [\beta, \lambda_+)$, $\lambda_1 < \infty$; if $\lambda_1 = \lambda_+$, then additionally we assume that $\varphi'(\lambda_+) < \infty$. Denote by α_1 a finite solution of $\lambda(\alpha) = \lambda_1$. For any $\lambda \in [0, \lambda_1)$, let the initial distribution of the chain have the finite exponential moment $\mathbf{E}e^{\lambda X(0)} < \infty$ and let the increments of the chain satisfy the condition*

$$\sup_{u<0} \mathbf{E}e^{\lambda(u+\xi(u))} < \infty, \qquad \sup_{u\geqq 0} \mathbf{E}e^{\lambda\xi(u)} < \infty. \tag{3.8}$$

Then, as $x \to \infty$, the relation

$$x^{-1} \log \pi_n(x) \leqq -\widetilde{V}_1\left(\frac{x}{n}\right) + o(1) \tag{3.9}$$

is valid uniformly over all values of n such that the ratio x/n is bounded from above. Here $\widetilde{V}_1(\alpha)$ is a continuous function specified by the equality

$$\widetilde{V}_1(\alpha) = \begin{cases} \beta & \text{if } \alpha \leqq \alpha_0, \\ \dfrac{\Lambda(\alpha)}{\alpha} & \text{if } \alpha_0 \leqq \alpha \leqq \alpha_1, \\ \dfrac{\Lambda(\alpha_1) + (\alpha - \alpha_1)\lambda_1}{\alpha} & \text{if } \alpha \geqq \alpha_1. \end{cases}$$

Remark 4. By definition, $\widetilde{V}_1(\alpha)$ coincides with $\widetilde{V}(\alpha)$ on the set $\alpha \leqq \alpha_1$. If $\lambda_+ < \infty$, $\varphi'(\lambda_+) < \infty$, and $\lambda_1 = \lambda_+$, then $\alpha_1 = \alpha_+ < \infty$ and $\widetilde{V}_1(\alpha) = \widetilde{V}(\alpha)$ for all α. In the general case, $\widetilde{V}_1 \leqq \widetilde{V}$.

Proof of Lemma 9. Let λ_2 be an arbitrary number less than λ_1, let α_2 solve $\lambda(\alpha) = \lambda_2$, and let $\widetilde{V}_2(\alpha)$ be a function constructed as $\widetilde{V}_1(\alpha)$ with replacement of λ_1 and α_1 by λ_2 and α_2, respectively.

For any $U > 0$ we specify a random variable $\Xi(U)$ with distribution

$$\mathbf{P}\{\Xi(U) \geqq y\} = \max\left(\sup_{u<U} \mathbf{P}\{u + \xi(u) \geqq U + y\},\ \sup_{u\geqq U} \mathbf{P}\{\xi(u) \geqq y\}\right).$$

Since $\xi(u) \Rightarrow \xi$ weakly as $u \to \infty$, it follows that $\Xi(U)$ weakly converges to ξ as $U \to \infty$, and by condition (3.8) the exponential moments converge as well: for any $\lambda \in [0, \lambda_2]$

$$\mathbf{E}e^{\lambda\Xi(U)} \longrightarrow \mathbf{E}e^{\lambda\xi} \quad \text{as } U \to \infty.$$

Denote $\varphi(U,\lambda) = \mathbf{E}e^{\lambda\Xi(U)}$, $\beta(U) = \sup\{\lambda\colon \varphi(U,\lambda) \leqq 1\}$, $\alpha_0(U) = \varphi'(U,\beta(U))$. Let $\Lambda(U,\alpha)$ be the deviation function of the random variable $\Xi(U)$, $\lambda(U,\alpha) = \Lambda'(U,\alpha)$, and let $\alpha_2(U)$ solve $\lambda(U,\alpha) = \lambda_2$ and

$$\widetilde{V}_2(U,\alpha) = \begin{cases} \beta(U) & \text{if } \alpha \leqq \alpha_0(U), \\ \dfrac{\Lambda(U,\alpha)}{\alpha} & \text{if } \alpha_0(U) \leqq \alpha \leqq \alpha_2(U), \\ \dfrac{\Lambda(U,\alpha_2(U)) + (\alpha - \alpha_2(U))\,\lambda_2}{\alpha} & \text{if } \alpha \geqq \alpha_2(U). \end{cases}$$

All these quantities are well defined for sufficiently large values of U. The deviation function $\Lambda(U,\alpha)$ converges to $\Lambda(\alpha)$ on the set $\alpha \leqq \alpha_2$ (α_2 solves $\lambda(\alpha) = \lambda_2$) and, respectively,

$$\widetilde{V}_2(U,\alpha) \longrightarrow \widetilde{V}_2(\alpha) \quad \text{as } U \to \infty. \tag{3.10}$$

Consider a homogeneous Markov chain $Y_n(U) = (Y_{n-1}(U) + \Xi_n(U))^+$ where $\Xi_n(U)$ are independent copies of $\Xi(U)$. By our construction the inequality $v+\xi(v) \leqq_{\rm st} u + \Xi(U)$ is valid for any initial states v and u subject to the conditions $v \leqq u$ and $u \geqq U$. Hence the homogeneous chain $U + Y_n(U)$ majorizes in probability the original chain X_n. In particular,

$$\mathbf{P}\bigl\{X(n) \geqq x\bigr\} \leqq \mathbf{P}\bigl\{U + Y_n(U) \geqq x\bigr\}.$$

This inequality and (3.5) yield

$$\mathbf{P}\bigl\{X(n) \geqq x\bigr\} \leqq e^{-(x-U)\widetilde{V}(U,(x-U)/n)}.$$

The statement of the lemma follows now from the inequality $\widetilde{V}_2(U,\alpha) \leqq \widetilde{V}(U,\alpha)$, convergence (3.10), and from the possibility of choosing arbitrary λ_2 less than λ_1.

In the case when ξ is a random variable bounded from above, we can deduce from Lemma 9 the following statement by letting $\lambda_1 \to \infty$, (respectively, $\alpha_1 \to \alpha_+$).

Corollary 6. *Let $\lambda_+ = \infty$ and $\alpha_+ < \infty$; that is, ξ is bounded from above by α_+. For any $\lambda > 0$, let condition* (3.8) *be valid and $\mathbf{E}e^{\lambda X(0)}$ be finite. Then for any $\varepsilon > 0$ the following relation is valid as $x \to \infty$:*

$$x^{-1}\log \pi_n(x) \leqq -\widetilde{V}\left(\frac{x}{n}\right) + o(1)$$

uniformly in n such that $x/n \leqq \alpha_+ - \varepsilon$, and the convergence

$$x^{-1}\log \pi_n(x) \to -\infty$$

is uniform in n such that $x/n \geqq \alpha_+ + \varepsilon$.

