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LARGE-DEVIATION PROBABILITIES FOR ONE-DIMENSIONAL MARKOV CHAINS. PART 3: PRESTATIONARY DISTRIBUTIONS IN THE SUBEXPONENTIAL CASE*

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Abstract. This paper continues investigations of A. A. Borovkov and D. A. Korshunov [*Theory Probab. Appl.*, 41 (1996), pp. 1–24 and 45 (2000), pp. 379–405]. We consider a time-homogeneous Markov chain $\{X(n)\}$ that takes values on the real line and has increments which do not possess exponential moments. The asymptotic behavior of the probability $\mathbf{P}\{X(n) \geqq x\}$ is studied as $x \to \infty$ for fixed values of time n and for unboundedly growing n as well.

Key words. Markov chain, asymptotic behavior of large-deviation probabilities, subexponential distribution, invariant measure, integrated distribution tail

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1. Introduction. Let $X(n) = X(y,n)$, $n = 0, 1, \dots$, be a time-homogeneous Markov chain with values on the real line $\mathbf{R}$ and with initial state $y \equiv X(y,0)$. Denote by $P(y,B) = \mathbf{P}\{X(y,1) \in B\}$, where B is a Borel set in $\mathbf{R}$, the transition probability of the chain.

Let $\xi(y)$ be the increment of the chain X in one step at point $y \in \mathbf{R}$, that is, $\xi(y) = X(y,1) - y$.

One of the main objects under study in the present paper consists of *asymptotically homogeneous* in space Markov chains, that is, chains for which the distribution of $\xi(y)$ converges weakly as $y \to \infty$ to the distribution of some random variable ξ; we denote it by $\xi(y) \Rightarrow \xi$. We assume everywhere that $m = \mathbf{E}\xi < 0$ and $\mathbf{P}\{\xi > 0\} > 0$. Also, we extend our analysis to a more general class of chains with asymptotically homogeneous drift, that is, chains which are satisfying only the property $\mathbf{E}\,\xi(y) \to m$ as $y \to \infty$.

In [2], we classified the asymptotically homogeneous-in-space Markov chains with respect to the behavior of the Laplace transform $\varphi(\lambda) = \mathbf{E}e^{\lambda\,\xi}$. In terms of that classification, in the present paper we investigate the third case (c) which corresponds to the situation $\varphi(\lambda) = \infty$ for any $\lambda > 0$. Of course, we assume regular behavior of the tail $\mathbf{P}\{\xi \geqq x\}$ of the distribution ξ as $x \to \infty$. Let us recall some definitions.

A positive function g is called *long-tailed* if, for any fixed t, the limit of the ratio $g(u+t)/g(u)$ is equal to 1 as $u \to \infty$. We say that *a distribution G is long-tailed* if the tail $G(x) \equiv G([x,\infty))$ of this distribution is long-tailed.

Note that, for any random variable ξ with long-tailed distribution, $\mathbf{E}e^{\lambda\xi} = \infty$ for any $\lambda > 0$.

We say [3] that a distribution G on $\mathbf{R}^+$ with unbounded support belongs to the class $\mathcal{S}$ (and is called a *subexponential* distribution) if the convolution $G * G$ satisfies

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the equivalence $G*G(x)\sim 2G(x)$ as $x\to\infty$.

It is shown in [3] that any subexponential distribution G is long-tailed with necessity. Sufficient conditions for subexponentiality may be found, for example, in [3], [8]. In particular, the class $\mathcal{S}$ includes distributions with the tail $G(x)=x^{-\alpha}\,\varepsilon(x)$, where $\alpha>0$ and $\varepsilon(x)$ is a slowly-varying-at-infinity function. Moreover, the class $\mathcal{S}$ also contains the so-called *upper power* distributions, that is, the long-tailed distributions satisfying the property $\sup_x G(x/2)/G(x)<\infty$.

Let G be an arbitrary distribution on $\mathbf{R}$ with support unbounded from above and with finite mean value. For any $t\in(0,\infty]$, define the distribution G_t on $\mathbf{R}^+$ with the distribution tail

$$G_t(x)\equiv \min\biggl(1,\ \int_x^{x+t} G(u)\,du\biggr),\qquad x>0. \tag{1.1}$$

Note that any long-tailed distribution G satisfies the following relation, for any fixed $s>0$:

$$G_s(x)=o\bigl(G_t(x)\bigr)\qquad \text{as}\quad t,x\to\infty. \tag{1.2}$$

We say (see [6]) that the subexponential distribution G on $\mathbf{R}^+$ is *strongly subexponential* (and write $G\in\mathcal{S}_*$) if the convolution of the distribution G_t with itself satisfies the equivalence $G_t*G_t(x)\sim 2G_t(x)$ as $x\to\infty$ uniformly in $t\in[1,\infty]$.

Criteria for strong subexponentiality are given in [6]. In particular, the class $\mathcal{S}_*$ includes the following distributions:

(i) the upper power distributions and, in particular, all distributions with regularly-varying-at-infinity tails;

(ii) the lognormal distribution with the density $\exp\{-(\log x-\log\alpha)^2/2\sigma^2\}/x\sigma\sqrt{2\pi}$, $x>0$, where σ, $\alpha>0$;

(iii) the Weibull distribution with the tail $G([x,\infty))=e^{-x^\alpha}$, $x\geqq 0$, where $\alpha\in(0,1)$.

In Part 1 (see [1]) we assumed that X was a chain possessing an invariant measure π, that is, a measure solving the equation

$$\pi(\cdot)=\int_{\mathbf{R}}\pi(du)\,P(u,\cdot),\qquad \pi(\mathbf{R})=1. \tag{1.3}$$

We have proved the following result about the asymptotic behavior of the tail $\pi(x)$ of the invariant measure π.

THEOREM 1. *Let X be an asymptotically homogeneous-in-space Markov chain such that the family* (*with respect to u*) *of jumps $\{\xi(u)\}$ is uniformly integrable. Let $m=\mathbf{E}\xi<0$ and the distribution F of the random variable ξ be such that the distribution F_∞* (*for a definition, see* (1.1)) *is upper power. If, for some bounded function $c(u)$, the convergence*

$$\frac{\mathbf{P}\{\xi(u)\geqq t\}}{\mathbf{P}\{\xi\geqq t\}}\longrightarrow c(u)\qquad as\quad t\to\infty$$

holds uniformly in u, then

$$\pi(x)\sim\frac{F_\infty(x)}{|m|}\int_{\mathbf{R}}c(u)\,\pi(du)\qquad as\quad x\to\infty.$$

Remark 1. Unfortunately, the boundedness condition for the function $c(u)$ was missed in the statement of Theorem 6 in [1].

In this third part we investigate the asymptotic behavior of the tail $\pi_n(x) = \mathbf{P}\{X(n) \geqq x\}$ of the distribution π_n of the random variable $X(n)$ as $x \to \infty$ for fixed values of the time parameter n and for unboundedly growing n as well.

