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Ergodicity of a polling network with an infinite number of stations *

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A Markov polling system with infinitely many stations is studied. The topic is the ergodicity of the infinite-dimensional process of queue lengths. For the infinite-dimensional process, the usual type of ergodicity cannot prevail in general and we introduce a modified concept of ergodicity, namely, weak ergodicity. It means the convergence of finite-dimensional distributions of the process. We give necessary and sufficient conditions for weak ergodicity. Also, the "usual" ergodicity of the system is studied, as well as convergence of functionals which are continuous in some norm.

Keywords: polling system with infinitely many stations, ergodicity, weak ergodicity, monotonicity properties of a polling system, bounds

1. Introduction and results

In this paper we study a polling system with infinitely many stations, numbered $1, 2, \ldots$. At station i groups of customers arrive from outside the system at the instants of a Poisson process with rate μ_i, $i = 1, 2, \ldots$. The size of a group is drawn from a station specific distribution; it is greater than or equal to 1 and such that $\mathbf{E}[\xi_i(1)] = \lambda_i < \infty$, where $\xi_i(t)$ is the number of customers arriving at station i in $[0, t]$, and where λ_i is a given parameter. At each station the customers wait until they have received service. Thereafter they depart from the system.

A single server is polling the stations in Markovian fashion: given the server has been polling station i the next one to be polled is station j, with probability p_{ij}. Let $S = S(n)$, $n = 0, 1, \ldots$, be the corresponding Markov chain, i.e., let $S(n)$ denote the nth station polled by the server. We assume that the chain S is irreducible and ergodic with stationary distribution $\{\pi_i\}$. Given a switch from station i to j, the server will spend a finite time to walk from i to j, distributed as a random variable W_{ij} with arbitrary distribution on the positive real numbers and with positive finite expectation w_{ij}.

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When the server arrives at station i for the mth time there will be a number of consecutive services, distributed as $\min(x_i, D_i(m))$, where x_i is the number of customers present at station i at the moment of server's arrival and the $D_i(m)$ are independent integer valued random variables distributed as D_i, $\mathbf{E}D_i{=}d_i \in (0,\infty)$. The service times in station i are drawn independently from an arbitrary distribution on the non-negative real numbers with finite mean b_i. The corresponding random variable shall be denoted by B_i.

Arrival streams, routing, walking, and services are mutually independent. Let $X_i(n)$, $n = 0,1,\ldots$, denote the number of customers present at station i at the moment the server finishes the walk from the $(n-1)$st to the nth station polled. If the system encompasses only finitely many stations, say N, then the process $(S;X) = \{(S(n);X_1(n),\ldots,X_N(n))\}$ is an irreducible Markov chain, for which necessary and sufficient conditions for ergodicity were given in [2,4] (for a non-Markovian case, see [5]). The polling models with cyclic routing were studied in [7]. The topic of the present paper is also ergodicity.

Put $\mathcal{X} = (\mathbb{Z}^+)^\infty$, where $\mathbb{Z}^+ = \{0,1,2,\ldots\}$. For every vector $\boldsymbol{x} \in \mathcal{X}$ let $\boldsymbol{x}_i$ be its ith coordinate and let $\mathbf{0} \equiv (0,0,\ldots)$. The process $Y = (S;X) = \{(S(n);X_1(n),X_2(n),\ldots)\}$ is valued in $\mathcal{Y} \equiv \mathbb{N}\times\mathcal{X}$, $\mathbb{N} = \{1,2,\ldots\}$.

Let $i \in \mathbb{N}$, $B \subseteq (\mathbb{Z}^+)^i$, and let

$$C_B \equiv \bigl\{\boldsymbol{x} \in \mathcal{X}\colon\ (\boldsymbol{x}_1,\ldots,\boldsymbol{x}_i) \in B\bigr\} \tag{1.1}$$

be the finite-dimensional cylinder with base B. Denote by $\sigma(\mathcal{X})$ the σ-algebra generated by the finite-dimensional cylinders. The process $(S;X)$ is defined on the measurable state space $(\mathcal{Y},\sigma(\mathcal{Y}))$, where $\sigma(\mathcal{Y})$ is generated by $\sigma(\mathbb{N})\times\sigma(\mathcal{X})$, $\sigma(\mathbb{N})$ denoting the set of all subsets of $\mathbb{N}$.

For stations i, j, vector $\boldsymbol{x}$ and cylinder C_B, $B \subseteq (\mathbb{Z}^+)^l$, $l \in \mathbb{N}$, set

$$P\bigl((i;\boldsymbol{x}),(j;C_B)\bigr) \equiv \sum_{\boldsymbol{y}\colon\ (y_1,\ldots,y_l)\in B} \mathbf{P}\bigl\{S(1)=j,\ X_1(1)=y_1,\ \ldots,\ X_l(1)=y_l \mid S(0)=i,\ X(0)=\boldsymbol{x}\bigr\},$$

where the conditional probabilities in the sum are defined in the obvious way.

After extending the function $P((i;\boldsymbol{x}),\cdot)$ to a probability measure on $(\mathcal{Y},\sigma(\mathcal{Y}))$ $P(\cdot,\cdot)$ is a transition probability, i.e., for every $C \in \sigma(\mathcal{Y})$ the function $P(\cdot,C)$ is measurable in the first argument. If C is a cylinder, then this is clear. Consider the family of measurable sets C for which $P(\cdot,C)$ is a measurable function in the first argument. Since it contains the cylinders and is a σ-algebra, it coincides with $\sigma(\mathcal{Y})$. Hence, $P(\cdot,\cdot)$ is indeed a transition probability and $(S(n);X(n))$ is a Markov chain with the measurable state space $(\mathcal{Y},\sigma(\mathcal{Y}))$.

Let $\mathcal{X}^* = \{\boldsymbol{x} \in \mathcal{X}\colon\ \sum_i \boldsymbol{x}_i < \infty\}$, $\mathcal{Y}^* \equiv \mathbb{N}\times\mathcal{X}^*$, and let $\sigma(\mathcal{Y}^*)$ denote the power set of $\mathcal{Y}^*$. This is identical with the restriction of $\sigma(\mathcal{Y})$ to $\mathcal{Y}^*$. If $X(0) \in \mathcal{X}^*$ and $\sum \lambda_i < \infty$, then Y is an irreducible Markov chain with the countable state space $\mathcal{Y}^*$, and we can study the question of its ergodicity in the usual meaning of this notion.

This will be one thing to be done below. In general, however, Y is not irreducible, as there are clearly states of Y which do not communicate.

For instance:

(a) If $\sum_{k=1}^{\infty} |\boldsymbol{x}_k - \boldsymbol{y}_k| = \infty$, then one of the vectors, say $\boldsymbol{x}$, is such that $x_k \geqslant y_k + 1$ for infinitely many k. Therefore, the state $(i;\boldsymbol{y})$ cannot be reached from the state $(j;\boldsymbol{x})$ in finitely many steps.

(b) If the summary input stream is finite, i.e., $\sum_{i=1}^{\infty} \lambda_i < \infty$, then the equivalence relation of communicability between states generates a continuum number of classes of communicating states. If $(i;\boldsymbol{x}) \in \mathcal{Y}$, then the class containing the element $(i;\boldsymbol{x})$ consists of all $(j;\boldsymbol{y}) \in \mathcal{Y}$ such that the vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ have only a finite number of different coordinates, i.e.,

$$\sum_{i=1}^{\infty} |\boldsymbol{x}_i - \boldsymbol{y}_i| < \infty,$$

because only these states can be reached from $(i;\boldsymbol{x})$ in finitely many steps.

Thus the usual type of ergodicity cannot prevail in general. Therefore, we introduce a modified concept of ergodicity, which turns out to be useful.

We say that the chain $(S(n); X(n))$ is *weakly ergodic* if there exists a distribution π on the measurable space $(\mathcal{Y}, \sigma(\mathcal{Y}))$ such that, for all i, j, all $\boldsymbol{x} \in \mathcal{X}^*$, and all finite-dimensional cylinders C_B

$$\mathbf{P}\bigl\{S(n) = j,\ X(n) \in C_B \mid S(0) = i,\ X(0) = \boldsymbol{x}\bigr\} \to \pi(j; C_B)$$

holds as $n \to \infty$. Note that we restrict this requirement to starting points in $\mathcal{Y}^*$. Call π *the limiting distribution associated with weak ergodicity*.