3.3. Lower bounds for the large-deviation probabilities of an asymptotically homogeneous Markov chain. In this subsection we consider an asymptotically homogeneous Markov chain X with values in $\mathbf{R}$. The initial distribution of the chain is assumed to be arbitrary.

Lemma 10. *Let α_1 be a number exceeding α_0 (in the case when ξ is bounded we additionally assume that $\alpha_1 < \alpha_+$). For any level $u \in \mathbf{R}$, let there be $n_0 = n_0(u)$ such that*

$$\mathbf{P}\bigl\{X(n_0) \geqq u\bigr\} > 0. \tag{3.11}$$

Then, as $x \to \infty$

$$x^{-1} \log \pi_n(x) \geqq -\widetilde{V}\left(\frac{x}{n}\right) + o(1)$$

uniformly in n *such that* $\alpha_0 \leqq x/n \leqq \alpha_1$.

If, in addition, for any $u \in \mathbf{R}$ *there exists* n_0 *such that*

$$\inf_{n \geqq n_0} \mathbf{P}\big\{X(n) \geqq u\big\} > 0, \tag{3.12}$$

then, as $x \to \infty$, *we have*

$$x^{-1} \log \pi_n(x) \geqq -\beta + o(1)$$

uniformly in n *such that* $x/n \leqq \alpha_0$.

Remark 5. In view of the weak convergence $\xi(u) \Rightarrow \xi$ as $u \to \infty$ and condition $\mathbf{P}\{\xi > 0\} > 0$ there exists a level U such that for any $u \geqq U$ the inequality $\mathbf{P}\{\xi(u) \geqq \delta\} \geqq \delta$ holds for some $\delta > 0$. Therefore, in order that condition (3.11) be valid it is necessary and sufficient that the event $\{X(n) \geqq U\}$ has positive probability for some n. Respectively, condition (3.12) is equivalent to the fact that the probabilities of the events $\{X(n) \geqq U\}$ are separated from zero uniformly in $n \geqq N$ for some N.

Remark 6. A sufficient condition for (3.12) to be valid is the weak convergence of the distribution of $X(n)$ (ergodicity) to the distribution of a random variable $X(\infty)$ which is not bounded from above.

Proof of Lemma 10. For any u consider a random variable $\eta(u)$ with distribution $\mathbf{P}\{\eta(u) \geqq y\} = \inf_{v \geqq u} \mathbf{P}\{\xi(v) \geqq y\}$. By our construction $\eta(u) \leqq_{\rm st} \xi(v)$ for any u and v related by the inequality $u \leqq v$. Since the chain is asymptotically homogeneous, we have $\eta(u) \Rightarrow \xi$ as $u \to \infty$. Denote by $T_i(u) = \eta_1(u) + \cdots + \eta_i(u)$ the sum of i independent copies of $\eta(u)$. By construction the following inequality is valid for any $u \leqq v \leqq y$ and m:

$$\begin{aligned} &\mathbf{P}\big\{X(v, k) \geqq v \text{ for all } k < m,\ X(v,m) \geqq v + y\big\} \\ &\qquad \geqq \mathbf{P}\big\{T_k(u) \geqq 0 \text{ for all } k < m,\ T_m(u) \geqq y\big\}. \end{aligned} \tag{3.13}$$

For any $N < n$ the probability that $X(n)$ exceeds a level x can be evaluated as follows ($U \geqq 0$):

$$\begin{aligned} \pi_n(x) &\geqq \mathbf{P}\big\{X(N) \geqq U,\ X(k) \geqq X(N) \text{ for } N < k < n, X(n) - X(N) \geqq x\big\} \\ &= \mathbf{P}\big\{X(N) \geqq U\big\}\, \mathbf{P}\big\{X(k) \geqq X(N) \text{ for } N < k < n, \\ &\qquad X(n) - X(N) \geqq x \mid X(N) \geqq U\big\} \equiv p_1 p_2. \end{aligned} \tag{3.14}$$

Let us estimate p_1 and p_2 from below. By condition (3.11) for any fixed U there exists an N such that $p_1 > 0$. Applying (3.13) with $y = x$ and $m = n - N$, we deduce the following estimate for p_2:

$$p_2 \geqq \mathbf{P}\big\{T_k(U) \geqq 0 \text{ for all } k < n - N,\ T_{n-N}(U) \geqq x\big\}.$$

Substituting the obtained estimates in (3.14) we arrive at the inequality

$$\pi_n(x) \geqq p_1 \mathbf{P}\big\{T_k(U) \geqq 0 \text{ for all } k < n - N,\ T_{n-N}(U) \geqq x\big\}. \tag{3.15}$$

Hence by Lemma 5 we derive the following estimate as $x \to \infty$:

$$x^{-1} \log \pi_n(x) \geqq \widetilde{V}_{\eta(U)}\left(\frac{x}{n-N}\right) + o(1) \tag{3.16}$$

uniformly in n such that $\alpha_0 \leqq x/(n-N) \leqq \alpha_1$, where $\widetilde{V}_{\eta(U)}$ is the function $\widetilde{V}$, corresponding to the random variable $\eta(U)$.

We know that the family of random variables $\eta(u)$ is stochastically nondecreasing and weakly converges to ξ. Therefore, $\widetilde{V}_{\eta(u)}(\alpha) \to \widetilde{V}(\alpha)$ as $u \to \infty$ uniformly in $\alpha \in [\alpha_0, \alpha_1]$. Now the first statement of the lemma follows from (3.16), since the level U is arbitrary.

To prove the second statement of the lemma we put $N = n - x/\alpha_0$ (with the known agreement that N should be integer). Let U be an arbitrary level. By condition (3.12) there exists n_0 such that for all $N \geqq n_0$ the probability $p_1 = p_1(N)$ is not less than a positive number δ. Inequality (3.15) transforms in the following one:

$$\pi_n(x) \geqq \delta \mathbf{P}\left\{T_k(U) \geqq 0 \text{ for all } k < \frac{x}{\alpha_0},\ T_{x/\alpha_0}(U) \geqq x\right\}.$$

Evaluating the last probability by Lemma 5 we obtain, as $x \to \infty$, that $x^{-1} \log \pi_n(x) \geqq \widetilde{V}_{\eta(U)}(\alpha_0) + o(1)$ uniformly over n satisfying $x/n \leqq \alpha_0$. This, as before, completes the proof.

3.4. Rough (logarithmic) asymptotics of the large-deviation probabilities of an asymptotically homogeneous chain. In this subsection we also assume that X is an asymptotically homogeneous Markov chain with values in $\mathbf{R}$. The next theorem follows from Lemmas 9 and 10 and Corollary 6.