Let $\{\xi_k\}$ be a tuple of independent copies of ξ. Put $S_0 = 0$, $S_k = \xi_1 + \cdots + \xi_k$, and $M_n = \max_{0 \leqq k \leqq n} S_k$. It is known (see, for example, [5, Chap. VI, section 9]) that the distribution of the homogeneous-in-space (see [1]) Markov chain $X(n) = (X(n+1) + \xi_n)^+$ with zero initial state $X(0) = 0$ coincides with the distribution of M_n, that is,

$$\mathbf{P}\bigl\{X(0,n) \geqq x\bigr\} = \mathbf{P}\{M_n \geqq x\}. \tag{1.4}$$

Below we need the following theorem proved in [6].

THEOREM 2. *Let $m < 0$ and let the distribution of the random variable $\xi\,\mathbf{I}\{\xi \geqq 0\}$ be strongly subexponential. Then*

$$\mathbf{P}\{M_n \geqq x\} \sim \frac{F_{n|m|}(x)}{|m|}$$

as $x \to \infty$ uniformly in $n \geqq 1$.

As was mentioned above, one of the main topics in this paper is the investigation of Markov chains with asymptotically homogeneous drift. For such chains we obtain in section 2 the lower bound for the probability $\pi_n(x) = \mathbf{P}\{X(n) \geqq x\}$. In section 3, the upper bounds are given for this probability. Combining these results in section 4 we get the theorem on the large deviation asymptotics for the asymptotically homogeneous-in-space Markov chain X.

2. Lower bound for large deviation probabilities for a prestationary chain. In this section we estimate from below the probability $\pi_n(x)$ for large values of n and x. This estimate is asymptotically correct. We start with some auxiliary results.

2.1. SLLN-type statements for a Markov sequence. Consider a nonhomogeneous in time Markov chain $Y = \{Y_n\}$. The initial distribution of this chain is assumed to be arbitrary. Let $\eta_{n+1}(u)$ be a random variable corresponding to the jump of the chain Y at time n from the state u, i.e.,

$$\mathbf{P}\{Y_{n+1} \in \cdot \mid Y_n = u\} = \mathbf{P}\bigl\{u + \eta_{n+1}(u) \in \cdot\bigr\}.$$

LEMMA 1. *Let the drift of the Markov chain Y_n be bounded from below by $\widehat{a}$: $\mathbf{E}\eta_n(u) \geqq \widehat{a}$ for any time $n \geqq 1$ and any state $u \in \mathbf{R}$. Moreover, let the family of random variables $\{|\eta_n(u)|,\, n \geqq 1,\, u \in \mathbf{R}\}$ admit an integrable majorant, that is, there exists a random variable η with finite mean value such that $|\eta_n(u)| \leqq_{\mathrm{st}} \eta$ for each n and u. Then, for any initial distribution Y_0,*

$$\liminf_{n\to\infty} \frac{Y_n - Y_0}{n} \geqq \widehat{a} \quad a.s.$$

Proof. Fix $A > 0$. Define a threshold of the jump $\eta_n(u)$ at the level An as follows:

$$\eta_n^{[An]}(u) \equiv \eta_n(u)\,\mathbf{I}\bigl\{|\eta_n(u)| < An\bigr\}.$$

Let us consider a nonhomogeneous in time Markov chain Z_n, $Z_0 = Y_0$, with jumps $\eta_n^{[An]}(u)$:

$$\mathbf{P}\{Z_{n+1} \in \cdot \mid Z_n = u\} = \mathbf{P}\Big\{u + \eta_n^{[An]}(u) \in \cdot\Big\}.$$

By the construction of Z_n, we may estimate the probability of the event that the trajectories of Z_n and Y_n are different in the following way:

$$\begin{aligned}\mathbf{P}\Big\{\sup_n |Z_n - Y_n| \neq 0\Big\} &\leqq \sum_{n=0}^{\infty} \mathbf{P}\big\{|Y_{n+1} - Y_n| \geqq An\big\} \\ &\leqq \sum_{n=1}^{\infty} \mathbf{P}\big\{|\eta| \geqq An\big\} \leqq \mathbf{E}\,\frac{|\eta|}{A}. \end{aligned} \tag{2.1}$$

Put $\Delta_n^0 = \mathbf{E}\{Z_n - Z_{n-1} | Z_{n-1}\}$ and $\Delta_n^1 = Z_n - Z_{n-1} - \Delta_n^0$. Then

$$Z_n - Z_0 = \sum_{k=1}^{n} \Delta_k^0 + \sum_{k=1}^{n} \Delta_k^1 \equiv Z_n^0 + Z_n^1.$$

For every u we have the following equality and inequality:

$$\begin{aligned}\mathbf{E}\big\{\Delta_n^0 \mid Z_{n-1} = u\big\} &= \mathbf{E}\eta_n^{[An]}(u) = \mathbf{E}\eta_n(u) - \mathbf{E}\Big\{\eta_n(u);\ \big|\eta_n(u)\big| \geqq An\Big\} \\ &\geqq \widehat{a} - \mathbf{E}\big\{|\eta|;\ |\eta| \geqq An\big\},\end{aligned}$$

in view of the conditions on the jumps $\eta_n(u)$. Therefore,

$$\frac{1}{n}\sum_{k=1}^{n} \Delta_k^0 \geqq \widehat{a} - \frac{1}{n}\sum_{k=1}^{n} \mathbf{E}\big\{|\eta|;\ |\eta| \geqq Ak\big\}.$$

In view of the existence of $\mathbf{E}|\eta|$, the last inequality implies that uniformly in all elementary events ω,

$$\liminf_{n\to\infty} \frac{Z_n^0(\omega)}{n} \geqq \widehat{a}. \tag{2.2}$$

Since $\mathbf{E}\{\Delta_n^1 | Z_n\} = 0$, the process Z_n^1 is a martingale with respect to σ-fields $\sigma(Z_0, \dots, Z_{n-1})$. Let us prove that the increments of this martingale satisfy the condition

$$\sum_{n=1}^{\infty} \frac{\mathbf{E}(\Delta_n^1)^2}{n^2} < \infty. \tag{2.3}$$

For every u, by the construction of Δ_n^1 and in view of the lemma conditions, we have

$$\begin{aligned}\mathbf{E}\big\{(\Delta_n^1)^2 \mid Z_{n-1} = u\big\} &= \mathbf{E}\Big(\eta_n^{[An]}(u) - \mathbf{E}\eta_n^{[An]}(u)\Big)^2 \\ &\leqq \mathbf{E}\Big(\eta_n^{[An]}(u)\Big)^2 \leqq \mathbf{E}\big\{\eta^2;\ |\eta| < An\big\}.\end{aligned}$$

Thus,

$$\sum_{n=1}^{\infty} \frac{\mathbf{E}(\Delta_n^1)^2}{n^2} \leqq \sum_{n=1}^{\infty} \frac{\mathbf{E}\{\eta^2; |\eta| < An\}}{n^2}.$$

The last series converges for any A, since

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\mathbf{E}\{\eta^2;|\eta|<An\}}{n^2}&=\sum_{n=1}^{\infty}\frac{A^2}{n^2}\mathbf{E}\biggl\{\Bigl(\frac{\eta}{A}\Bigr)^2;\ \Bigl|\frac{\eta}{A}\Bigr|<n\biggr\}\\&\leqq\sum_{n=1}^{\infty}\frac{A^2}{n^2}\sum_{k=1}^{n}k^2\mathbf{P}\biggl\{k-1\leqq\Bigl|\frac{\eta}{A}\Bigr|<k\biggr\}\\&=A^2\sum_{k=1}^{\infty}k^2\mathbf{P}\biggl\{k-1\leqq\Bigl|\frac{\eta}{A}\Bigr|<k\biggr\}\sum_{n=k}^{\infty}\frac{1}{n^2}<\infty,\end{aligned}$$

by virtue of the equivalence $\sum_{n=k}^{\infty}1/n^2\sim 1/k$ and the existence of $\mathbf{E}|\eta|$.