As usual, we say that the measure $\pi = \{\pi(i;\cdot)\}$ on the space $(\mathcal{Y}, \sigma(\mathcal{Y}))$ is *invariant* if for every $i \in \mathbb{N}$ and $C \in \sigma(\mathcal{X})$

$$\pi(i;C) = \sum_{j=1}^{\infty} \int_{\mathcal{X}} \pi(j;\mathrm{d}\boldsymbol{x}) P\bigl((j;\boldsymbol{x}),(i;C)\bigr).$$

Note that this is equivalent to the following assertion. For every $i \in \mathbb{N}$ and every finite-dimensional cylinder C_B, $B \in (\mathbb{Z}^+)^l$,

$$\begin{aligned}
&\pi(i;C_B) \\
&\quad = \sum_{j=1}^{\infty} \sum_{x_1,\dots,x_l,x_j=0}^{\infty} \pi\bigl(j;\{\boldsymbol{x}\colon\ \boldsymbol{x}_1 = x_1,\ \dots,\ \boldsymbol{x}_l = x_l,\ \boldsymbol{x}_j = x_j\}\bigr) \\
&\qquad \times \mathbf{P}\bigl\{S(1) = i,\ X(1) \in C_B \mid S(0) = j,\ X_1(0) = x_1,\ \dots,\ X_l(0) = x_l,\ X_j(0) = x_j\bigr\}.
\end{aligned}$$

The latter probability here is well defined since the conditional probability

$$\mathbf{P}\bigl\{S(1) = i,\ X(1) \in C_B \mid S(0) = j,\ X(0) = \boldsymbol{x}\bigr\}$$

depends only on $\boldsymbol{x}_1, \dots, \boldsymbol{x}_l$, and $\boldsymbol{x}_j$.

If the Markov chain $(S; X)$ *is weakly ergodic, then the associated distribution* π *is invariant*. Indeed, for all n, j, and $B \in (\mathbb{Z}^+)^l$

$$\begin{aligned}
&\mathbf{P}\big\{S(n+1)=i,\ X(n+1)\in C_B\big\}\\
&\quad=\sum_{j=1}^{\infty}\sum_{x_1,\ldots,x_l,x_j=0}^{\infty}\mathbf{P}\big\{S(n)=j,\ X_1(n)=x_1,\ \ldots,\ X_l(n)=x_l,\ X_j(n)=x_j\big\}\\
&\qquad\times\mathbf{P}\big\{S(1)=i,\ X(1)\in C_B \mid S(0)=j,\ X_1(0)=x_1,\ \ldots,\ X_l(0)=x_l,\ X_j(0)=x_j\big\}\\
&\quad\equiv\sum_{j=1}^{\infty}a_j(n).
\end{aligned}$$

Since $a_j(n) \leqslant \mathbf{P}\{S(n)=j\} \to \pi_j$ as $n \to \infty$, by dominated convergence

$$\lim_{n\to\infty}\sum_j a_j(n)=\sum_j\lim_{n\to\infty}a_j(n).$$

Therefore, by weak ergodicity, we finally get

$$\begin{aligned}
&\pi(i, C_B)\\
&\quad\equiv\lim_{n\to\infty}\mathbf{P}\big\{S(n+1)=i,\ X(n+1)\in C_B\big\}\\
&\quad=\sum_{j=1}^{\infty}\sum_{x_1,\ldots,x_l,x_j=0}^{\infty}\lim_{n\to\infty}\mathbf{P}\big\{S(n)=j,\ X_1(n)=x_1,\ \ldots,\ X_l(n)=x_l,\ X_j(n)=x_j\big\}\\
&\qquad\times\mathbf{P}\big\{S(1)=i,\ X(1)\in C_B \mid S(0)=j,\ X_1(0)=x_1,\ \ldots,\ X_l(0)=x_l,\ X_j(0)=x_j\big\}\\
&\quad=\sum_{j=1}^{\infty}\sum_{x_1,\ldots,x_l,x_j=0}^{\infty}\pi\big(j;\{\boldsymbol{x}\colon\ \boldsymbol{x}_1=x_1,\ \ldots,\ \boldsymbol{x}_l=x_l,\ \boldsymbol{x}_j=x_j\}\big)\\
&\qquad\times\mathbf{P}\big\{S(1)=i;\ X(1)\in C_B \mid S(0)=j;\ X_1(0)=x_1,\ \ldots,\ X_l(0)=x_l,\ X_j(0)=x_j\big\}.
\end{aligned}$$

So we can call a limit distribution associated with weak ergodicity also *an invariant distribution associated with weak ergodicity*, or simply *associated invariant distribution*.

Weak ergodicity does not exclude the convergence

$$\sum_{i=1}^{\infty}X_i(n)\to\infty\quad\text{as } n\to\infty, \tag{1.2}$$

given $\sum_{i=1}^{\infty}X_i(0)<\infty$. Equivalently, weak ergodicity does not exclude the convergence of the number of busy stations at step n to infinity as $n\to\infty$.

Note that the invariant distribution associated with weak ergodicity has positive mass on $\mathcal{Y}^*$ if and only if Y is ergodic on $\mathcal{Y}^*$ in the usual sense. Note also that, in the presence of weak ergodicity, invariant distributions may exist besides the one associated with the weak ergodicity.

The following two theorems give necessary and sufficient conditions for weak ergodicity of the Markov chain $(S; X)$; as before, $\{\pi_i\}$ stand for the invariant probabilities of S.

Theorem 1 (Necessity). If the Markov chain $(S; X)$ admits an invariant distribution π (this is true if $(S; X)$ is weakly ergodic) then

$$w \equiv \sum_i \pi_i \sum_j p_{ij} w_{ij} < \infty \tag{1.3}$$

and, for all i,

$$\lambda_i w < (1 - \rho)\pi_i d_i, \tag{1.4}$$

where $\rho = \sum_i \lambda_i b_i$. If, in addition,

$$b \equiv \sum_i \pi_i d_i b_i < \infty, \tag{1.5}$$

then

$$\sum_{x=0}^{\infty} \pi\bigl(i; \{\boldsymbol{x}\colon\ \boldsymbol{x}_i = x\}\bigr)\mathbf{E}\bigl\{\min(x, D_i)\bigr\} = \lambda_i \frac{w}{1-\rho}, \tag{1.6}$$

for all i.

Theorem 2 (Sufficiency). Let conditions (1.3) and (1.4) be fulfilled. Then the Markov chain $(S; X)$ is weakly ergodic. If, in addition, (1.5) holds, then the chain has no invariant distribution other than the one associated with the weak ergodicity.

Theorems 1 and 2 imply the following:

Corollary 1. The following assertions are equivalent:

(i) The Markov chain $(S; X)$ is weakly ergodic.

(ii) The Markov chain $(S; X)$ admits an invariant measure.

(iii) The Markov chain $(S; X)$ satisfies conditions (1.3) and (1.4).

In the case of finitely many stations condition (1.4) was shown to be also sufficient for ergodicity (see [2,4]). In the present case of infinitely many stations (1.4) implies weak ergodicity, only, and this may still allow (1.2).

To obtain the "usual" ergodicity of the system we have to introduce an additional condition which excludes (1.2). The following theorem points out sufficient and necessary conditions for the invariant measure to be concentrated on $\mathcal{Y}^*$ and, hence, gives conditions for ergodicity of the chain $(S(n); X(n))$ in the space $\mathcal{Y}^*$.

Let

$$q_i \equiv {}_1p_{1i} \equiv \mathbf{P}\bigl\{S(1) \neq 1,\ \ldots,\ S(n-1) \neq 1,\ S(n) = i \text{ for some } n \mid S(0) = 1\bigr\}.$$

Theorem 3.

(a) Sufficiency: Let $X(0) \in \mathcal{X}^*$ a.s. If conditions (1.3)–(1.5) are fulfilled, and if

$$\sum_i \frac{\lambda_i}{q_i} < \infty, \tag{1.7}$$

then the Markov chain $(S; X)$ is ergodic in $\mathbb{N} \times \mathcal{X}^*$.

(b) Necessity: If the Markov chain $(S; X)$ is ergodic in $\mathbb{N} \times \mathcal{X}^*$, then (1.3) and (1.4) hold and

$$\sum_i \frac{\mu_i}{q_i} < \infty. \tag{1.8}$$

In particular, $\sum_i \mu_i/\pi_i < \infty$.

Remark. It is well known that the expected number of visits to i between two visits to 1 equals π_i/π_1, and hence we have the inequality $q_i \leqslant \pi_i/\pi_1$. This implies the very last statement in part (b) of the last theorem. In lemma 6 we prove that, under broad conditions, $q_i \geqslant \delta\pi_i$ for some $\delta > 0$ and all i. If this holds, then (1.7) is equivalent to the convergence of the series $\sum_i \lambda_i/\pi_i$.