THEOREM 1. *Let $\lambda_1 \in [\beta, \lambda_+)$, $\lambda_1 < \infty$, and if $\lambda_1 = \lambda_+$, then we additionally assume that $\varphi'(\lambda_1) < \infty$. Denote by α_1 the solution of $\lambda(\alpha) = \lambda_1$. For any $\lambda \in (0, \lambda_1)$, let the initial distribution of the chain have finite exponential moment $\mathbf{E}e^{\lambda X(0)}$ and let the increments of the chain meet condition* (3.8). *Let condition* (3.12) *be valid. Then, as $x \to \infty$, the equality*

$$x^{-1} \log \pi_n(x) = -\widetilde{V}\left(\frac{x}{n}\right) + o(1) \tag{3.17}$$

is valid uniformly in n such that $x/n \leqq \alpha_1$.

If $\lambda_1 = \lambda_+$, then equality (3.17) *takes place if $x \to \infty$ uniformly in n such that* $\sup x/n < \infty$.

Note that the condition on $X(0)$ may be weakened depending on the growth of n and x.

4. The Cramér case: The exact asymptotics of $\pi_n(x)$ for a partially homogeneous chain. Let X be a U-partially homogeneous Harris chain with values in $\mathbf{R}$ and $0 < \beta < \lambda_+$. We assume that the sequence of measures π_n converges in the metric of total variation to a measure π (being invariant with necessity) which is unbounded to the right, that is,

$$\sup_{B \subseteq \mathbf{R}} \left|\pi_n(B) - \pi(B)\right| \longrightarrow 0, \quad n \to \infty, \tag{4.1}$$

where the supremum is taken over all Borel sets B. For a countable chain X, the convergence in variation takes place automatically if the chain is irreducible, nonperiodic, and admits an invariant distribution (in this case π is the invariant distribution); the conditions of convergence in variation for real-valued chains can be found, for example, in [12] and [15].

In this section we assume that the increments of the chain satisfy for some λ_1 from the interval (β, λ_+) the condition

$$\sup_{u<U} \int_0^\infty e^{\lambda_1 v}\, P(u,\, dv) < \infty, \tag{4.2}$$

and that the initial distribution π_0 has finite exponential moment of the same order:

$$\mathbf{E} e^{\lambda_1 X(0)} < \infty. \tag{4.3}$$

Let α_1 solve $\lambda(\alpha) = \lambda_1$. The following theorem is valid.

THEOREM 2. *Let F be a nonlattice distribution.*

(a) *Let $y = y_n(x) \in \mathbf{R}$ be such that $n = x/\alpha_0 + y\sqrt{x}$; let $z(x)$ be an arbitrary function such that $z(x) \to \infty$, $z(x) = o(x^{1/6})$ as $x \to \infty$. Then the following relation is valid, as $x \to \infty$, uniformly for all n such that $y_n(x) \geqq -z(x)$:*

$$\pi_n(x) = \bigl(c(\alpha_0) + o(1)\bigr)\, e^{-\beta x}\, \Phi\biggl(\frac{y\alpha_0^{3/2}}{\sigma^{(\alpha_0)}}\biggr),$$

where $\Phi(v)$ is the distribution function of the standard normal law, and $c(\alpha_0) > 0$ is a constant independent of the initial distribution of the chain X (see (4.10)).

In particular, if $\widehat{y}(x) \to \infty$, then the following asymptotics takes place, as $x \to \infty$, uniformly in $n \geqq x/\alpha_0 + \widehat{y}(x)\sqrt{x}$:

$$\pi_n(x) \sim c(\alpha_0)\, e^{-\beta x}.$$

(b) *If $x/n \to \alpha \in (\alpha_0, \alpha_1)$, then as $x \to \infty$*

$$\pi_n(x) \sim c(\alpha)\, n^{-1/2} e^{-xV(x/n)},$$

where $c(\alpha)$ is a continuous function in α depending on the initial distribution of the chain X and specifying by formula (4.14) for $\alpha \in (\alpha_0, \alpha_1)$.

Proof. To simplify notation we use the symbol π_n to denote the distribution of $X(n)$:

$$\pi_n(B) = \mathbf{P}\bigl\{X(n) \in B\bigr\}.$$

Thus, $\pi_n(x) \equiv \pi_n([x, \infty))$. This agreement leads to no confusion. The basis for our subsequent arguments is the total probability formula with respect to the last entry of the chain into the set $(-\infty, U]$:

$$\begin{aligned} \pi_n(x) = {} & \int_U^\infty \pi_0(dv)\, q_n(U - v,\, x - v) \\ & + \sum_{k=0}^{n-1} \int_{-\infty}^U \pi_k(du) \int_U^\infty P(u,\, dv)\, q_{n-k-1}(U - v,\, x - v), \end{aligned} \tag{4.4}$$

where

$$\begin{aligned} q_k(U - v,\, x - v) &\equiv \mathbf{P}\bigl\{X(v,\, l) \geqq U,\ l = 1, \dots, k-1;\ X(v, k) \geqq x\bigr\} \\ &= \mathbf{P}\{S_l \geqq U - v,\ l = 1, \dots, k-1;\ S_k \geqq x - v\}, \end{aligned}$$

since the chain is U-partially homogeneous. $\int_U^\infty$ in (4.4) is understood as $\int_{U+0}^\infty$. The notation $q_k(v,x)$ has been introduced in Lemma 5.

Splitting the integration domain in the integrals with respect to the variable v into two parts $(U,U_1]$ and (U_1,∞), we obtain the inequality

$$\begin{aligned}\Bigl|\pi_n(x) - \int_U^{U_1} \pi_0(dv)\, q_n(U-v,\, x-v) \\ - \sum_{k=0}^{n-1} \int_{-\infty}^{U} \pi_k(du) \int_U^{U_1} P(u,\, dv) q_{n-k-1}(U-v, x-v)\Bigr| \\ \leqq \int_{U_1}^{\infty} \pi_0(dv)\, q_n(U-v,\, x-v) \\ + \sum_{k=0}^{n-1} \sup_{u<U} \int_{U_1}^{\infty} P(u,\, dv) q_{n-k-1}(U-v,\, x-v).\end{aligned} \tag{4.5}$$

Our immediate goal is to select a level $U_1 = U_1(x)$, growing sufficiently slowly and in such a way that the estimate in the right-hand side would be infinitesimal with respect to the claimed asymptotics of $\pi_n(x)$ in the whole spectrum of deviations.