So, the martingale Z_n^1 really satisfies condition (2.3) and we may apply Corollary 2 from [7, p. 534]. By this corollary, the following SLLN is valid for this martingale:

$$\lim_{n\to\infty}\frac{Z_n^1}{n}=0\quad\text{a.s.}\tag{2.4}$$

Relations (2.2) and (2.4) imply the inequality $\liminf_{n\to\infty}(Z_n-Z_0)/n\geqq\widehat{a}$ almost surely. Now the assertion of the lemma follows from (2.1) in view of the arbitrary choice of A.

LEMMA 2. *Let the drift of the Markov chain Y_n above some space level U be bounded below by $\widehat{a}<0$: $\mathbf{E}\eta_n(u)\geqq\widehat{a}$ for every $n\geqq 1$ and $u\geqq U$. Moreover, let the family of the random variables $\{|\eta_n(u)|,\ n\geqq 1,\ u\geqq U\}$ admit an integrable majorant; that is, there exists a random variable η with finite mean value such that $|\eta_n(u)|\leqq_{\rm st}\eta$ for every n and $u\geqq U$. Then, for each $\varepsilon>0$, the following convergence holds*:

$$\mathbf{P}\Bigl\{Y_k\geqq u-n\bigl(|\widehat{a}|+\varepsilon\bigr)\ \textit{for all}\ k\leqq n\mid Y_0=u\Bigr\}\longrightarrow 1$$

as $n\to\infty$ uniformly in $u\geqq U+n(|\widehat{a}|+\varepsilon)$.

Proof. It is sufficient to consider a new, also nonhomogeneous in time, Markov chain $\widetilde{Y}_n$ with jumps $\widetilde{\eta}_n(u)$, where $\widetilde{\eta}_n(u)$ coincides with $\eta_n(u)$ if $u\geqq U$, and $\widetilde{\eta}_n(u)$ is equal to $\widehat{a}$ if $u<U$. Applying Lemma 1 we complete the proof.

2.2. Lower bound for the large deviation probabilities. The initial distribution π_0 of the chain X is assumed to be arbitrary. Let f be a positive nonincreasing long-tailed function. Let the function $\underline{c}(u)\geqq 0$ be such that, for every $U>0$,

$$\frac{\mathbf{P}\{\xi(u)\geqq x\}}{f(x)}\geqq\underline{c}(u)+o(1)$$

as $x\to\infty$ uniformly in $|u|\leqq U$. Denote

$$c_\infty\equiv\lim_{U\to\infty}\liminf_{n\to\infty}\int_{-U}^{U}\underline{c}(u)\,\pi_n(du)\in[0,\infty].$$

In particular, if the distribution π_n converges in the total variation norm to some (invariant with necessity) measure π, i.e., if the convergence

$$\sup_B\bigl|\pi_n(B)-\pi(B)\bigr|\longrightarrow 0\tag{2.5}$$

holds as $n\to\infty$, where the supremum is taken over all Borel sets on the real line, then $c_\infty=\int_{\mathbf{R}}\underline{c}(u)\,\pi(du)$. If the function $\underline{c}(u)$ is continuous, the last equality remains true in the case of weak convergence $\pi_n\Rightarrow\pi$.

Put

$$\underline{a} \equiv \liminf_{u\to\infty} \mathbf{E}\,\xi(u).$$

It was established in Lemma 1 in [2] that the existence of the invariant measure together with the condition $\sup_u \mathbf{E}|\xi(u)| < \infty$ implies $\underline{a} \leqq 0$. Note that in the present section we do not put any restrictions on the sign of $\underline{a}$.

The following lemma is valid.

LEMMA 3. *Put $a = -\underline{a}$ if $\underline{a} < 0$, and let a be an arbitrary positive real number otherwise. Let there exist a level U such that the family $\{|\xi(u)|,\ u \geqq U\}$ admits an integrable majorant. Then the following estimate holds*:

$$\liminf_{n,x\to\infty} \frac{\pi_n(x)}{\int_x^{x+na} f(y)\,dy} \geqq \frac{c_\infty}{a}.$$

Proof. It is sufficient to consider the case $c_\infty > 0$ only. Let $c' < c'' < c_\infty$. The definitions of $\underline{c}(u)$ and c_∞ imply the existence of $U' > 0$ such that

$$\liminf_{n\to\infty} \int_{-U'}^{U'} \frac{\mathbf{P}\{\xi(u) \geqq x\}}{f(x)}\,\pi_n(du) \geqq c''$$

uniformly in all sufficiently large x. Since the function $f(x)$ is long-tailed and non-increasing, $f(x-u)/f(x) \to 1$ as $x \to \infty$ uniformly in $|u| \leqq U'$. Therefore, there exist N and x_0 such that

$$\int_{-U'}^{U'} \frac{\mathbf{P}\{u+\xi(u) \geqq x\}}{f(x)}\,\pi_n(du) \geqq c' \tag{2.6}$$

uniformly in $n \geqq N$ and $x \geqq x_0$.

Consider the event $A_{i,n} \equiv A_{i,n}(x)$, $i \in [1,n]$, which occurs if $X(i-1) < x$ and $X(j) \geqq x$ for any $j \in [i,n]$. First, the events $A_{i,n}$, $i \in [1,n]$, are disjoint. Second, $\cup_{i=1}^n A_{i,n} = \{X(n) \geqq x\}$. Thus

$$\pi_n(x) = \sum_{i=1}^n \mathbf{P}\{A_{i,n}\}. \tag{2.7}$$

For $v \in [x,\infty)$, introduce the probability $p_i(v)$ by the equality

$$p_i(v) = \mathbf{P}\big\{X(v,j) \geqq x \text{ for any } j \leqq i\big\}.$$

Fix $\varepsilon > 0$ and put $b = a + \varepsilon$. Since the event $A_{i,n}(U') \equiv \{X(i-1) \in [-U',U')$, $X(i) \geqq x + (n-i)\,b$, and $X(j) \geqq x$ for any $j \in [i+1,n]\}$ implies the event $A_{i,n}$, the following inequality holds:

$$\mathbf{P}\{A_{i,n}\} \geqq \int_{-U'}^{U'} \pi_{i-1}(du) \int_{x+(n-i)b}^{\infty} \mathbf{P}\big\{u+\xi(u) \in dv\big\}\,p_{n-i}(v).$$

Therefore,

$$\mathbf{P}\{A_{i,n}\} \geqq \int_{-U'}^{U'} \pi_{i-1}(du)\mathbf{P}\big\{u+\xi(u) \geqq x+(n-i)\,b\big\} \min_{v\geqq x+(n-i)b} p_{n-i}(v). \tag{2.8}$$