Some estimates for the probability of the ith queue to be non-empty can be given when the polling system is in stationary regime. Let the chain $(S; X)$ be weakly ergodic with the corresponding invariant measure π, and let $(S(\infty); X(\infty))$ be a random element with distribution π. If $\overline{w} \equiv \sup_{i,j}(w_{ij} + b_i d_i) < \infty$ and $\mathbf{E}\tau^2 < \infty$, where τ is the number of steps required to reach 1 given start in 1, for the chain S, then for each station i the inequality

$$\mathbf{P}\bigl\{X_i(\infty) \geqslant 1\bigr\} \leqslant \frac{w + b}{\pi_1^2}\frac{\lambda_i}{q_i} + \overline{w}\mathbf{E}\tau^2\lambda_i \tag{1.9}$$

holds (for the proof see section 4). Note that the above results can be useful for an analysis of systems with a finite but large number of stations. For instance, we can construct bounds for the number of busy stations in large time.

Now return to weak ergodicity under the conditions of theorem 2. We can consider a richer class of functionals of Y for which weak convergence holds. Along with the finite-dimensional functionals, we may consider functionals which are continuous in some norm. Let $\alpha = (\alpha_1, \alpha_2, \ldots)$ be a sequence of positive numbers. Consider the norm

$$|\boldsymbol{x}|_\alpha = \sum_{i=1}^{\infty} \alpha_i |\boldsymbol{x}_i|,$$

and let $\mathcal{X}_\alpha$ be the collection of all $\boldsymbol{x}$ such that $|\boldsymbol{x}|_\alpha < \infty$. Put

$$\left|(j;\boldsymbol{x})\right|_\alpha = |j| + |\boldsymbol{x}|_\alpha.$$

Theorem 4. Let $S(0)$ have arbitrary distribution, and let $X(0)$ be distributed in $\mathcal{X}^*$. Let

$$\mathbf{E}\tau^2 < \infty \quad \text{and} \quad \sup_{i,j}\left(\mathbf{E}W_{ij}^2 + \mathbf{E}B_i^2\mathbf{E}D_i^2\right) < \infty. \tag{1.10}$$

Let condition (1.4) be fulfilled and

$$\limsup_{i\to\infty} \frac{\lambda_i}{q_i} < \frac{\pi_1}{w+b}.$$

Put $\sigma_i \equiv \mathbf{E}(\xi_i(1) - \lambda_i)^2 = \mathbf{D}\xi_i(1)$. If for some sequence $\beta_i > 0$, $\sum_i \beta_i < \infty$,

$$\sum_{i=1}^{\infty} \frac{\alpha_i}{\beta_i}\,\frac{\sigma_i}{q_i} < \infty, \tag{1.11}$$

then Y is weakly ergodic and the distributions of $f(Y(n))$ converge weakly for any bounded functional $f : \mathbb{N}\times\mathcal{X}_\alpha \to \mathbb{R}$ which is continuous in the norm $|\cdot|_\alpha$.

Remarks.

(i) Condition (1.10) implies that $\sup_{i,j}(w_{ij}+b_i d_i) < \infty$. In particular, conditions (1.3) and (1.5) are fulfilled.

(ii) If the input stream $\xi_i(t)$ is simple, i.e., if $\mu_i = \lambda_i$, then $\sigma_i = \lambda_i$.

Corollary 2. Let $\gamma > 0$. If $\sum \sigma_i/i^\gamma q_i < \infty$ then $\{Y(n)\}$ converges in distribution in the space $\mathcal{X}_\alpha$ with $\alpha_i = 1/i^{\gamma+1+\varepsilon}$, $\varepsilon > 0$.

Corollary 3. If $\sup \sigma_i/q_i < \infty$ then the sequence $\{Y(n)\}$ is convergent in distribution in the space $\mathcal{X}_\alpha$ with $\alpha_i = 1/i^{2+\varepsilon}$, $\varepsilon > 0$.

If the input stream is simple, if $\sup \lambda_i/q_i < \infty$, and if there exist some exponential moments for the random variables τ, W_{ij}, B_i, D_i, then it can be shown that $\{Y(n)\}$ converges in distribution in the space $\mathcal{X}_\alpha$ with $\alpha_i = 1/i \ln^{2+\varepsilon} i$, $\varepsilon > 0$.

Condition (1.7) in theorem 3 is in fact the condition for the family of distributions of $\{Y(n),\ n \in \mathbb{N}\}$ to be tight in the space $\mathcal{Y}^* = \mathbb{N} \times \mathcal{X}_{(1,1,\ldots)}$ ($\mathcal{X}_{(1,1,\ldots)}$ is the space l_1). This condition implies the convergence in total variation and, therefore, the convergence in distribution of any bounded functional of $Y(n)$.

Some analog of theorem 3 for the non-Markovian polling system with simple input is proved in [6]. An infinite number of nodes approach to the ergodicity study of queueing networks is known in the literature. For example, ergodicity results for some queueing networks with random routing and with a very large or infinite number of nodes may be found in [1]. The uniqueness issues for invariant measure of Jackson networks on denumerable graphs were studied in [8].

2. Necessary conditions for weak ergodicity

In the present section we prove theorem 1. For two polling systems $(\widetilde{S};\widetilde{X})$ and $(S;X)$ with identical walking schemes (i.e., $\widetilde{S} =_{\rm st} S$), we say that $\widetilde{X}(n) \geqslant_{\rm st} X(n)$, iff, for every station i, $\widetilde{X}(n)$ is greater or equal in distribution than/to $X(n)$, given $S(n) = i$. We need the following two monotonicity properties of our polling systems.

Lemma 1. Let $(\widetilde{S};\widetilde{X})$ be a polling system such that $\widetilde{S}(0) =_{\rm st} S(0)$, $\widetilde{p}_{ij} = p_{ij}$, $\widetilde{W}_{ij} \geqslant_{\rm st} W_{ij}$, $\widetilde{\lambda}_i \geqslant \lambda_i$, $\widetilde{B}_i \geqslant_{\rm st} B_i$, and $\widetilde{D}_i =_{\rm st} D_i$ for every i and j. If $\widetilde{X}(0) \geqslant_{\rm st} X(0)$ then $\widetilde{X}(n) \geqslant_{\rm st} X(n)$ for every n.

Proof is straightforward from the definition of the polling systems.

Lemma 2. Let $S(0)$ have distribution $\{\pi_i\}$, and let $X(0) = \mathbf{0}$. Then the polling system $(S(n);X(n))$ is increasing in distribution, i.e., for every moment of time n, there exist copies $(S'(n);X'(n))$ and $(S'(n+1);X'(n+1))$ of the random elements $(S(n);X(n))$ and $(S(n+1);X(n+1))$, respectively, such that $S'(n) = S'(n+1)$ and $X'_i(n) \leqslant X'_i(n+1)$ a.s. for all i.

Proof. The proof follows by induction with respect to n. Indeed, for $n = 0$, the assertion is true, because $S(0)$ and $S(1)$ are equal in distribution and $X_i(1) \geqslant 0 = X_i(0)$.

In view of lemma 1 the random element $X(2)$ is greater or equal in distribution than/to $X(1)$, because $X(1)$ is greater or equal in distribution than/to $X(0)$ and because $S(1) =_{\rm st} S(0)$. The transition $n \mapsto n+1$ can be carried out in the same way. The lemma is proved. □

We need also the following equilibrium-type inequality and identity for a Markov chain in stationary regime.