Since $q_k(U-v, x-v) \leqq \mathbf{P}\{S_k \geqq x-v\}$, it follows by the exponential Chebyshev inequality that for any $\lambda \geqq \beta$, the estimate

$$\sum_{k=0}^{n-1} q_{n-k-1}(U-v,\, x-v) \leqq e^{-(x-v)\lambda} \sum_{k=0}^{n-1} \varphi^{n-k-1}(\lambda) \leqq n e^{v\lambda} e^{-x\lambda} \varphi^n(\lambda)$$

is valid in view of $\varphi(\lambda) \geqq 1$. If $x/n \geqq \alpha_0$, then letting $\lambda = \lambda(x/n)$ here we obtain for such n the inequality

$$\sum_{k=0}^{n-1} q_{n-k-1}(U-v,\, x-v) \leqq n e^{v\lambda(x/n)} e^{-xV(x/n)} \leqq c_1 x e^{v\lambda(x/n)} e^{-xV(x/n)}.$$

If $x/n \leqq \alpha_0$, we apply inequality (2.11), according to which

$$\sum_{k=0}^{n-1} q_{n-k-1}(U-v,\, x-v) \leqq c_2 e^{v\beta} e^{-\beta x}.$$

Thus, for any n the right-hand side in (4.5) does not exceed

$$\begin{aligned} c_3 x\, e^{-x\widetilde{V}(x/n)} \left(\int_{U_1}^{\infty} \pi_0(dv) + \sup_{u<U} \int_{U_1}^{\infty} P(u,\, dv) \right) e^{-v\widetilde{\lambda}(x/n)} \\ \leqq c_4 x\, e^{-x\widetilde{V}(x/n)}\, e^{-U_1(\lambda_1 - \widetilde{\lambda}(x/n))}\end{aligned}$$

in view of conditions (4.2) and (4.3).

In each of the cases (a) and (b) there exists an $\varepsilon > 0$ such that $\lambda_1 - \widetilde{\lambda}(x/n) > \varepsilon$ for all sufficiently large x. Put $U_1 = 2\varepsilon^{-1} \log x$. Then the right-hand side of the previous estimate does not exceed $c_4 e^{-x\widetilde{V}(x/n)}/x$. Taking this fact into account in (4.5), we obtain, as $x \to \infty$, the relation

$$\begin{aligned}\pi_n(x) = \int_U^{U_1} \pi_0(dv)\, q_n(U-v,\, x-v) \\ + \sum_{k=0}^{n-1} \int_{-\infty}^{U} \pi_k(du) \int_U^{U_1} P(u, dv)\, q_{n-k-1}(U-v,\, x-v) + o\biggl(\frac{e^{-x\widetilde{V}(x/n)}}{\sqrt{n}} \biggr).\end{aligned} \tag{4.6}$$

The remaining calculations differ essentially for cases (a) and (b) of the theorem.

Case (a). Splitting the sum in equality (4.6) into two and taking into account the inequality $\widetilde{V}(x/n) \geqq \beta$, we obtain

$$\pi_n(x) = \int_U^{U_1} \pi_0(dv) q_n(U-v, x-v) + \left(\sum_{k=0}^{\log x} + \sum_{k=\log x}^{n-1} \right) \times \int_{-\infty}^{U} \pi_k(du) \int_U^{U_1} P(u,\, dv)\, q_{n-k-1}(U-v,\, x-v) + o(e^{-x\beta})$$

$$\equiv I_0 + \Sigma_1 + \Sigma_2 + o(e^{-x\beta}). \tag{4.7}$$

The second sum gives the principal contribution to the asymptotics of the probability $\pi_n(x)$, whereas the contribution of the remaining summands is negligible. Let us check this fact. Convergence in variation π_k to π as $x \to \infty$ takes place uniformly in $k > \log x$. Therefore, the equivalence

$$\Sigma_2 \sim \int_{-\infty}^{U} \pi(du) \int_U^{U_1} P(u,\, dv) \sum_{k=\log x}^{n-1} q_{n-k-1}(U-v,\, x-v)$$

$$\sim \int_{-\infty}^{U} \pi(du) \int_U^{U_1} P(u, dv)\, b(U-v,\, \alpha_0)\, c_H e^{-\beta(x-v)} \Phi\left(\frac{y\alpha_0^{3/2}}{\sigma^{(\alpha_0)}} \right) \tag{4.8}$$

takes place in view of Corollary 5 where, as before, $b(U-v, \alpha) = \mathbf{P}\{M_-^{(\alpha)} \geqq U - v\}$.

Let us show the relative smallness of the integral I_0 and the sum Σ_1. Using sequentially the second statement of Corollary 5 and conditions (4.3) and (4.2), we arrive at the relations

$$I_0 + \Sigma_1 = o(e^{-\beta x}) \left(\int_U^{U_1} \pi_0(dv)\, e^{\beta v} + \sup_{u<U} \int_U^{U_1} P(u,\, dv) \right) e^{\beta v} = o(e^{-\beta x}). \tag{4.9}$$

Substituting (4.9) and (4.8) in (4.7) we obtain case (a) of the theorem with constant

$$c(\alpha_0) = \frac{1}{\beta\alpha_0} \int_{-\infty}^{U} \pi(du) \int_U^{\infty} b(U-v,\, \alpha_0)\, e^{\beta v} P(u,\, dv). \tag{4.10}$$

Case (b). We split the sum in equality (4.6) into two:

$$\pi_n(x) = \int_U^{U_1} \pi_0(dv)\, q_n(U-v,\, x-v) + \left(\sum_{k=0}^{n^{1/7}} + \sum_{k=n^{1/7}}^{n-1} \right) \int_{-\infty}^{U} \pi_k(du) \int_U^{U_1} P(u,\, dv)\, q_{n-k-1}(U-v,\, x-v)$$

$$+\, o\left(\frac{e^{-xV(x/n)}}{\sqrt{n}} \right) \equiv I_0 + \Sigma_1 + \Sigma_2 + o\left(\frac{e^{-xV(x/n)}}{\sqrt{n}} \right). \tag{4.11}$$

Since $x/n \to \alpha \in (\alpha_0, \alpha_1)$, Lemma 5 shows that, as $x \to \infty$, the following relation takes place uniformly in $k = o(n)$ and $v = o(x)$:

$$q_{n-k}(U-v,\, x-v) = \mathbf{P}\{S_{n-k} \geqq x - v\} \big(b(U-v,\, \alpha) + o(1) \big). \tag{4.12}$$