By the definition of a and b, we have the inequality $\mathbf{E}\xi(v) \geqq -b+\varepsilon/2$ for all sufficiently large v. Thus, by virtue of Lemma 2,

$$\min_{v\geqq x+(n-i)b} p_{n-i}(v) \to 1 \qquad \text{as} \quad x \text{ and } n-i\to\infty.$$

Substituting this convergence into (2.8), we obtain the inequality

$$\liminf_{n-i,x\to\infty} \frac{\mathbf{P}\{A_{i,n}\}}{f(x+(n-i)b)} \geqq \liminf_{n-i,x\to\infty} \int_{-U'}^{U'} \frac{\mathbf{P}\{u+\xi(u)\geqq x+(n-i)b\}}{f(x+(n-i)b)}\, \pi_{i-1}(du) \geqq c'$$

in view of (2.6). Using the last inequality, we derive from (2.7) the estimate

$$\pi_n(x) \geqq (c'-\varepsilon)\sum_{i=N}^{n-I} f\bigl(x+(n-i)\,b\bigr),$$

which is valid for arbitrary slow growing I and for all sufficiently large x. Since the function f is long-tailed and nonincreasing, for every fixed I,

$$\sum_{i=N}^{n-I} f\bigl(x+(n-i)\,b\bigr) \sim \frac{1}{b}\int_x^{x+nb} f(y)\,dy \qquad \text{as} \quad x\to\infty.$$

Therefore,

$$\liminf_{n,x\to\infty} \frac{\pi_n(x)}{\int_x^{x+nb} f(y)\,dy} \geqq \frac{c'-\varepsilon}{b}.$$

Since $b=a+\varepsilon$, $c'<c_\infty$, and $\varepsilon>0$ were chosen arbitrarily, the lemma proof is complete.

3. Upper bounds for the tails of prestationary and stationary distributions.

3.1. Upper bound with a "nonexact" multiple constant. The following lemma generalizes Lemma 2 from [1] in the part which is related to subexponential distributions.

LEMMA 4. *Let ζ be an unbounded-from-above random variable with distribution G and with negative mean $\mathbf{E}\zeta<0$. Let there exist a level U such that, for each $t\geqq U$,*

(3.1) $$\xi(u) \leqq_{\rm st} \zeta \qquad \text{if} \quad u\geqq U,$$

(3.2) $$\mathbf{P}\bigl\{u+\xi(u)\geqq t\bigr\} \leqq G(t) \qquad \text{if} \quad u<U.$$

Then the following assertions are true:

(i) *If the distribution G_∞ (for a definition, see (1.1)) is subexponential and the chain X admits (not unique, in general) an invariant measure π, then the following estimate holds*:

(3.3) $$\limsup_{x\to\infty} \frac{\pi(x)}{G_\infty(x)} \leqq \frac{1}{|\mathbf{E}\zeta|};$$

(ii) *if the distribution G is strongly subexponential and the initial distribution $X(0)$ is bounded from above, then the following estimate holds*:

(3.4) $$\limsup_{x\to\infty} \sup_{n\geqq 1} \frac{\pi_n(x)}{G_{n|\mathbf{E}\zeta|}(x)} \leqq \frac{1}{|\mathbf{E}\zeta|}.$$

Proof. Without loss of generality we assume that $X(0) \leqq 0$ and $U = 0$. Consider the homogeneous-in-space Markov chain $\{Y_n\}$ with nonnegative values defined by the equalities $Y_{n+1} = (Y_n + \zeta_{n+1})^+$, where ζ_n are independent copies of ζ. By virtue of conditions (3.1) and (3.2), the Markov chain Y_n dominates the chain $X(n)$ and, therefore,

$$\pi(x) \leqq \pi^Y(x), \tag{3.5}$$

where π^Y is the invariant measure of the chain Y.

Since the distribution G_∞ is subexponential, it follows from Theorem 2(B) in [9] that

$$\pi^Y(x) \sim \frac{G_\infty(x)}{|\mathbf{E}\zeta|} \qquad \text{as} \quad x \to \infty.$$

Combined with (3.5) this implies (3.3).

Inequality (3.4) also follows from the majorization of the chain $X(n)$ by the chain Y_n. By majorization, for every n and x we have the estimate $\pi_n(x) \leqq \pi_n^Y(x) = \mathbf{P}\{Y_n \geqq x\}$. It remains to apply Theorem 2. The lemma is proved.

3.2. Some auxiliary results. Let $\zeta_1, \zeta_2, \ldots$ be i.i.d. random variables.

LEMMA 5. *Let $\mathbf{E}\zeta_1 < 0$. Then the series*

$$\sum_{n=1}^{\infty} \mathbf{P}\{\zeta_1 + \cdots + \zeta_k \geqq 0 \textit{ for all } k \leqq n\}$$

converges.

Proof. Consider the homogeneous-in-space Markov chain defined by the equalities $Y_{n+1} = (Y_n + \zeta_{n+1})^+$ with the initial value $Y_0 = 1$. We have the inequality and the equality $\mathbf{P}\{\zeta_1 + \cdots + \zeta_k \geqq 0 \text{ for all } k \leqq n\} \leqq \mathbf{P}\{Y_k > 0 \text{ for all } k \leqq n | Y_0 = 1\} = \mathbf{P}\{\eta > n\}$, where η is the first hitting time of the state 0 by the chain $\{Y_n\}$. Since $\mathbf{E}\zeta_1 < 0$, the chain $\{Y_n\}$ is positive recurrent, that is, $\mathbf{E}\eta < \infty$. Thus, the series $\sum_{n=1}^{\infty} \mathbf{P}\{\eta > n\}$ converges and the lemma is proved.

In the following theorem, the local behavior of the function is studied which is dominated by the long-tailed function. Let a positive nonincreasing integrable function $f(x)$ be long-tailed. Since the function f does not increase, it is long-tailed if and only if there exists a sequence $\Delta(x) \to \infty$ such that

$$\frac{f(x)}{f(x - \Delta(x))} \longrightarrow 1 \qquad \text{as} \quad x \to \infty.$$

Let c be an arbitrary positive constant. Put

$$f_n(x) = \int_x^{x+cn} f(y)\, dy.$$

We have the equivalence

$$f_n\big(x - \Delta(x)\big) \sim f_n(x) \tag{3.6}$$

as $x \to \infty$ uniformly in all n. Let a nonnegative function $h_n(x)$ be such that, for each n, it is nonincreasing in x.

LEMMA 6. *Let $h_n(x) \leqq f_n(x)$ for any n and x. Then there exists a sequence of intervals $[x_n^-, x_n^+] \subseteq [x-\Delta(x), x]$ such that $x_n^+ - x_n^- \to \infty$ and $h_n(x_n^-) - h_n(x_n^+) = o(f_n(x))$ as n, $x \to \infty$.*

Proof. Choose a sequence $u(x) \to \infty$ such that $u(x) = o(\Delta(x))$ as $x \to \infty$ and $l(x) = \Delta(x)/u(x)$ is a natural number. By the choice of $u(x)$ we have the convergence $l(x) \to \infty$ as $x \to \infty$.