Lemma 3 (see [3,9]). Let $Y(n)$ be a Markov chain valued in a measurable space $\mathcal{Y}$ and admitting a stationary distribution π. Let $G(y)$ be a non-negative measurable function and put $g(y) = \mathbf{E}\{G(Y(1)) - G(y) \mid Y(0) = y\}$. If

$$\int_{\mathcal{Y}} \max\bigl(0, g(y)\bigr)\pi(\mathrm{d}y) < \infty,$$

then

$$\int_{\mathcal{Y}} g(y)\pi(\mathrm{d}y) \geqslant 0.$$

In addition, if there exists $c < \infty$ such that for every y

$$\mathbf{E}\{|G(Y(1)) - G(y)| \mid Y(0) = y\} \leqslant c(1 + |g(y)|),$$

then

$$\int_{\mathcal{Y}} g(y)\pi(\mathrm{d}y) = 0.$$

Proof of theorem 1. Let the random element $(S(0); X(0))$ have distribution π. Then for every n the random element $(S(n); X(n))$ also has distribution π. Let $N > 0$ and let us consider an auxiliary polling system $(S(n); X^{(N)}(n))$ which differs from the original one only in its initial positions and in its walking and service times; we put

$$W_{ij}^{(N)} \equiv \min(W_{ij}, N), \qquad B_i^{(N)} = B_i \quad \text{for } i \leqslant N, \qquad B_i^{(N)} = 0 \quad \text{for } i > N.$$

Denote $w^{(N)} \equiv \sum_i \pi_i \sum_j p_{ij} w_{ij}^{(N)}$; by definition,

$$w^{(N)} \leqslant N. \tag{2.1}$$

Let $X^{(N)}(0) = \mathbf{0}$, and let $S(0)$ have stationary distribution $\{\pi_i\}$. Since $X^{(N)}(0) = \mathbf{0} \leqslant X(0)$, by lemma 1 we have that for every n

$$X^{(N)}(n) \leqslant_{\mathrm{st}} X(n). \tag{2.2}$$

It follows from lemma 2 that the random element $(S(n); X^{(N)}(n))$ has a weak limit as $n \to \infty$. In view of (2.2) and stationarity of $X(n)$ each coordinate of this weak limit is finite with probability 1. Let $\pi^{(N)}$ be an invariant distribution for the system $(S; X^{(N)})$ and let $(S(\infty); X^{(N)}(\infty))$ be a random element with distribution $\pi^{(N)}$. It follows from lemma 1 that

$$X^{(N)}(\infty) \leqslant_{\mathrm{st}} X^{(N+1)}(\infty). \tag{2.3}$$

For $(S; X^{(N)})$ let $\alpha_k^{(N)}(i, j; \boldsymbol{x})$ denote the expected increment in the number of customers at station k between two consecutive arrivals of the server at a station, given the first one is station i, the second station j, and given the customer numbers at all stations at the start of the first visit are given by $\boldsymbol{x}$. Then

$$\alpha_k^{(N)}(i, j; \boldsymbol{x}) = (w_{ij}^{(N)} + d_i(\boldsymbol{x}_i) b_i^{(N)}) \lambda_k - d_i(\boldsymbol{x}_i)\delta_{ik},$$

where $d_i(x) = \mathbf{E} \min(x, D_i)$, $b_i^{(N)} = b_i$ for $i \leqslant N$, $b_i^{(N)} = 0$ for $i > N$, $\delta_{ii} = 1$, and $\delta_{ik} = 0$ for $i \neq k$.

Put

$$\begin{aligned} L_k^{(N)}(i; \boldsymbol{x}) &\equiv \mathbf{E}\{X_k^{(N)}(1) - X_k^{(N)}(0) \mid S(0) = i,\ X^{(N)}(0) = \boldsymbol{x}\} \\ &= \sum_j p_{ij} \alpha_k^{(N)}(i, j; \boldsymbol{x}) \\ &= \sum_j p_{ij} w_{ij}^{(N)} \lambda_k + d_i(\boldsymbol{x}_i) b_i^{(N)} \lambda_k - d_i(\boldsymbol{x}_i)\delta_{ik}. \end{aligned}$$

We have

$$\mathbf{E}\big\{\big|X_k^{(N)}(1) - X_k^{(N)}(0)\big| \mid S(0) = i,\ X^{(N)}(0) = \boldsymbol{x}\big\}$$
$$\leqslant \sum_j p_{ij} w_{ij}^{(N)} \lambda_k + d_i b_i^{(N)} \lambda_k + d_i \delta_{ik},$$

and in view of (2.1)

$$\sum_i \int_{\mathcal{X}} \pi^{(N)}(i; \mathrm{d}\boldsymbol{x})\, \mathbf{E}\big\{\big|X_k^{(N)}(1) - X_k^{(N)}(0)\big| \mid S(0) = i,\ X^{(N)}(0) = \boldsymbol{x}\big\}$$
$$\leqslant \lambda_k w^{(N)} + \lambda_k \sum_{i=1}^{N} \pi_i d_i b_i + \pi_k d_k < \infty,$$

where

$$\int_{\mathcal{X}} \mu(i; \mathrm{d}\boldsymbol{x}) g(\boldsymbol{x}_i, \boldsymbol{x}_k) \equiv \sum_{x=0}^{\infty} \mu\big(i; \{\boldsymbol{x}\colon\ \boldsymbol{x}_i = x\}\big) g(x, \boldsymbol{x}_k)$$

for any measure μ and function g. So, the increments of the chain $(S; X^{(N)})$ satisfy the conditions of lemma 3, and hence, for every $k \in \mathbb{N}$,

$$\sum_i \int_{\mathcal{X}} \pi^{(N)}(i; \mathrm{d}\boldsymbol{x}) L_k^{(N)}(i; \boldsymbol{x}) = 0.$$

This equality implies the equation

$$\left(w^{(N)} + \sum_{i=1}^{N} b_i \int_{\mathcal{X}} \pi^{(N)}(i; \mathrm{d}\boldsymbol{x}) d_i(\boldsymbol{x}_i)\right) \lambda_k = \sum_{i=1}^{\infty} \int_{\mathcal{X}} \pi^{(N)}(i; \mathrm{d}\boldsymbol{x}) d_i(\boldsymbol{x}_i) \delta_{ik}$$
$$= \int_{\mathcal{X}} \pi^{(N)}(k, \mathrm{d}\boldsymbol{x}) d_k(\boldsymbol{x}_k).$$

Letting

$$c_i^{(N)} = \int_{\mathcal{X}} \pi^{(N)}(i; \mathrm{d}\boldsymbol{x}) d_i(\boldsymbol{x}_i)$$

it follows that

$$\left(w^{(N)} + \sum_{i=1}^{N} c_i^{(N)} b_i\right) \lambda_k = c_k^{(N)}, \quad k = 1, 2, \ldots. \tag{2.4}$$

Multiplying with b_k and summing up yields

$$\rho^{(N)} \left(w^{(N)} + \sum_{i=1}^{N} c_i^{(N)} b_i\right) = \sum_{k=1}^{N} c_k^{(N)} b_k,$$

where

$$\rho^{(N)} \equiv \sum_{k=1}^{N} \lambda_k b_k,$$

and this implies $\rho^{(N)} < 1$ (recall that $w^{(N)} > 0$ for N large enough). Hence,

$$\sum_{i=1}^{N} c_i^{(N)} b_i = \frac{\rho^{(N)} w^{(N)}}{1 - \rho^{(N)}}.$$

From the last equality and (2.4) we have for every k

$$\lambda_k w^{(N)} = c_k^{(N)} - \lambda_k \frac{\rho^{(N)} w^{(N)}}{1 - \rho^{(N)}}.$$

So, $\lambda_k w^{(N)} = c_k^{(N)}(1 - \rho^{(N)})$, or, equivalently,

$$\int_{\mathcal{X}} \pi^{(N)}(k; \mathrm{d}\boldsymbol{x}) d_k(\boldsymbol{x}_k) = \frac{\lambda_k w^{(N)}}{1 - \rho^{(N)}}. \tag{2.5}$$

Letting $N \to \infty$ we obtain from (2.3) that

$$\int_{\mathcal{X}} \pi^{(N)}(k; \mathrm{d}\boldsymbol{x}) d_k(\boldsymbol{x}_k) \uparrow \int_{\mathcal{X}} \pi^{(\infty)}(k; \mathrm{d}\boldsymbol{x}) d_k(\boldsymbol{x}_k) \leqslant \pi_k d_k < \infty, \tag{2.6}$$

where $\pi^{(\infty)}$ is a weak limit for $\pi^{(N)}$. Hence, there exists a finite limit for the right-hand side in (2.5):

$$\frac{\lambda_k w^{(N)}}{1 - \rho^{(N)}} \uparrow \frac{\lambda_k w}{1 - \rho} < \infty. \tag{2.7}$$

In particular, $w < \infty$ and (1.3) are proved. Using (2.6) and (2.7) in (2.5) implies the equality

$$\int_{\mathcal{X}} \pi^{(\infty)}(k; \mathrm{d}\boldsymbol{x}) d_k(\boldsymbol{x}_k) = \frac{\lambda_k w}{1 - \rho}.$$

Since $d_k(x) \leqslant d_k$, $d_k(0) = 0$, and $\pi^{(\infty)}(k; \{\boldsymbol{x}\colon\ \boldsymbol{x}_k{=}0\}) > 0$; the integral here is less than $\pi_k d_k$. Therefore, the last equality implies the assertion (1.4).