In addition, by Corollary 2 we have for $k = o(n)$ and uniformly in $v \in [U, U_1)$ (respectively, $v^2/n \leqq U_1^2/n \longrightarrow 0$)

$$\begin{aligned}\mathbf{P}\{S_{n-k} \geqq x - v\} &= \frac{\widehat{c}(\alpha) + o(1)}{\sqrt{n-k}}\, e^{-(n-k)\Lambda((x-v)/(n-k))} \\ &= \big(\widehat{c}(\alpha) + o(1)\big)\, n^{-1/2} e^{-(n-k)\Lambda(x/(n-k)) + v\lambda(x/(n-k)) + o(1)} \\ &\sim \widehat{c}(\alpha)\, n^{-1/2} e^{-xV(x/(n-k)) + v\lambda(\alpha) + o(v)},\end{aligned}$$

where $\widehat{c}(\alpha) = 1/\sqrt{2\pi}\sigma^{(\alpha)}\lambda(\alpha)$ is a function continuous in α. Continuing our calculations, we deduce the relation

$$\mathbf{P}\{S_{n-k} \geqq x - v\} \sim \big(\widehat{c}(\alpha) + o(1)\big)\, n^{-1/2} e^{-xV(x/n) - \alpha^2 kV'(x/n) + v\lambda(\alpha) + o(v)},$$

valid uniformly in $k \leqq n^{1/7}$ and $v \leqq U_1 = O(\log n)$. As was mentioned in section 2.7, $V'(\alpha) = \alpha^{-2} \log \varphi(\lambda(\alpha))$. Hence, taking into account (4.12), we have the asymptotics

$$I_0 + \Sigma_1 \sim c(\alpha)\, n^{-1/2} e^{-xV(x/n)}, \tag{4.13}$$

where

$$c(\alpha) = \frac{1}{\sqrt{2\pi}\sigma^{(\alpha)}\lambda(\alpha)} \bigg(\int_U^\infty \pi_0(dv) + \sum_{k=0}^\infty \frac{1}{\varphi^k(\lambda(\alpha))} \int_{-\infty}^U \pi_k(du) \int_U^\infty P(u, dv) \bigg)$$
$$\times\, b(U - v,\, \alpha)\, e^{\lambda(\alpha)v}. \tag{4.14}$$

We show that the sum Σ_2 in (4.11) is negligible in comparison with Σ_1. For $k \geqq n^{1/7}$ and $v \leqq U_1 = O(\log n)$ the inequality $(x-v)/(n-k) \geqq x/n + n^{1/8}/x$ holds for sufficiently large n. Since $x/n \to \alpha > \alpha_0$, it follows, in particular, that $(x-v)/(n-k) > \alpha_0$. In view of this fact, the exponential Chebyshev inequality, and the inequality, $V((x-v)/(n-k)) \geqq V(x/n) + V'(x/n)\, n^{1/8}/x$ (valid in view of the convexity of V), we have for $k \geqq n^{1/7}$ and $v \leqq U_1$ the estimates

$$\mathbf{P}\{S_{n-k} \geqq x - v\} \leqq e^{v\lambda((x-v)/(n-k))} e^{-xV((x-v)/(n-k))} \leqq e^{v\lambda_1}\, e^{-xV(x/n)}\, e^{-n^{1/9}}.$$

This and condition (4.2) imply that $\Sigma_2 = o(e^{-xV(x/n)}/\sqrt{n})$. Substituting this estimate and asymptotics (4.13) in (4.11) we obtain statement (b) of the theorem.

Let F be a lattice distribution with minimal lattice $\{kh,\ k \in \mathbf{Z}\}$, and let X be a chain taking only the values from this lattice in the domain $[U, \infty)$. Then the statements of the theorem for the probabilities $\pi_n(x)$ remain valid if we restrict ourselves to the values x from this lattice and multiply the constants $c(\alpha_0)$ and $c(\alpha)$ by $\beta h/(1 - e^{-\beta h})$ and $\lambda(\alpha)\, h/(1 - e^{-\lambda(\alpha)h})$, respectively.

5. The intermediate case: Exact asymptotics of $\pi_n(x)$ for an asymptotically homogeneous chain. In this section we investigate an asymptotically homogeneous Markov chain in the intermediate case. Here the picture is absolutely different in comparison with the Cramér case, and the asymptotic behavior of the large-deviation probabilities is found in an explicit form for all values of time variable n.

In comparison with the preceding sections we consider here a more general situation. Namely, we assume in this section that there exists a random variable η with distribution G and Laplace transform $\psi(\lambda) = \mathbf{E}\, e^{\lambda\eta}$ such that, for some $V \in \mathbf{R}$, we have

$$v + \xi(v) \leqq_{\mathrm{st}} V + \eta \quad \text{for } v < V, \qquad \xi(v) \leqq_{\mathrm{st}} \eta \quad \text{for } v \geqq V. \tag{5.1}$$

The principal conditions are imposed on η: $\mathbf{E}\eta < 0$ and $\psi(\beta) < 1$, where $\beta = \sup\{\lambda\colon\ \psi(\lambda) \leqq 1\}$. Note that now *parameter* β, playing an important role everywhere, *is attributed to the distribution* G *of the majorant* η rather than to the distribution F of ξ.

Let us recall the definitions of certain classes of functions and distributions playing an important role in studying the large-deviation probabilities in the intermediate case.

Following [5], we say that a positive function g behaves like a *local power* function if, for any fixed t, the limit of the ratio $g(u+t)/g(u)$ is equal to 1 as $u \to \infty$. Similar to the agreements above we set for brevity $G(x) = G([x,\infty))$, $\pi(x) = \pi([x,\infty))$.

DEFINITION 2. *We say that a nonlattice distribution* G *in* $\mathbf{R}$ *belongs to the class* $\mathcal{S}(\gamma)$, $\gamma \geqq 0$, *if*

(a) $e^{\gamma u} G(u)$ *is a local power function*;

(b) $\psi(\gamma) \equiv \int_{\mathbf{R}} e^{\gamma u} G(du) < \infty$;

(c) *the convolution of the distribution* G *with itself satisfies the condition of asymptotic equivalence* $G * G(u) \sim 2\psi(\gamma)\, G(u)$ *as* $u \to \infty$.

We say that a distribution G *in* $\mathbf{Z}$ *belongs to the class* $\mathcal{S}(\gamma)$, $\gamma \geqq 0$, *if properties* (a)–(c) *hold for* $u \in \mathbf{Z}$.

Class $\mathcal{S}(\gamma)$ includes, in particular, the distributions with the tails of type $e^{-\gamma u} g(u)$, where $g(u)$ is an integrable function regularly varying at infinity. Note that if the distribution of a random variable η belongs to class $\mathcal{S}(\gamma)$ and $\psi(\gamma) \leqq 1$, then $\beta = \gamma$.

5.1. Upper bounds for the tails of the prestationary and stationary distributions of a chain. This subsection contains a generalization of Lemma 2 in [5] related to an upper bound of the tail of the distribution of a chain with values in $\mathbf{R}$ in the intermediate case.