By virtue of (3.6), it is sufficient to prove that, for any n and x, there exists a point $x_n^- \in [x-\Delta(x),\, x-u(x)]$ such that $h_n(x_n^-) - h_n(x_n^- + u(x)) = o(f_n(x_n^-))$ as $n, x \to \infty$ (and put $x_n^+ = x_n^- + u(x)$). We argue by the rule of contraries and assume that the last relation does not hold. Then there exist a number $\varepsilon > 0$, subsequences $n_k \to \infty$, $k \to \infty$, and $x_k \to \infty$, $k \to \infty$, such that, for any $y \in [x_k - \Delta(x_k), x_k - u(x_k)]$, the following inequality holds:

$$h_{n_k}(y) - h_{n_k}\big(y + u(x_k)\big) \geqq \varepsilon f_{n_k}(y). \tag{3.7}$$

In particular, in view of the inequality $h_{n_k} \leqq f_{n_k}$,

$$h_{n_k}\big(y + u(x_k)\big) \leqq h_{n_k}(y) - \varepsilon f_{n_k}(y) \leqq (1-\varepsilon)\, h_{n_k}(y).$$

Therefore,

$$\begin{aligned} h_{n_k}\big(x_k - u(x_k)\big) &\leqq (1-\varepsilon)^{l(x_k)-1} h_{n_k}\big(x_k - \Delta(x_k)\big) \\ &\leqq (1-\varepsilon)^{l(x_k)-1} f_{n_k}\big(x_k - \Delta(x_k)\big) = o\Big(f_{n_k}\big(x_k - u(x_k)\big)\Big) \end{aligned}$$

as $k \to \infty$ by virtue of $l(x_k) \to \infty$ and (3.6). It contradicts (3.7) with $y = x_k - u(x_k)$. This contradiction proves the lemma.

3.3. Upper bound with an "exact" multiple constant. Here we assume that the initial distribution π_0 is concentrated on the set bounded from above.

LEMMA 7. *Let an unbounded-from-above random variable ζ with distribution G be such that the distribution of the random variable $\zeta \mathbf{I}\{\zeta \geqq 0\}$ is strongly subexponential and $\mathbf{E}\zeta < 0$. Let there exist a level U such that* (3.1) *holds. Moreover, let, for some function $c(v) \leqq 1$, the inequality*

$$\mathbf{P}\big\{\xi(u) \geqq t\big\} \leqq c(u)\, G(t) \tag{3.8}$$

hold for $u \geqq U$, $t \geqq U$ and

$$\mathbf{P}\big\{u + \xi(u) \geqq t\big\} \leqq c(u)\, G(t) \tag{3.9}$$

for $u < U$, $t \geqq U$. If, in addition, convergence in total variation (2.5) *holds, then*

$$\pi_n(x) \leqq \big(1 + o(1)\big) \sum_{k=1}^{n} c_{k-1} G\big(x + (n-k)\, |\mathbf{E}\zeta|\big), \tag{3.10}$$

as $x \to \infty$ uniformly in all $n \geqq 1$, where $c_k \equiv \int_{\mathbf{R}} c(u)\, \pi_k(du) \geqq 0$. In particular,

$$\limsup_{n,x\to\infty} \frac{\pi_n(x)}{G_{n|\mathbf{E}\zeta|}(x)} \leqq \frac{c_\infty}{|\mathbf{E}\zeta|}, \tag{3.11}$$

where $c_\infty \equiv \lim_{k\to\infty} c_k = \int_{\mathbf{R}} c(u)\, \pi(du) \geqq 0$.

Proof. Since the time parameter n takes its values in the countable set, the distribution G is long-tailed, and $c_n \to c_\infty$, it is sufficient to check the following two relations: For any *fixed* n,

$$\pi_n(x) \leqq \big(1+o(1)\big)\, G(x) \sum_{k=1}^{n} c_{k-1} \qquad \text{as} \quad x \to \infty \tag{3.12}$$

and (3.11). Let us prove the first relation by induction. For every $U' \in (U, x)$ we have the equality

$$\pi_{n+1}(x) = \left(\int_{-\infty}^{U'} + \int_{U'}^{x-U'} + \int_{x-U'}^{\infty} \right) \mathbf{P}\{u+\xi(u) \geqq x\} \pi_n(du) \equiv I_1 + I_2 + I_3.$$

In view of conditions (3.8) and (3.9), for any fixed U', we have the estimate

$$\limsup_{x\to\infty} \frac{I_1}{G(x)} \leqq \int_{-\infty}^{U'} c(u)\, \pi_n(du) \leqq c_n.$$

In view of $c(v) \leqq 1$ and condition (3.8), the second term I_2 admits the following estimate:

$$I_2 \leqq \int_{U'}^{x-U'} G(x-u)\, \pi_n(du).$$

Integrating by parts, we arrive at the inequality

$$\begin{aligned} I_2 &\leqq -G(x-u)\, \pi_n(u) \,\Big|_{U'}^{x-U'} + \int_{U'}^{x-U'} \pi_n(x-u)\, G(du) \\ &\leqq G(x-U')\, \pi_n(U') + \int_{U'}^{x-U'} \pi_n(x-u)\, G(du). \end{aligned}$$

Using the inductive hypothesis, we get the estimate

$$\limsup_{x\to\infty} \frac{I_2}{G(x)} \leqq cG(U') + c \limsup_{x\to\infty} \int_{U'}^{x-U'} \frac{G(x-u)}{G(x)}\, G(du).$$

Since the distribution G is subexponential, it follows from relation (2) in [4] that the value of lim sup on the right-hand side of the last inequality may be made arbitrarily small by the appropriate choice of U'. Thus, $\lim_{U'\to\infty} \limsup_{x\to\infty} I_2/G(x) = 0$.

The third term I_3 does not exceed $\pi_n(x-U')$. Thus, by the induction assumption and by the fact that the distribution G is long-tailed, the estimate

$$\limsup_{x\to\infty} \frac{I_3}{G(x)} \leqq \sum_{k=1}^{n} c_{k-1}$$

is valid for any fixed U'. Combining the estimates for I_1, I_2, and I_3, we deduce the induction step $n \to n+1$.