Let now (1.5) be fulfilled. In view of (1.3) and (1.5) we have

$$\sum_i \int_{\mathcal{X}} \pi(i; \mathrm{d}\boldsymbol{x}) \mathbf{E}\{|X_k(1) - X_k(0)| \mid S(0) = i,\ X(0) = \boldsymbol{x}\}$$
$$\leqslant \lambda_k w + \pi_k d_k + \lambda_k \sum_i \pi_i d_i b_i < \infty,$$

and we can apply lemma 3 to the chain (S, X) with stationary distribution π. As above, we obtain for this chain the equality $\lambda_i w = c_i(1 - \rho)$, where $c_i \equiv \int_{\mathcal{X}} \pi(i; \mathrm{d}\boldsymbol{x}) d_i(\boldsymbol{x}_i)$, and the proof of theorem 1 is complete. □

3. Sufficient conditions for weak ergodicity

This section is devoted to the proof of theorem 2. First we construct a minorant for the process $X(n)$. Let $(S; \underline{X}^{(k)})$ be an auxiliary polling system, which differs from $(S; X)$ only in its service times at the stations $k+1, k+2, \ldots$; we assume that

$$\underline{B}_i^{(k)} = B_i \quad \text{for } i \leqslant k \quad \text{and} \quad \underline{B}_i^{(k)} = 0 \quad \text{for } i > k.$$

The distribution of the process $(S(n); \underline{X}_1^{(k)}(n), \ldots, \underline{X}_k^{(k)}(n))$ is the same as of the process $(S(n); X_1(n), \ldots, X_k(n))$, given $X_{k+1}(0) = X_{k+2}(0) = \cdots = 0$ and $\lambda_{k+1}(0) = \lambda_{k+2}(0) = \cdots = 0$. The finite-dimensional process $(S; \underline{X}_1^{(k)}, \ldots, \underline{X}_k^{(k)})$ is a Markov chain. It follows from conditions (1.3), (1.4) and from the results in [2,4] that the chain $(S; \underline{X}^{(k)})$ is ergodic in $\mathcal{Y}^*$. Let $\underline{\pi}^{(k)}$ be the invariant distribution for the chain $(S; \underline{X}^{(k)})$. In view of theorem 1 the measure $\underline{\pi}^{(k)}$ satisfies the equality

$$\sum_{x=0}^{\infty} \underline{\pi}^{(k)}\big(i; \{\boldsymbol{x}\colon\ \boldsymbol{x}_i = x\}\big)\mathbf{E}\big\{\min(x, D_i)\big\} = \lambda_i \frac{w}{1 - \underline{\rho}^{(k)}}, \tag{3.1}$$

where

$$\underline{\rho}^{(k)} = \sum_{j=1}^{k} \lambda_j b_j.$$

Let $(S(\infty); \underline{X}^{(k)}(\infty))$ be the random element with distribution $\underline{\pi}^{(k)}$. By virtue of lemma 1 the random elements $(S(\infty); \underline{X}^{(k)}(\infty))$, $k \in \mathbb{N}$, are increasing in distribution. Therefore, $\underline{\pi}^{(k)}$ has a weak limit $\underline{\pi}$ in the enlarged space $\overline{\mathcal{Y}} = \mathbb{N} \times \overline{\mathcal{X}}$, where $\overline{\mathcal{X}} = (\mathbb{Z}^+ \cup \{\infty\})^\infty$. Let $(S(\infty); \underline{X}(\infty))$ be a random element in $\overline{\mathcal{Y}}$ with distribution $\underline{\pi}$.

The measure $\underline{\pi}$ is invariant for the chain $(S; X)$ in $\overline{\mathcal{Y}}$. If $S(0)$ has distribution $\{\pi_i\}$ and $X(0) = \mathbf{0}$ then the distribution of the random element $(S(n); X(n))$ converges to $\underline{\pi}$ in the space $\overline{\mathcal{Y}}$.

Fix any $i \in \mathbb{N}$. Taking the limit in (3.1) with respect to k, $k \to \infty$, we obtain the equality

$$\sum_{x=0}^{\infty} \underline{\pi}\big(i; \{\boldsymbol{x}\colon\ \boldsymbol{x}_i = x\}\big)\mathbf{E}\big\{\min(x, D_i)\big\} + \underline{\pi}\big(i; \{\boldsymbol{x}\colon\ \boldsymbol{x}_i = \infty\}\big)d_i = \lambda_i \frac{w}{1 - \rho}. \tag{3.2}$$

Assume that $\mathbf{P}\{\underline{X}_i = \infty\} = 1$. Then $\underline{\pi}(i; \{\boldsymbol{x}\colon\ \boldsymbol{x}_i = \infty\}) = \pi_i$ and it follows from (3.2) that $\pi_i d_i = \lambda_i w/(1 - \rho)$, which is in contradiction to (1.4). So,

$$\mathbf{P}\{\underline{X}_i < \infty\} > 0. \tag{3.3}$$

Let μ be the conditional distribution of the random element $(S(\infty); \underline{X}(\infty))$, given $\underline{X}_i(\infty) < \infty$. The measure μ in the enlarged space $\overline{\mathcal{Y}}$ is well defined in view of (3.3). Since the sets $\{\boldsymbol{x}\colon\ \boldsymbol{x}_i = \infty\}$ and $\{\boldsymbol{x}\colon\ \boldsymbol{x}_i < \infty\}$ do not communicate, the measure μ is invariant (in the space $\overline{\mathcal{Y}}$) for the chain $(S; X)$.

Let $S'(0)$ have distribution $\{\pi_j\}$, and let $X'(0) = \mathbf{0}$. If the random element $(\widetilde{S}(0); \widetilde{X}(0))$ has distribution μ then for every n the random element $(\widetilde{S}(n); \widetilde{X}(n))$ also has distribution μ. Since $X'(0) \leqslant_{\mathrm{st}} \widetilde{X}(0)$, by lemma 1 $X'(n) \leqslant_{\mathrm{st}} \widetilde{X}(n)$ for every n. Letting $n \to \infty$, we obtain the inequality

$$\begin{aligned}\mathbf{P}\{\underline{X}_i = \infty\} &= \lim_{y\to\infty} \lim_{n\to\infty} \mathbf{P}\big\{X_i'(n) \geqslant y\big\} \\ &\leqslant \lim_{y\to\infty} \lim_{n\to\infty} \mathbf{P}\big\{\widetilde{X}_i(n) \geqslant y\big\} = \mu\big(\mathbb{N}; \{\boldsymbol{x}\colon\ \boldsymbol{x}_i = \infty\}\big) = 0.\end{aligned}$$

We have verified that $\mathbf{P}\{\underline{X}_i < \infty\} = 1$ for every $i \in \mathbb{N}$ and, therefore, the random element $(S(\infty); \underline{X}(\infty))$ is valued in $\mathcal{Y}$. So, the measure $\underline{\pi}$ is invariant for the chain $(S; X)$ and the first part of theorem 2 is proved.

Now let us assume that the system $(S; X)$ satisfies, in addition, condition (1.5) and has one more invariant measure, say $\overline{\pi}$. Denote by $(S(\infty); \overline{X}(\infty))$ the random element with distribution $\overline{\pi}$. We are going to prove that

$$\underline{\pi} = \overline{\pi}. \tag{3.4}$$

Let $S(0)$ have distribution $\{\pi_i\}$, and let $\underline{X}(0) = \mathbf{0}$. If the random element $(S(0); \overline{X}(0))$ has distribution $\overline{\pi}$, then for every n the random element $(S(n); \overline{X}(n))$ also has distribution $\overline{\pi}$. Since $\underline{X}(0) \leqslant_{\mathrm{st}} \overline{X}(0)$, by lemma 1 $\underline{X}(n) \leqslant_{\mathrm{st}} \overline{X}(n)$ for every n. Letting $n \to \infty$, we obtain the inequality

$$\underline{X}(\infty) \leqslant_{\mathrm{st}} \overline{X}(\infty). \tag{3.5}$$

Let both $(S(n); \underline{X}(n))$ and $(S(n); \overline{X}(n))$ be in stationary regime with distributions $\underline{\pi}$ and $\overline{\pi}$, respectively. In view of (3.5) equality (3.4) is equivalent to the following system of equalities:

$$\mathbf{P}\big\{\underline{X}_i(0) \geqslant x\big\} = \mathbf{P}\big\{\overline{X}_i(0) \geqslant x\big\}, \quad i \in \mathbb{N},\ x \in \mathbb{Z}^+. \tag{3.6}$$

Let us assume that, to the contrary, (3.6) is not true. Then there exist i', i, and x such that

$$\mathbf{P}\big\{S(0) = i';\ \underline{X}_i(0) \geqslant x\big\} < \mathbf{P}\big\{S(0) = i';\ \overline{X}_i(0) \geqslant x\big\}. \tag{3.7}$$

We assume that x is the minimal number with the property that there exist i' and i for which (3.7) holds.