LEMMA 11. *Let the distribution* G *of a random variable* η *belong to class* $\mathcal{S}(\beta)$. *Then if* X *is a chain having an invariant measure* π (*generally speaking*, *nonunique*), *then*

$$\sup_x \frac{\pi(x)}{G(x)} < \infty. \tag{5.2}$$

If the initial distribution π_0 *of the chain is such that* $\pi_0(x) \leqq c' G(x)$ *for some* c', *then*

$$\sup_{n,x} \frac{\pi_n(x)}{G(x)} < \infty. \tag{5.3}$$

Proof. Consider a space homogeneous chain $Y = \{Y(n)\}$ with nonnegative values specified by the equality $Y(n+1) = (Y(n) + \eta_{n+1})^+$, where η_n are random variables which are independent copies of η. By (5.1) the Markov chain $V + Y(n)$ majorizes the chain $X(n)$ and, therefore,

$$\pi(x) \leqq \pi_Y(x - V), \tag{5.4}$$

where π_Y is an invariant measure of Y. Since $\mathbf{E}e^{\beta\eta} < 1$ and the distribution of the random variable η belongs to the class $\mathcal{S}(\beta)$, it follows by Theorem 1 in [11] that

$\pi_Y(x) \sim c_1\mathbf{P}\{\eta \geqq x\}$ as $x \to \infty$ for some c_1. The function $e^{\beta t}G(t)$ being local power, $\pi_Y(x-V) \sim c_1 e^{\beta V} G(x)$. Substituting the derived equivalence in (5.4) we arrive at estimate (5.2).

We prove (5.3). Let $Y(0) = 0$ and, therefore, $Y(1) = \eta_1^+$. In view of the condition $\pi_0(x) \leqq c'G(x)$ and the fact that $e^{\beta x}G(x)$ is a local power function, there exists a sufficiently large level V' such that $X(0) \leqq_{\rm st} V' + Y(1)$. From this and condition (5.1) we conclude that $X(n) \leqq_{\rm st} V' + Y(n+1)$ for any n and, respectively,

$$\pi_n(x) \leqq \mathbf{P}\bigl\{Y(n+1) \geqq x - V'\bigr\}.$$

The last inequality implies (5.3), since, as we know, the homogeneous chain $Y(n)$ with zero initial condition does not decrease in distribution and, therefore, $\mathbf{P}\{Y(n+1) \geqq x - V'\} \leqq \pi_Y(x - V')$ for any n. The lemma is proved.

5.2. Exact asymptotics of the prestationary distributions.

THEOREM 3. *Let the distribution G of a random variable η be a nonlattice and belong to the class $\mathcal{S}(\beta)$. Let condition* (5.1) *be valid and, for any $u \in \mathbf{R}$, there exist $c(u)$ such that*

$$\frac{\mathbf{P}\{\xi(u) \geqq t\}}{G(t)} \longrightarrow c(u) \tag{5.5}$$

as $t \to \infty$ uniformly in u from any compact. In addition, we assume that convergence (4.1) *takes place. Then*

(i) *if the initial distribution π_0 is such that $\pi_0(x)/G(x) \longrightarrow c_0 \geqq 0$ then, uniformly in $n \geqq 0$, the relation*

$$\pi_n(x) = \bigl(c_n + o(1)\bigr)\, G(x) \quad \text{as } x \to \infty \tag{5.6}$$

is valid, where c_n is specified for $n \geqq 1$ by the recurrence $c_n = \alpha_{n-1} + \varphi(\beta)c_{n-1}$,

$$\alpha_{n-1} = \int_{\mathbf{R}} c(u)\, e^{\beta u}\, \pi_{n-1}(du) \geqq 0;$$

(ii) *if $\pi_0(x) \leqq c'G(x)$, then as n, $x \to \infty$*

$$\pi_n(x) = \bigl(c_\infty + o(1)\bigr)\, G(x), \tag{5.7}$$

where

$$c_\infty = \lim_{n\to\infty} c_n = \frac{\alpha_\infty}{1-\varphi(\beta)}, \quad \alpha_\infty = \lim_{n\to\infty} \alpha_n = \int_{\mathbf{R}} c(u)\, e^{\beta u}\, \pi(du) \geqq 0.$$

Remark 7. Let G be a distribution concentrated on the lattice of integer numbers and let this lattice be minimal. Then the statements of the theorem remain in force if the values of the chain X above some level and the values of the parameters t and x in (5.5), (5.6), and (5.7) are integer.

Remark 8. Since $\varphi(\beta) \leqq \psi(\beta) < 1$ and $c(u) \leqq c$, all constants α_n are finite by Lemma 11 and, moreover, the sequences $\{\alpha_n\}$ and $\{c_n\}$ are uniformly bounded.

Point (ii) of the theorem implies that the asymptotics of the probability $\pi_n(x)$ coincides, as n, $x \to \infty$, with the asymptotics of the tail of the invariant measure: $\pi_n(x) = (1 + o(1))\, \pi(x)$.

Let ξ_n be independent random variables having the same distribution as ξ, $\varphi(\beta) < 1$ and let the distribution of ξ belong to the class $\mathcal{S}(\beta)$. Let X be a homogeneous chain in $\mathbf{R}^+$, that is, $X(n+1) = (X(n) + \xi_{n+1})^+$. Applying Theorem 3 for this particular case we obtain that, for homogeneous chains, $\pi_n(x) = (c_n + o(1))\, F(x)$, $F(x) = F([x,\infty))$ as $x \to \infty$ uniformly in $n \geqq 1$, where $c_n = \sum_{k=0}^{n-1} \varphi^k(\beta)\, \mathbf{E} e^{\beta X(n-1-k)}$.

In view of (1.2) the results of this section for homogeneous chains related to the case of fixed n repeat the results of [16]; however, we give another expression for the coefficients c_n. In addition, we prove that

$$\mathbf{P}\Big\{ \sup_{k \leqq n} S_k \geqq x \Big\} \sim \mathbf{P}\Big\{ \sup_{k} S_k \geqq x \Big\} \sim c_\infty F(x) \quad \text{as } n,\, x \to \infty.$$

Proof. Proof of Theorem 3 is given for the nonlattice case only (to handle the lattice case it is necessary to make only minor changes related to the fact that some parameters involved should be integer). Since the time parameter n takes on only a countable number of values, to prove the theorem it suffices to check that

(a) equation (5.6) holds as $x \to \infty$ for any fixed n;

(b) condition (5.7) fulfills.

We fix $U > V$. For any $x > 2U$, we have the equality

$$\begin{aligned} \frac{\pi_{n+1}(x)}{G(x)} &= \left(\int_{-\infty}^{-U} + \int_{-U}^{U} + \int_{U}^{x-U} + \int_{x-U}^{\infty} \right) \frac{\mathbf{P}\{u + \xi(u) \geqq x\}}{G(x)}\, \pi_n(du) \\ &\equiv I_1(U,n,x) + \cdots + I_4(U,n,x). \end{aligned} \tag{5.8}$$

Consider separately each of these four summands.