Now let us prove relation (3.11). Without loss of generality, assume $X(0) \leqq U$. Choose the sequence of points x_k, $k = 1, 2, \ldots,$ such that $x_{k+1} - x_k \to \infty$ and

$G(x_{k+1}) \sim G(x_k)$ as $k \to \infty$. In particular, $G_{n|\mathbf{E}\zeta|}(x_{k+1}) \sim G_{n|\mathbf{E}\zeta|}(x_k)$ as $k \to \infty$ uniformly in all n. So, relation (3.11) is equivalent to the relation

$$\limsup_{n,k\to\infty} \frac{\pi_n(x_k)}{G_{n|\mathbf{E}\zeta|}(x_k)} \leqq \frac{c_\infty}{|\mathbf{E}\zeta|}. \tag{3.13}$$

By virtue of Theorem 4, for any fixed i, the function $h_n(x) \equiv \pi_{n-i}(x)$ satisfies the conditions of Lemma 6 with $f(x) \equiv \text{const} \cdot G(x)$ and $c = |\mathbf{E}\zeta|$. Hence, there exist sequences x_{kn}^- and x_{kn}^+ such that $[x_{kn}^-, x_{kn}^+] \subseteq [x_k, x_{k+1}]$, $x_{kn}^+ - x_{kn}^- \to \infty$, and $\pi_{n-i}(x_{kn}^-) - \pi_{n-i}(x_{kn}^+) = o(G_{n|\mathbf{E}\zeta|}(x_{k+1}))$ as n, $k \to \infty$. Therefore, without loss of generality, for convenience, we may assume that there exists (for any fixed i) the function x_n^- such that $x_n^- \in [0, x]$, $x - x_n^- \to \infty$, and

$$\pi_{n-i}\big([x_n^-, x_n^+)\big) = \pi_{n-i}(x_n^-) - \pi_{n-i}(x) = o\big(G_{n|\mathbf{E}\zeta|}(x)\big), \qquad n, x \to \infty. \tag{3.14}$$

Condition (3.9) implies (3.2). Consider the homogeneous Markov chain $\{Y_n\}$ with nonzero states defined in the proof of Lemma 4. Since the chain $\{U + Y_n\}$ dominates the chain $\{X(n)\}$, given $X(0) \leqq U + Y_0$, the chains $\{X(n)\}$ and $\{U + Y_n\}$ can be constructed on the same probability space in such a way that

$$X(n) \leqq U + Y_n \tag{3.15}$$

for any n with probability 1. All characteristics of the chain $\{X(n)\}$ will be completed by an upper index X and the characteristics of the chain $\{U+Y_n\}$ by an upper index Y.

The events $A_{i,n}$ and the probabilities $p_i(v)$ were defined in the proof of Lemma 3. Let us turn again to equality (2.7). For any fixed i, estimates (3.4) and (1.2) imply the following relations:

$$\mathbf{P}\{A_{i,n}^X\} \leqq \mathbf{P}\big\{X(i) \geqq x\big\} = \pi_i(x) = O\big(G_{i|\mathbf{E}\zeta|}(x)\big) = o\big(G_{n|\mathbf{E}\zeta|}(x)\big)$$

as n, $x \to \infty$. Therefore, there exists an unboundedly growing sequence $I = I(n, x)$ such that

$$\pi_n(x) = \sum_{i=I}^{n} \mathbf{P}\{A_{i,n}^X\} + o\big(G_{n|\mathbf{E}\zeta|}(x)\big) \qquad \text{as} \quad n, x \to \infty. \tag{3.16}$$

It turns out that any finite number of the last summands in this sum has order $o(G_{n|\mathbf{E}\zeta|}(x))$. This more delicate observation will be checked at the end of the proof.

For the probability of the event $A_{i,n}^X$, we have the equality

$$\mathbf{P}\{A_{i,n}^X\} = \int_{-\infty}^{x} \pi_{i-1}^X(du) \int_x^\infty P^X(u, dv)\, p_{n-i}^X(v).$$

Using (3.15) with $X(0) = U + Y_0 = v$, we obtain $p_{n-i}^X(v) \leqq p_{n-i}^Y(v)$. Hence,

$$\mathbf{P}\{A_{i,n}^X\} \leqq \int_{-\infty}^{x} \pi_{i-1}^X(du) \int_x^\infty P^X(u, dv)\, p_{n-i}^Y(v). \tag{3.17}$$

Being space homogeneous, the chain $U + Y_n$ is stochastically increasing. Thus, the function $p_{n-i}^Y(v)$ does not decrease in v and, therefore, is a function of bounded variation. Integrating by parts, we may rewrite the internal integral in the following way:

$$\int_x^\infty P^X(u, dv)\, p_{n-i}^Y(v) = P^X\big(u, [x, \infty)\big)\, p_{n-i}^Y(x) + \int_x^\infty P^X\big(u, [v, \infty)\big)\, dp_{n-i}^Y(v).$$

Estimating $P^X(u,[v,\infty))$ on the right-hand side of the last inequality via condition (3.9) and integrating back by parts, we obtain recursively the estimate and the equality, for any $u\in(-\infty,U)$,

$$\int_x^\infty P^X(u,dv)\,p^Y_{n-i}(v)\leqq c(u)\left(G(x)\,p^Y_{n-i}(x)+\int_x^\infty G(v)\,dp^Y_{n-i}(v)\right)$$
$$=c(u)\int_x^\infty G(dv)\,p^Y_{n-i}(v). \tag{3.18}$$

In the same manner, in view of condition (3.8) and inequality $c(u)\leqq 1$ we obtain, for any $u\in[U,x)$,

$$\int_x^\infty P^X(u,\,dv)\,p^Y_{n-i}(v)\leqq c(u)\int_x^\infty G(dv-u)\,p^Y_{n-i}(v) \tag{3.19}$$
$$\leqq\int_x^\infty G(dv-u)\,p^Y_{n-i}(v). \tag{3.20}$$

Since the function $G(t)$ is long-tailed, there exists an unboundedly growing (as $x\to\infty$) level V such that $G(v-u)\sim G(v)$ as $u\in[U,V)$ and $v\geqq x$. Then inequality (3.19), for $u\in[U,V)$, can be rewritten as

$$\int_x^\infty P^X(u,dv)\,p^Y_{n-i}(v)\;\leqq\big(c(u)+o(1)\big)\int_x^\infty G(dv)\,p^Y_{n-i}(v),\qquad x\to\infty. \tag{3.21}$$

Substituting (3.18), (3.20), and (3.21) into (3.17), we arrive at the inequality

$$\mathbf{P}\{A^X_{i,n}\}\leqq\big(1+o(1)\big)\int_{-\infty}^V\pi^X_{i-1}(du)\,c(u)\int_x^\infty G(dv)\,p^Y_{n-i}(v)$$
$$+\int_V^x\pi^X_{i-1}(du)\int_x^\infty G(dv-u)\,p^Y_{n-i}(v).$$

Recalling the definition of the constants c_i, we obtain

$$\mathbf{P}\{A^X_{i,n}\}\leqq\big(c_{i-1}+o(1)\big)\int_x^\infty G(dv)\,p^Y_{n-i}(v)+\int_V^x\pi^X_{i-1}(du)\int_x^\infty G(dv-u)\,p^Y_{n-i}(v)$$

as $x\to\infty$ uniformly in $i\in[1,n]$. Summing up these inequalities with respect to i from $I(n,x)$ to n and taking into account the convergence $c_i\to c_\infty$ as $i\to\infty$, we deduce from (3.16) the relation, as $n,\,x\to\infty$,

$$\pi^X_n(x)\leqq\big(c_\infty+o(1)\big)\int_x^\infty G(dv)\sum_{i=1}^n p^Y_{n-i}(v)$$
$$+\sum_{i=I}^n\int_V^x\pi^X_{i-1}(du)\int_x^\infty G(dv-u)\,p^Y_{n-i}(v)+o\big(G_{n|\mathbf{E}\zeta|}(x)\big). \tag{3.22}$$