Let us prove now that (3.7) is fulfilled with $i' = i$, i.e., that

$$\mathbf{P}\big\{S(0) = i;\ \underline{X}_i(0) \geqslant x\big\} < \mathbf{P}\big\{S(0) = i;\ \overline{X}_i(0) \geqslant x\big\}. \tag{3.8}$$

Indeed, if $i' \neq i$, let n_0 be the minimal $n \geqslant 1$ such that $\mathbf{P}\{S(n) = i \mid S(0) = i'\} > 0$. Put

$$f(j, y, z) = \mathbf{P}\big\{S(n_0) = i;\ X_i(n_0) \geqslant x \mid S(0) = j;\ X_j(0) = y,\ X_i(0) = z\big\}.$$

By the stationarity of $(S(n); \underline{X}(n))$

$$p \equiv \mathbf{P}\{S(n_0)=i;\ \underline{X}_i(n_0)\geqslant x\} = \sum_{j,y,z}\mathbf{P}\{S(0)=j;\ \underline{X}_j(0)=y,\ \underline{X}_i(0)=z\}f(j,y,z).$$

Because the function $f(j,y,z)$ is nondecreasing in y and z, in view of $\underline{X}_j(0) \leqslant \overline{X}_j(0)$ and $\underline{X}_i(0) \leqslant \overline{X}_i(0)$ a.s. we have

$$\begin{aligned} p \leqslant & \sum_{j\neq i',y,z}\mathbf{P}\{S(0)=j;\ \overline{X}_j(0)=y,\ \overline{X}_i(0)=z\}f(j,y,z) \\ & + \sum_{y,z}\mathbf{P}\{S(0)=i';\ \underline{X}_{i'}(0)=y,\ \underline{X}_i(0)=z\}f(i',y,z). \end{aligned}$$

Since $\mathbf{P}\{S(n)=i \mid S(0)=i'\}=0$ for every $n<n_0$, the function $f(i',y,z)$ is strictly increasing in $z\in[0,x]$. Therefore, by (3.7),

$$\begin{aligned} & \sum_{y,z}\mathbf{P}\{S(0)=i';\ \underline{X}_{i'}(0)=y,\ \underline{X}_i(0)=z\}f(i',y,z) \\ & \quad < \sum_{y,z}\mathbf{P}\{S(0)=i';\ \overline{X}_{i'}(0)=y,\ \overline{X}_i(0)=z\}f(i',y,z). \end{aligned}$$

Hence,

$$\begin{aligned} p & < \sum_{j,y,z}\mathbf{P}\{S(0)=j;\ \overline{X}_j(0)=y,\ \overline{X}_i(0)=z\}f(j,y,z) \\ & = \mathbf{P}\{S(n_0)=i;\ \overline{X}_i(n_0)\geqslant x\} \\ & = \mathbf{P}\{S(0)=i;\ \overline{X}_i(0)\geqslant x\}. \end{aligned}$$

So (3.8) is proved.

Now we prove that $\mathbf{P}\{D_i \geqslant x\} > 0$. If, to the contrary, $D_i \leqslant x-1$ a.s., then it follows from (3.8) that for every j such that $p_{ij}>0$ we have

$$\mathbf{P}\{S(1)=j;\ \underline{X}_i(1)\geqslant y\} < \mathbf{P}\{S(1)=j;\ \overline{X}_i(1)\geqslant y\}$$

for some $y<x$. This contradicts the property for x to be the minimal number for which (3.7) is fulfilled. So, $\mathbf{P}\{D_i \geqslant x\}>0$ and, therefore, (3.8) implies

$$\mathbf{E}\{\min(\underline{X}_i(0), D_i); S(0)=i\} < \mathbf{E}\{\min(\overline{X}_i(0), D_i); S(0)=i\}.$$

Since the measures $\underline{\pi}$ and $\overline{\pi}$ are invariant, by virtue of equation (1.6)

$$\mathbf{E}\{\min(\underline{X}_i(0), D_i); S(0)=i\} = \int_{\mathcal{X}}\underline{\pi}(i;\mathrm{d}\boldsymbol{x})\mathbf{E}\{\min(x_i, D_i)\} = \lambda_i\frac{w}{1-\rho}$$

and

$$\mathbf{E}\{\min(\overline{X}_i(0), D_i); S(0)=i\} = \int_{\mathcal{X}}\overline{\pi}(i;\mathrm{d}\boldsymbol{x})\mathbf{E}\{\min(x_i, D_i)\} = \lambda_i\frac{w}{1-\rho}.$$

We arrive at a contradiction to the previous inequality. The proof of theorem 2 is complete. □

4. Sufficient conditions for ergodicity

In this section we prove the sufficiency part (a) of theorem 3. We begin with the following lemma.

Lemma 4. Let the polling system $(S; X)$ be weakly ergodic with invariant measure π, and let $(S(\infty); X(\infty))$ be a random element with distribution π. Then, for every I and sequence $x_I, x_{I+1}, \ldots,$

$$\mathbf{P}\big\{X_i(\infty) \geqslant x_i \text{ for some } i \geqslant I\big\} \leqslant \limsup_{n\to\infty} \mathbf{P}\big\{X_i(n) \geqslant x_i \text{ for some } i \geqslant I\big\}.$$

Proof. We have that

$$\mathbf{P}\big\{X_i(\infty) \geqslant x_i \text{ for some } i \geqslant I\big\} = \lim_{J\to\infty} \mathbf{P}\big\{X_i(\infty) \geqslant x_i \text{ for some } i \in [I, J]\big\}.$$

Hence, the assertion follows from the convergence

$$\mathbf{P}\big\{X_i(n) \geqslant x_i \text{ for some } i \in [I, J]\big\} \to \mathbf{P}\big\{X_i(\infty) \geqslant x_i \text{ for some } i \in [I, J]\big\}$$

as $n \to \infty$. □

We need a further lemma. Let $T(0) = \min\{n \geqslant 0\colon\ S(n) = 1\}$ and $T(l+1) = \min\{n > T(l)\colon\ S(n) = 1\}$, for $l \geqslant 0$. We have that for every l the difference $T(l{+}1) - T(l)$ is equal in distribution to τ. First we prove the following lemma:

Lemma 5. For any initial distribution of $(S(0); X(0))$, and, for every station i,

$$\limsup_{l\to\infty} \mathbf{P}\big\{X_i\big(T(l)\big) \geqslant 1\big\} \leqslant \frac{(w+b)\lambda_i}{\pi_1 q_i}.$$

Proof. We need to consider the case where

$$(w+b)\lambda_i < \pi_1 q_i. \tag{4.1}$$

Let us define an auxiliary polling system $(S; \overline{X})$ with the following characteristics:

(1) $\overline{X}(0) = X(0)$;

(2) $\overline{W}_{kj} = W_{kj} + B_k(1) + \cdots + B_k(D_k)$ for all k, j;

(3) $\overline{\lambda}_k = \lambda_k$, $\overline{B}_k \equiv 0$, $\overline{D}_k \equiv D_k$ for all k.

By definition the process $\overline{X}$ can be constructed in such a way that for every n

$$X_i(n) \leqslant \overline{X}_i(n) \quad \text{a.s.} \tag{4.2}$$

Put $G_i(l) = \overline{X}_i(T(l))$. The process $G_i(l)$ is a Markov chain with the following increments:

$$a_i(0) \equiv \mathbf{E}\big\{G_i(1) - G_i(0) \mid G_i(0) = 0\big\} = \lambda_i \mathbf{E}\eta,$$
$$a_i(x) \equiv \mathbf{E}\big\{G_i(1) - G_i(0) \mid G_i(0) = x\big\} \leqslant \lambda_i \mathbf{E}\eta - q_i, \quad x = 1, 2, \dots,$$

where η is the period of time (in real time) between two consecutive arrivals of the server at the first station in system $(S; \overline{X})$. By a standard result $\mathbf{E}\eta = (w+b)/\pi_1$ and, hence,

$$a_i(0) = \frac{(w+b)\lambda_i}{\pi_1}, \tag{4.3}$$
$$a_i(x) \leqslant \frac{(w+b)\lambda_i}{\pi_1} - q_i, \quad x \geqslant 1. \tag{4.4}$$

In view of (4.4) and (4.1) $a_i(x) \leqslant -\delta < 0$ for $x \geqslant 1$. So, by [10, theorem 1] the Markov chain $G_i(l)$ is ergodic with invariant probabilities denoted by $\{\pi_i(x)\}_{x=0}^{\infty}$. It follows from lemma 3 that

$$0 \leqslant \sum_{x=0}^{\infty} \pi_i(x) a_i(x).$$

Substituting (4.3) and (4.4) we obtain that

$$0 \leqslant \frac{(w+b)\lambda_i}{\pi_1} - q_i \sum_{x=1}^{\infty} \pi_i(x).$$

Hence,

$$\pi_i\big([1,\infty)\big) \leqslant \frac{(w+b)\lambda_i}{\pi_1 q_i}.$$

The last inequality and (4.2) imply the assertion of the lemma. □

Theorem 2 implies that the Markov chain $(S(n); X(n))$ is weakly ergodic with unique invariant measure π. In particular, for every i there exists a limit of the probability $\mathbf{P}\{S(n) = 1,\ X_i(n) \geqslant 1\}$ as $n \to \infty$.