Since $e^{\beta x}\mathbf{P}\{\eta \geqq x\}$ is a local power function, condition (5.5) provides the convergence

$$\frac{\mathbf{P}\{\xi(u) \geqq x - u\}}{G(x)} \longrightarrow c(u)\, e^{\beta u} \quad \text{as } x \to \infty \tag{5.9}$$

uniformly in $|u| \leqq U$. Therefore, uniformly in n

$$I_2(U,\, n,\, x) \longrightarrow \int_{-U}^{U} c(u)\, e^{\beta u} \pi_n(du) \quad \text{as } x \to \infty. \tag{5.10}$$

Let us show that the summands $I_1(U,n,x)$ and $I_3(U,n,x)$ are, in a sense, negligibly small. By condition (5.1), $I_1(U,n,x)$ admits the estimate

$$I_1(U,n,x) \leqq \frac{G(x-V)}{G(x)} \big(1 - \pi_n(-U)\big).$$

Since $e^{\beta x}G(x)$ is a local power function and the family of distributions $\{\pi_n\}$ is weakly compact, it follows from the last estimate that

$$\begin{gathered} \limsup_{x\to\infty} \Big(\sup_n I_1(U,n,x) \Big) \leqq e^{\beta V} \sup_n \big(1 - \pi_n(-U)\big) \equiv I_1(U), \\ \lim_{U\to\infty} I_1(U) = 0. \end{gathered} \tag{5.11}$$

We evaluate $I_3(U,n,x)$. Using condition (5.1) and integrating by parts we obtain

$$
\begin{aligned}
I_3(U,\,n,\,x) &\leqq \int_U^{x-U} \frac{G(x-u)}{G(x)}\,\pi_n(du) = -\,\frac{\pi_n(u)\,G(x-u)}{G(x)}\bigg|_U^{x-U} \\
&\quad + \int_U^{x-U} \frac{\pi_n(u)\,d_uG(x-u)}{G(x)} \leqq \frac{\pi_n(U)\,G(x-U)}{G(x)} \\
&\quad + \int_U^{x-U} \frac{\pi_n(x-v)\,G(dv)}{G(x)} \equiv I_{31}(U,\,n,\,x) + I_{32}(U,\,n,\,x).
\end{aligned} \tag{5.12}
$$

For any fixed U

$$
\limsup_{x\to\infty}\left(\sup_n I_{31}(U,\,n,\,x)\right) \leqq \sup_n \pi_n(U)\,e^{\beta U},
$$

and, therefore, by Lemma 11

$$
\begin{gathered}
\limsup_{x\to\infty}\left(\sup_n I_{31}(U,\,n,\,x)\right) \leqq \widehat{c}\,G(U)\,e^{\beta U} = I_{31}(U), \\
\lim_{U\to\infty} I_{31}(U) = 0.
\end{gathered} \tag{5.13}
$$

Using Lemma 11 once again we evaluate I_{32}:

$$
I_{32}(U,n,x) \leqq \widehat{c}\int_U^{x-U} \frac{G(x-v)\,G(dv)}{G(x)}.
$$

Consequently, as follows from [13, relation (2)],

$$
\lim_{U\to\infty}\,\limsup_{x\to\infty}\left(\sup_n I_{32}(U,\,n,\,x)\right) = 0. \tag{5.14}
$$

Substituting (5.13) and (5.14) in (5.12), we arrive at the relations

$$
\limsup_{x\to\infty}\left(\sup_n I_3(U,\,n,\,x)\right) \leqq I_3(U), \qquad \lim_{U\to\infty} I_3(U) = 0. \tag{5.15}
$$

Let us estimate $I_4(U,n,x)$. Denote

$$
\overline{c}_n = \limsup_{x\to\infty} \frac{\pi_n(x)}{G(x)}.
$$

According to Lemma 11, $\overline{c}_n < \infty$ for any n. Since the condition of weak convergence of the distributions of increments is equivalent to convergence in the Lévy–Prokhorov metric, for any $\varepsilon > 0$ there exists a $u_0 = u_0(\varepsilon)$ such that

$$
F(v+\varepsilon) - \varepsilon \leqq \mathbf{P}\big\{\xi(u) \geqq v\big\} \leqq F(v-\varepsilon) + \varepsilon \tag{5.16}
$$

for any $u \geqq u_0$ and v. Therefore, $I_4(U,n,x)$ in (5.8) admits the following estimate for $u \geqq u_0 + U$:

$$
\begin{aligned}
I_4(U,\,n,\,x) &\leqq \int_{x-U}^{\infty} \big(F(x-u-\varepsilon) + \varepsilon\big)\,\frac{\pi_n(du)}{G(x)} \\
&= \int_{-\infty}^{U-\varepsilon} F(v)\,\frac{d_v\pi_n(x-\varepsilon-v)}{G(x)} + \varepsilon\,\frac{\pi_n(x-U)}{G(x)} \\
&\equiv I_{41}(U,\,n,\,x) + I_{42}(U,\,n,\,x).
\end{aligned} \tag{5.17}
$$

Integrating by parts we evaluate $I_{41}(U,\, n,\, x)$:

$$I_{41}(U,\, n,\, x) = \int_{-\infty}^{U-\varepsilon} \frac{\pi_n(x-\varepsilon-v)}{G(x)}\, dF(v) + \frac{F(U-\varepsilon)\pi_n(x-U)}{G(x)}.$$

Combining this fact with (5.17) and recalling the definition of $\overline{c}_n$ we have

$$\limsup_{x\to\infty} I_4(U,\, n,\, x) \leqq \overline{c}_n\, e^{\beta\varepsilon} \int_{-\infty}^{U} e^{\beta v}\, F(dv) + O\left(F(U-\varepsilon)\, e^{\beta U} + \varepsilon\right). \tag{5.18}$$

Substituting (5.11), (5.10), (5.15), and (5.18) in (5.8), we arrive at the relation

$$\begin{aligned}\overline{c}(n+1) \leqq \int_{-U}^{U} c(u)\, e^{\beta u}\, \pi_n(du) + \overline{c}_n\, e^{\beta\varepsilon} \int_{-\infty}^{U} e^{\beta v}\, F(dv) \\ + O\left(I_1(U) + I_3(U) + F(U-\varepsilon)\, e^{\beta U} + \varepsilon\right).\end{aligned}$$

Hence, recalling that $\varepsilon > 0$ is arbitrary, we derive the inequality

$$\overline{c}(n+1) \leqq \int_{-U}^{U} c(u)\, e^{\beta u}\, \pi_n(du) + \overline{c}_n \int_{-\infty}^{U} e^{\beta v}\, F(dv) + O\left(I_1(U) + I_3(U) + F(U)\, e^{\beta U}\right).$$

Letting U go to infinity and using (5.10), (5.11), and (5.15), deduce the estimate $\overline{c}(n+1) \leqq \alpha_n + \overline{c}_n\varphi(\beta)$. Hence, by induction we conclude that the lim sup of the ratio $\pi_{n+1}(x)/G(x)$, as $x \to \infty$, does not exceed c_{n+1}.

In the same way, one can establish that the lim inf of the same ratio is not less than c_{n+1}. This proves point (a).