For the homogeneous-in-space Markov chain $U+Y$, we have the inequality

$$\mathbf{P}\{A^Y_{i,n}\}\geqq\int_U^V\pi^Y_{i-1}(du)\int_x^\infty P^Y(u,\,dv)\,p^Y_{n-i}(v)$$
$$=\int_U^V\pi^Y_{i-1}(du)\int_x^\infty G(dv-u)\,p^Y_{n-i}(v).$$

By virtue of the choice of the level V, we have

$$\mathbf{P}\{A_{i,n}^Y\} \geqq \int_U^V \pi_{i-1}^Y(du) \int_x^\infty G(dv)\, p_{n-i}^Y(v) = \pi_{i-1}^Y([U,V)) \int_x^\infty G(dv)\, p_{n-i}^Y(v).$$

Since $V \to \infty$,

$$\mathbf{P}\{A_{i,n}^Y\} \geqq \bigl(1+o(1)\bigr) \int_x^\infty G(dv)\, p_{n-i}^Y(v)$$

as $x \to \infty$ uniformly in $i \in [1,n]$. Summing up these inequalities with respect to i from 1 to n, we deduce from (2.7) the relation

$$\pi_n^Y(x) = \sum_{i=1}^n \mathbf{P}\{A_{i,n}^Y\} \geqq \bigl(1+o(1)\bigr) \int_x^\infty G(dv) \sum_{i=1}^n p_{n-i}^Y(v).$$

It follows from (1.4) and Theorem 2 that, as $x \to \infty$ uniformly in $n \geqq 1$,

$$\pi_n^Y(x) \leqq \frac{1+o(1)}{|\mathbf{E}\zeta|}\, G_{n|\mathbf{E}\zeta|}(x).$$

The last two inequalities imply the estimate

$$\int_x^\infty G(dv) \sum_{i=1}^n p_{n-i}^Y(v) \leqq \frac{1+o(1)}{|\mathbf{E}\zeta|}\, G_{n|\mathbf{E}\zeta|}(x) \qquad \text{as} \quad x \to \infty.$$

Substituting it into (3.22), we arrive at the relation

$$\pi_n^X(x) \leqq \frac{c_\infty + o(1)}{|\mathbf{E}\zeta|}\, G_{n|\mathbf{E}\zeta|}(x) + \sum_{i=I}^n \int_V^x \pi_{i-1}^X(du)\, q_{n-i}(u) \tag{3.23}$$

as n, $x \to \infty$, where $q_{n-i}(u) = \int_x^\infty G(dv-u)\, p_{n-i}^Y(v)$.

The previous considerations also imply the following estimate for the chain $U+Y$, which is valid as $x \to \infty$ uniformly in $n \geqq 1$:

$$\sum_{i=1}^n \int_V^x \pi_{i-1}^Y(du) \int_x^\infty G(dv-u)\, p_{n-i}^Y(v) = o(G_{n|\mathbf{E}\zeta|}(x)). \tag{3.24}$$

It remains to estimate the general term of the sum in (3.23). Integration by parts implies the equality

$$\int_V^x \pi_{i-1}^X(du)\, q_{n-i}(u) = -\pi_{i-1}^X(u)\, q_{n-i}(u)\,\Big|_V^x + \int_V^x \pi_{i-1}^X(u)\, dq_{n-i}(u).$$

In view of Lemma 4 and Theorem 2, $\pi_{i-1}^X(u) \leqq c\pi_{i-1}^Y(u)$ for some $c < \infty$ and for all u and i. Hence,

$$\int_V^x \pi_{i-1}^X(du)\, q_{n-i}(u) \leqq c\pi_{i-1}^Y(V)\, q_{n-i}(V) + c\int_V^x \pi_{i-1}^Y(u)\, dq_{n-i}(u).$$

Again integrating by parts, we obtain

$$\int_V^x \pi_{i-1}^X(du)\, q_{n-i}(u)$$
$$\leqq c\pi_{i-1}^Y(V)\, q_{n-i}(V) + c\pi_{i-1}^Y(u)\, q_{n-i}(u)\Big|_V^x + c\int_V^x \pi_{i-1}^Y(du)\, q_{n-i}(u)$$
$$\leqq c\pi_{i-1}^Y(x)\, q_{n-i}(x) + c\int_V^x \pi_{i-1}^Y(du)\, q_{n-i}(u). \tag{3.25}$$

In the same way, we obtain the estimate

$$\int_V^x \pi_{i-1}^X(du)\, q_{n-i}(u) = \left(\int_V^{x_n^-} + \int_{x_n^-}^x\right)\pi_{i-1}^X(du)\, q_{n-i}(u) \leqq c\pi_{i-1}^Y(x_n^-)\, q_{n-i}(x_n^-)$$
$$+ c\int_V^{x_n^-} \pi_{i-1}^Y(du)\, q_{n-i}(u) + \pi_{i-1}^X\big([x_n^-, x)\big). \tag{3.26}$$

Fix a natural number J. Applying estimate (3.25) if $i \in [I, n-J]$ and estimate (3.26) if $i \in [n-J+1, n]$, we deduce the inequality

$$\sum_{i=I}^n \int_V^x \pi_{i-1}^X(du)\, q_{n-i}(u) \leqq c\sum_{i=I}^{n-J} \pi_{i-1}^Y(x)\, q_{n-i}(x) + c\sum_{i=n-J+1}^{n} \pi_{i-1}^Y(x_n^-)\, q_{n-i}(x_n^-)$$
$$+ c\sum_{i=I}^n \int_V^x \pi_{i-1}^Y(du)\, q_{n-i}(u) + \sum_{i=n-J+1}^n \pi_{i-1}^X\big([x_n^-,\, x)\big).$$

By virtue of (3.24), the third sum on the right-hand side of the estimate is of the order $o(G_{n|\mathbf{E}\zeta|}(x))$; the fourth sum is of the same order, for any fixed J, in view of (3.14). Therefore, for sufficiently slow-growing level $J \equiv J_n(x) \to \infty$, the following estimate holds:

$$\sum_{i=I}^n \int_V^x \pi_{i-1}^X(du)\, q_{n-i}(u)$$
$$\leqq c\sum_{i=I}^{n-J} \pi_{i-1}^Y(x)\, q_{n-i}(x) + c\sum_{i=n-J+1}^{n} \pi_{i-1}^Y(x_n^-)\, q_{n-i}(x_n^-) + o\big(G_{n|\mathbf{E}\zeta|}(x)\big)$$
$$\equiv c\,\Sigma_1 + c\Sigma_2 + o\big(G_{n|\mathbf{E}\zeta|}(x)\big) \qquad \text{as} \quad n,\, x \to \infty. \tag{3.27}$$

Since

$$q_{n-i}(x) = \int_x^\infty G(dv - x)\, p_{n-i}^Y(v)$$
$$= \int_x^\infty G(dv - x)\, \mathbf{P}\big\{Y_k \geqq x \text{ for all } k \in [2, n-i+1] \mid Y_1 = v\big\}$$
$$= \mathbf{P}\big\{Y_k \geqq x \text{ for all } k \in [1, n-i+1] \mid Y_0 = x\big\},$$

then

$$\Sigma_1 = \sum_{i=I}^{n-J} \pi_{i-1}^Y(x)\, \mathbf{P}\{\zeta_1 + \cdots + \zeta_k \geqq 0 \text{ for all } k \leqq n-i+1\}$$
$$= O\big(G_{n|\mathbf{E}\zeta|}(x)\big) \sum_{i=I}^{n-J} \mathbf{P}\{\zeta_1 + \cdots + \zeta_k \geqq 0 \text{ for all } k \leqq n-i+1\}.$$