Lemma 6. If $S(0)$ has distribution $\{\pi_i\}$ and $X(0) = \mathbf{0}$, then

$$\mathbf{P}\big\{S(n) = 1,\ X_i(n) \geqslant 1\big\} \leqslant \frac{(w+b)\lambda_i}{\pi_1 q_i},$$

for all i and n.

Proof. Since $T(k) \geqslant k$, it follows that

$$\lim_{k\to\infty} \mathbf{P}\big\{S(k) = 1,\ X_i(k) \geqslant 1\big\} = \lim_{k\to\infty} \frac{1}{k} \sum_{m=0}^{k} \mathbf{P}\big\{S(m) = 1,\ X_i(m) \geqslant 1\big\}$$
$$\leqslant \limsup_{k\to\infty} \frac{1}{k} \sum_{l=0}^{k} \mathbf{P}\big\{X_i\big(T(l)\big) \geqslant 1\big\}.$$

In view of lemma 2, for each n,

$$\mathbf{P}\big\{S(n)=1,\ X_i(n)\geqslant 1\big\}\leqslant \lim_{k\to\infty}\mathbf{P}\big\{S(k)=1,\ X_i(k)\geqslant 1\big\}.$$

Now lemma 5 finishes the proof. □

Let $S(0)$ have distribution $\{\pi_i\}$, and let $X(0)=\mathbf{0}$. Let $I\in\mathbb{N}$. Let us now estimate the probability that one of the queues i, $i\geqslant I$, is non-empty. For any m_1 and m_2, $m_1<m_2$, denote by $B_I[m_1,m_2]$ the event that during the (real-) time period between the server's m_1th and m_2th arrival times to a station at least one new customer arrives at one of the stations $i\geqslant I$. By the formula of total probability we get for any n

$$\begin{aligned}
&\mathbf{P}\big\{X_i(n)\geqslant 1 \text{ for some } i\geqslant I\big\}\\
&\quad\leqslant \sum_{m=0}^{n}\sum_{l=0}^{\infty}\Big(\mathbf{P}\big\{T(l)=n-m,\ T(l+1)>n,\ X_i(n-m)\geqslant 1 \text{ for some } i\geqslant I\big\}\\
&\qquad+\mathbf{P}\big\{T(l)=n-m,\ T(l+1)>n,\ B_I[n-m,n]\big\}\Big)+\mathbf{P}\big\{T(0)>n\big\}\\
&\quad=\sum_{m=0}^{n}\sum_{l=0}^{\infty}\Big(\mathbf{P}\big\{T(l)=n-m,\ X_i(n-m)\geqslant 1 \text{ for some } i\geqslant I\big\}\\
&\qquad\times\mathbf{P}\big\{T(l+1)-T(l)>m\big\}\\
&\qquad+\mathbf{P}\big\{T(l)=n-m,\ T(l+1)>n,\ B_I[n-m,n]\big\}\Big)+\mathbf{P}\big\{T(0)>n\big\}\\
&\quad=\sum_{m=0}^{n}\mathbf{P}\big\{S(n-m)=1,\ X_i(n-m)\geqslant 1 \text{ for some } i\geqslant I\big\}\mathbf{P}\{\tau>m\}\\
&\qquad+\sum_{m=0}^{n}\sum_{l=0}^{\infty}\mathbf{P}\big\{T(l)=n-m,\ T(l+1)>n,\ B_I[n-m,n]\big\}+\mathbf{P}\big\{T(0)>n\big\}\\
&\quad=R_1(I,n)+R_2(I,n)+\mathbf{P}\big\{T(0)>n\big\}.
\end{aligned}\tag{4.5}$$

Since

$$\sum_{m=0}^{\infty}\mathbf{P}\{\tau>m\}=\mathbf{E}\tau=\frac{1}{\pi_1}<\infty,$$

by lemma 6 we obtain that, for all I and n,

$$R_1(I,n)\leqslant\sum_{i\geqslant I}\frac{(w+b)\lambda_i}{\pi_1 q_i}\mathbf{E}\tau=\frac{w+b}{\pi_1^2}\sum_{i\geqslant I}\frac{\lambda_i}{q_i}.\tag{4.6}$$

The term $R_2(I,n)$ can be estimated in the following way:

$$R_2(I,n)=\sum_{m=0}^{n}\mathbf{P}\big\{S(n-m)=1,\ S(n-m+1)\neq 1,\ \ldots,\ S(n)\neq 1,\ B_I[n-m,n]\big\}$$

$$\leqslant \sum_{m=0}^{n} \mathbf{P}\bigl\{S(n-m+1) \neq 1,\ \ldots,\ S(n) \neq 1,\ B_I[n-m,n] \mid S(n-m)=1\bigr\}$$

$$= \sum_{m=0}^{n} \mathbf{P}\bigl\{S(1) \neq 1,\ \ldots,\ S(m) \neq 1,\ B_I[0,m] \mid S(0)=1\bigr\}. \tag{4.7}$$

Since

$$\mathbf{P}\bigl\{S(1) \neq 1,\ \ldots,\ S(m) \neq 1,\ B_I[0,m] \mid S(0)=1\bigr\} \leqslant \mathbf{P}\{\tau > m\},$$

and in view of $\mathbf{P}\{B_I[0,m] \mid S(0)=1\} \to 0$ as $I \to \infty$ for every m, by dominated convergence we obtain

$$\sup_n R_2(I,n) \to 0 \quad \text{as } I \to \infty. \tag{4.8}$$

Substituting estimate (4.6) into (4.5), we obtain

$$\mathbf{P}\bigl\{X_i(n) \geqslant 1 \text{ for some } i \geqslant I\bigr\} \leqslant \frac{w+b}{\pi_1^2} \sum_{i \geqslant I} \frac{\lambda_i}{q_i} + \sup_n R_2(I,n) + \mathbf{P}\bigl\{T(0) > n\bigr\}.$$

Hence,

$$\limsup_{n\to\infty} \mathbf{P}\bigl\{X_i(n) \geqslant 1 \text{ for some } i \geqslant I\bigr\} \leqslant \frac{w+b}{\pi_1^2} \sum_{i \geqslant I} \frac{\lambda_i}{q_i} + \sup_n R_2(I,n). \tag{4.9}$$

Let $(S(\infty), X(\infty))$ be a random element with distribution π. It follows from lemma 4 and (4.9) that, for all I,

$$\mathbf{P}\bigl\{X_i(\infty) \geqslant 1 \text{ for some } i \geqslant I\bigr\} \leqslant \frac{w+b}{\pi_1^2} \sum_{i \geqslant I} \frac{\lambda_i}{q_i} + \sup_n R_2(I,n).$$

The last inequality, condition (1.7), and convergence (4.8) imply that

$$\mathbf{P}\bigl\{X_i(\infty) \geqslant 1 \text{ for some } i \geqslant I\bigr\} \to 0 \quad \text{as } I \to \infty.$$

Therefore, π is concentrated on $\mathcal{Y}^*$, i.e., the Markov chain $(S(n), X(n))$ is ergodic in $\mathcal{Y}^*$. The proof of theorem 3 part (a) is complete. □

Now proceed to the proof of estimate (1.9). It follows from (4.9) that it is sufficient to prove that $R_2(i,n) \leqslant \overline{w}\,\mathbf{E}\tau^2\lambda_i$. Indeed, the mean number of new customers until the mth arrival time of the server at a station is less than or equal to $m\overline{w}\lambda_i$. Therefore, by estimate (4.7) and Chebyshev's inequality we have

$$R_2(I,n) \leqslant \sum_{m=0}^{n} \mathbf{P}\bigl\{S(1) \neq 1,\ \ldots,\ S(m) \neq 1 \mid S(0)=1\bigr\} m\overline{w}\lambda_i$$

$$= \overline{w}\lambda_i \sum_{m=0}^{n} \mathbf{P}\{\tau > m\} m \leqslant \overline{w}\lambda_i \mathbf{E}\tau^2.$$

□

5. Necessary conditions for ergodicity

In the present section we prove the necessity part (b) of theorem 3.