To prove point (b) we set

$$\overline{c} = \limsup_{n,x\to\infty} \frac{\pi_n(x)}{G(x)};$$

$\overline{c} < \infty$ by Lemma 11. Instead of (5.18), we derive as before the estimate

$$\limsup_{n,x\to\infty} I_4(U,\, n,\, x) \leqq \overline{c} e^{\beta\varepsilon} \int_{-\infty}^{U} e^{\beta v}\, F(dv) + O\left(F(U-\varepsilon)\, e^{\beta U} + \varepsilon\right). \tag{5.19}$$

Substituting (5.11), (5.10), (5.15), and (5.19) in (5.8), recalling that $\varepsilon > 0$ is arbitrary, and letting $U \to \infty$ we arrive at the relation $\overline{c} \leqq \limsup_{n\to\infty} \alpha_n + \overline{c}\,\varphi(\beta) = \alpha_\infty + \overline{c}\,\varphi(\beta)$. Therefore, the lim sup as $n,\, x \to \infty$ of the ratio $\pi_{n+1}(x)/G(x)$ does not exceed $\alpha_\infty/(1-\varphi(\beta))$. By the same arguments one can show that the lim inf of the ratio is not less than $\alpha_\infty/(1-\varphi(\beta))$. Point (b) and, therefore, Theorem 3 are proved.

5.3. Exact asymptotics of stationary distributions. In this subsection we assume that X is a chain admitting an invariant measure π. The following refinement of Theorem 5 in [5] is valid for the large-deviation probabilities of a stationary distribution.

THEOREM 4. *Let the conditions of Theorem* 3 (*except* (4.1)) *be valid. Then*

$$\pi(x) = \frac{\alpha_\infty + o(1)}{1-\varphi(\beta)}\, G(x)$$

as $x \to \infty$, *where* $\alpha_\infty \equiv \int_{\mathbf{R}} c(u)\, e^{\beta u}\, \pi(du) \geqq 0$.

Remark 9. The formulated theorem generalizes Theorem 5 in [5] in that the class of the distributions of ξ is extended from the class of superexponential distributions to the class $\mathcal{S}(\beta)$ and, which is no less important, in the following. Theorem 5 in [5] uses the condition

$$\Big|\mathbf{P}\big\{\xi(u) \geqq t\big\} - \mathbf{P}\{\xi \geqq t\}\Big| \leqq \delta(u)\,\mathbf{P}\{\xi \geqq t\},$$

where $\delta(u) \downarrow 0$ as $u \to \infty$ which implies convergence of the distributions of $\xi(u)$ and ξ in the uniform metric. If X is a chain taking on values on a lattice, convergence in the uniform metric is equivalent to weak convergence $\xi(u) \Rightarrow \xi$. In the general case, when X is a real-valued chain, convergence in the uniform metric is essentially stronger, generally speaking, than the weak convergence $\xi(u) \Rightarrow \xi$, suggested by us.

Proof of Theorem 4. By Lemma 1 there exists a $\widehat{c}$ such that $\pi(x) \leqq \widehat{c}\,G(x)$ for any x. It remains to make use of Theorem 3 for $\pi_0 = \pi$.

Remarks on the first part of the paper. 1. As was noted by B. A. Rogozin, it is not correct to use the Lebesgue theorem in formula (7.7) in the proof of Lemma 5 in [5, section 7] while studying the distributions of convolutions of measures, since μ_2 is a signed measure.

Thus, the conditions of Lemma 5 in [5] need to be strengthened to allow us to apply the Lebesgue theorem. We do not give here the required modification of the statement of this lemma, since Lemma 5 is given in [5] to prove Theorem 5 in [5] only. However, Theorem 3 of this paper generalizes Theorem 5 in [5] and is supplied with an independent and more adequate proof.

2. To evaluate the second term of the asymptotics $\pi(x)$ in the Cramér case we used in [5, section 9] the estimates of the renewal function given in Stone's works. Stone uses a Cramér condition on the characteristic function of a particular summand. This condition was missed in the statement of Theorem 11 in [5, section 9]. It is necessary to complete the mentioned statement by the condition $\limsup_{|\lambda|\to\infty} |\mathbf{E}\, e^{i\lambda\widetilde{\xi}}| < 1$.

REFERENCES

[1] A. A. Borovkov, *New limit theorems in boundary problems for sums of independent terms*, Select. Trans. Math. Stat. Probab., 5 (1965), pp. 315–372.

[2] A. A. Borovkov, *Probability Theory*, Gordon and Breach, Amsterdam, 1998.

[3] A. A. Borovkov, *The Cramér transform, large deviations in boundary problems, and the conditional invariance principle*, Siberian Math. J., 36 (1995), pp. 417–434.

[4] A. A. Borovkov, *On conditional distributions associated with large deviations*, Siberian Math. J., 37 (1996), pp. 635–646.

[5] A. A. Borovkov and A. D. Korshunov, *Large-deviation probabilities for one-dimensional Markov chains*, *Part* 1: *Stationary distributions*, Theory Probab. Appl., 41 (1996), pp. 1–24.

[6] A. A. Borovkov and A. A. Mogulskii, *Large deviations and testing statistical hypotheses*, I. *Large deviations of sums of random vectors*, Siberian Adv. Math., 2 (1992), pp. 52–120.

[7] A. A. Borovkov and A. A. Mogulskii, *The second function of deviations and asymptotic problems of the reconstruction and attainment of a boundary for multi-dimensional random walks*, Siberian Math. J., 37 (1996), pp. 647–682.

[8] B. V. Gnedenko, *The Theory of Probability*, 4th ed., Chelsea Publishing Company, New York, 1984.

[9] V. V. Petrov, *On the probabilities of large deviations for sums of independent random variables*, Theory Probab. in Appl., 10 (1965), pp. 287–298.

[10] V. Feller, *An Introduction to Probability Theory and Its Applications*, Vol. 2, 2nd ed., Wiley, New York, 1971.

[11] J. Bertoin and R. A. Doney, *Some asymptotic results for transient random walks*, Adv. Appl. Probab., 28 (1996), pp. 207–226.

[12] A. A. Borovkov, *Ergodicity and Stability of Stochastic Processes*, Wiley, New York, 1998.
[13] P. Embrechts and C. M. Goldie, *On convolution tails*, Stochastic Process. Appl., 13 (1982), pp. 263–278.
[14] T. Höglund, *A unified formulation of the central limit theorem for small and large deviations from the mean*, Z. Wahrsch. Verw. Gebiete, 49 (1979), pp. 105–117.
[15] S. P. Meyn and R. L. Tweedie, *Markov Chains and Stochastic Stability*, Springer-Verlag, Berlin, 1993.
[16] M. S. Sgibnev, *On the distribution of the maxima of partial sums*, Statist. Probab. Lett., 28 (1996), pp. 235–238.