Since $J \to \infty$, it follows from Lemma 5 that $\sum_{i=I}^{n-J} \mathbf{P}\{\zeta_1 + \cdots + \zeta_k \geqq 0$ for all $k \leqq n-i+1\} \longrightarrow 0$. Therefore,

$$\Sigma_1 = o\bigl(G_{n|\mathbf{E}\zeta|}(x)\bigr) \qquad \text{as} \quad n,\, x \to \infty. \tag{3.28}$$

Now prove that

$$\Sigma_2 = o\bigl(G_{n|\mathbf{E}\zeta|}(x)\bigr) \qquad \text{as} \quad n,\, x \to \infty. \tag{3.29}$$

Since the sequence $J_n(x)$ may increase as slowly as possible, it is sufficient to check that, for any fixed i, $\pi^Y_{n-i}(x_n^-)\, q_{i-1}(x_n^-) = o(G_{n|\mathbf{E}\zeta|}(x))$ holds as n, $x \to \infty$. In view of (3.14) and Lemma 4, we have $\pi^Y_{n-i}(x_n^-) = \pi^Y_{n-i}(x) + o(G_{n|\mathbf{E}\zeta|}(x)) = O(G_{n|\mathbf{E}\zeta|}(x))$. Hence,

$$\begin{aligned}\pi^Y_{n-i}(x_n^-)\, q_{i-1}(x_n^-) &= O\bigl(G_{n|\mathbf{E}\zeta|}(x)\bigr)\, q_{i-1}(x_n^-) \leqq O\bigl(G_{n|\mathbf{E}\zeta|}(x)\bigr)\, P\bigl(x_n^-, [x,\, \infty)\bigr)\\ &= o\bigl(G_{n|\mathbf{E}\zeta|}(x)\bigr),\end{aligned}$$

since $x - x_n^- \to \infty$ and $P(x_n^-, [x,\infty)) \to 0$. So, relation (3.29) is proved.

Substituting (3.28) and (3.29) into (3.27), we arrive at the following relation:

$$\sum_{i=I}^{n} \int_V^x \pi^X_{i-1}(du)\, q_{n-i}(u) = o\bigl(G_{n|\mathbf{E}\zeta|}(x)\bigr) \qquad \text{as} \quad n, x \to \infty.$$

Taking into account the last relation and (3.23), we arrive at estimate (3.11) and, simultaneously, at the conclusion of the lemma.

4. Uniform-in-time theorem on large deviation probabilities. Now we consider the asymptotically homogeneous-in-space Markov chain, that is, we assume the weak convergence $\xi(u) \Rightarrow \xi$ as $u \to \infty$. Let $\mathbf{E}\xi < 0$. In the present section we also assume that the support of the initial distribution π_0 is bounded from above and that convergence in total variation (2.5) holds.

Let ζ be an unbounded-from-above random variable with distribution G and negative mean value. Assume that the distribution of the random variable $\zeta\mathbf{I}\{\zeta \geqq 0\}$ is strongly subexponential.

THEOREM 3. *Let there exist a level U such that the family of random variables $\{|\xi(u)|,\, u \geqq U\}$ admits an integrable majorant. Let condition*

$$\mathbf{P}\bigl\{u + \xi(u) \geqq t\bigr\} \leqq G(t) \qquad \textit{if} \quad u < U, \qquad \xi(u) \leqq_{\rm st} \zeta \qquad \textit{if} \quad u \geqq U$$

hold for $t \geqq U$. Suppose that, for some function $c(u) \leqq 1$, the uniform-in-u convergence $\mathbf{P}\{\xi(u) \geqq t\}/G(t) \to c(u)$ holds as $t \to \infty$. Then relation

$$\pi_n(x) = \bigl(1 + o(1)\bigr) \sum_{k=1}^{n} c_{k-1} G\bigl(x + (n-k)\, |\mathbf{E}\xi|\bigr)$$

holds as $x \to \infty$ uniformly in $n \geqq 1$, where the constants c_k are defined by the equalities $c_k = \int_{\mathbf{R}} c(u)\, \pi_k(du)$. In particular,

$$\pi_n(x) = \left(\frac{c_\infty}{|\mathbf{E}\xi|} + o(1)\right) G_{n|\mathbf{E}\xi|}(x) \qquad \textit{as} \quad n,\, x \to \infty,$$

where $c_\infty \equiv \lim_{k\to\infty} c_k = \int_{\mathbf{R}} c(u)\, \pi(du)$.

Remark 2. If the distribution tail of G is regularly varying at infinity and $c_\infty > 0$, then the last theorem implies the asymptotics $\pi_n(x) \sim c_\infty G_\infty(x)/|\mathbf{E}\xi|$ as $n/x \to \infty$. In particular, in the region $n/x \to \infty$ the asymptotical behavior of the probability $\pi_n(x)$ coincides with that of the invariant measure tail: $\pi_n(x) \sim \pi(x)$.

Proof of Theorem 3. By virtue of the theorem conditions, for any $\varepsilon > 0$, there exist a random variable ζ_ε with a distribution G_ε and level U_ε such that $\mathbf{E}\zeta_\varepsilon \leqq \mathbf{E}\xi + \varepsilon$, $\mathbf{P}\{\zeta_\varepsilon \geqq t\} = \mathbf{P}\{\zeta \geqq t\}$ for all sufficiently large t, and, for $t \geqq U_\varepsilon$,

$$\mathbf{P}\{u + \xi(u) \geqq t\} \leqq G_\varepsilon(t) \qquad \text{if} \quad u < U_\varepsilon, \qquad \xi(u) \leqq_{\mathrm{st}} \zeta_\varepsilon \qquad \text{if} \quad u \geqq U_\varepsilon.$$

It follows from Lemma 7 that

$$\pi_n(x) \leqq \bigl(1 + o(1)\bigr) \sum_{k=1}^{n} c_{k-1} G\Bigl(x + (n-k)\bigl(|\mathbf{E}\xi| - \varepsilon\bigr)\Bigr)$$

as $x \to \infty$ uniformly in $n \geqq 1$. Since the function $G(y)$ is long-tailed, it implies, by the arbitrary choice of $\varepsilon > 0$, the following upper estimate:

$$\pi_n(x) \leqq \bigl(1 + o(1)\bigr) \sum_{k=1}^{n} c_{k-1} G\bigl(x + (n-k)\,|\mathbf{E}\xi|\bigr).$$

Since the time parameter n takes only a countable number of values, the corresponding lower estimate follows from Lemma 3 and from the following relation: For any *fixed* n, $\pi_n(x) \geqq (1 + o(1))\, G(x) \sum_{k=1}^{n} c_{k-1}$ as $x \to \infty$. The latter relation may be verified by induction in the same way as relation (3.12). The proof of the theorem is complete.

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