Suppose that the chain $(S;X)$ is ergodic in $\mathbb{N} \times \mathcal{X}^*$. Then the state $(1;\mathbf{0})$ is positive recurrent, i.e., if

$$\widetilde{\tau} \equiv \min\bigl\{n\colon\ S(n) = 1,\ X(n) = \mathbf{0} \mid S(0) = 1,\ X(0) = \mathbf{0}\bigr\}$$

then

$$\mathbf{E}\widetilde{\tau} < \infty. \tag{5.1}$$

Put

$$\tau(i) \equiv \min\bigl\{n \geqslant 1\colon\ S(n) = 1,\ S\bigl(n'\bigr) = i \text{ for some } n' \leqslant n\colon\ S(0) = 1\bigr\}.$$

It follows from the definition of $\tau(i)$ that

$$\mathbf{E}\tau(i) = \sum_{s=1}^{\infty} \mathbf{P}\bigl\{\tau(i) \geqslant s\bigr\} \geqslant \sum_{s=1}^{\infty} (1 - q_i)^s = \frac{1}{q_i} - 1. \tag{5.2}$$

Ergodicity implies that $\sum \lambda_i < \infty$. Since $\mu_i \leqslant \lambda_i$,

$$\sum_{i=1}^{\infty} \mu_i < \infty. \tag{5.3}$$

Without loss of generality, we may assume that $p_{12} w_{12} > 0$. Hence, in view of (5.3) we have

$$\begin{aligned} r_i &\equiv \mathbf{P}\bigl\{S(1) = 2,\ X_i(1) \geqslant 1,\ X_k(1) = 0 \text{ for each } k \neq i \mid S(0) = 1,\ X(0) = \mathbf{0}\bigr\} \\ &= p_{12} \mathbf{P}\bigl\{X_i(1) \geqslant 1 \mid S(0) = 1,\ S(1) = 2,\ X(0) = \mathbf{0}\bigr\} \\ &\quad \times \prod_{k \neq i} \mathbf{P}\bigl\{X_k(1) = 0 \mid S(0) = 1,\ S(1) = 2,\ X(0) = \mathbf{0}\bigr\} \\ &= p_{12}\bigl(1 - \mathbf{E}\,\mathrm{e}^{-\mu_i W_{12}}\bigr) \prod_{k \neq i} \mathbf{E}\,\mathrm{e}^{-\mu_k W_{12}} \geqslant \delta \mu_i \end{aligned}$$

for some $\delta > 0$ and every $i \in \mathbb{N}$. Since

$$\mathbf{E}\widetilde{\tau} \geqslant \sum_i r_i \mathbf{E}\tau(i),$$

we have

$$\mathbf{E}\widetilde{\tau} \geqslant \delta \sum_i \mu_i \mathbf{E}\tau(i). \tag{5.4}$$

Theorem 3 part (b) follows now from (5.1), (5.2), and (5.4). □

6. Some estimates for the taboo probability q_i

In the following lemma we expose a condition on the chain S under which there exists $\delta > 0$ such that for every station i the inequality $q_i \geqslant \delta\pi_i$ holds. Under this inequality condition (1.7) is equivalent to the convergence of the series $\sum \lambda_i/\pi_i$.

Let ζ_i be a random variable with values in $\mathbb{Z}$ distributed according to

$$\mathbf{P}\{\zeta_i = k\} = p_{i,i+k}, \quad k \in \mathbb{Z}.$$

Lemma 7. Suppose that, for some random variable ζ with negative mean value and for some state i_*,

$$\zeta_i \leqslant_{\mathrm{st}} \zeta \tag{6.1}$$

for every $i \geqslant i_*$. Then there exists $\delta > 0$ such that $q_i \geqslant \delta\pi_i$ for all i.

Proof. Let ${}_1p_{ii}$ be the taboo probability of the transition from i to i avoiding the first station, and let ν_i denote the number of visits to i between two consecutive visits to 1. Then

$$\mathbf{E}\nu_i = \sum_{s=1}^{\infty} \mathbf{P}\{\nu_i \geqslant s\} = \sum_{s=1}^{\infty} {}_1p_{1i}({}_1p_{ii})^{s-1} = \frac{{}_1p_{1i}}{1 - {}_1p_{ii}}.$$

On the other hand,

$$\mathbf{E}\nu_i = \frac{\pi_i}{\pi_1},$$

hence,

$$\frac{\pi_i}{\pi_1} = \frac{{}_1p_{1i}}{1 - {}_1p_{ii}} = \frac{q_i}{1 - {}_1p_{ii}}. \tag{6.2}$$

Let $\zeta(n)$, $n = 1, 2, \ldots$, denote i.i.d. copies of ζ. By the Strong Law of Large Numbers and by the negativity of $\mathbf{E}\zeta$ there exists $\varepsilon > 0$ such that

$$p \equiv \mathbf{P}\bigl\{\zeta(1) + \cdots + \zeta(n) \leqslant -\varepsilon n \text{ for every } n\bigr\} > 0. \tag{6.3}$$

It follows from the condition (6.1) that, qiven $S(0) = i$, $i \geqslant i_*$, the chain S and the random variables $\zeta(n)$ can be constructed on the same probability space in such a way that

$$S(n) \leqslant i + \zeta(1) + \cdots + \zeta(n) \quad \text{a.s.}$$

on the event $S(1) \geqslant i_*, \ \ldots, \ S(n-1) \geqslant i_*$. The last property and (6.3) imply that for every $i \geqslant i_*$

$$\mathbf{P}\bigl\{S(1) \neq i, \ \ldots, \ S(n-1) \neq i, \ S(n) \leqslant i_* \text{ for some } n \mid S(0) = i\bigr\} \geqslant p > 0.$$

Since, in view of the ergodicity of the chain S,

$$\inf_{j \leqslant i_*} {}_ip_{j1} > 0,$$

it follows that

$$\inf_{i\geqslant i_*} {}_ip_{i1} \geqslant \mathbf{P}\bigl\{S(1)\neq i,\ \ldots,\ S(n-1)\neq i,\ S(n)\leqslant i_* \text{ for some } n \mid S(0)=i\bigr\} \\ \times \inf_{j\leqslant i_*} {}_ip_{j1} \\ = q > 0.$$

Hence, by the equality ${}_1p_{ii} + {}_ip_{i1} = 1$,

$$\inf_{i\geqslant i_*}(1 - {}_1p_{ii}) \geqslant q > 0.$$

Therefore,

$$\inf_i(1 - {}_1p_{ii}) > 0.$$

Substituting this into (6.2) yields the assertion of the lemma. □

7. Convergence in the space $\mathcal{X}_\alpha$

The present section is devoted to the proof of theorem 4. By theorem 2 the finite-dimensional distributions of $X(n)$ converge weakly. So, by Prokhorov's theorem, for proving the weak convergence of $Y(n)$ in the space with norm $|\cdot|_\alpha$ it remains to check the tightness condition.

The space $\mathcal{X}_\alpha$ is isomorphic to the Hilbert space l_1 with the isomorphism

$$(x_1, x_2, \ldots) \mapsto (\alpha_1 x_1, \alpha_2 x_2, \ldots).$$

Since the form of compact sets in l_1 is well known, we obtain that for every sequence $\varphi(I)\downarrow 0$, $I\to\infty$, the set

$$K_\varphi \equiv \left\{\boldsymbol{x}\colon \sum_{i\geqslant I}\alpha_i|\boldsymbol{x}_i| \leqslant \varphi(I),\ I = 1, 2, \ldots\right\}$$

is compact in the normed space $\mathcal{X}_\alpha$. Hence, if for every $\varepsilon > 0$ there is $\varphi(I)\downarrow 0$ such that $\mathbf{P}\{X(n)\notin K_\varphi\}\leqslant\varepsilon$ for each n, then tightness holds. The family $\{(X_1(n), \ldots, X_k(n)),\ n\in\mathbb{N}\}$ is tight for each fixed k because of weak convergence of the finite-dimensional distributions. Taking this into account we obtain that the family $\{X(n),\ n\in\mathbb{N}\}$ is tight in $\mathcal{X}_\alpha$ if

$$\limsup_{n\to\infty}\mathbf{P}\bigl\{X_i(n) > \beta_i/\alpha_i \text{ for some } i\geqslant I\bigr\}\to 0 \quad \text{as } I\to\infty. \tag{7.1}$$

Indeed, in this case we have with high probability

$$\sum_{i\geqslant I}\alpha_i X_i(n) \leqslant \sum_{i\geqslant I}\beta_i,$$

where $\sum_{i\geqslant I}\beta_i \to 0$ as $I\to\infty$. So, to prove theorem 4 it is sufficient to verify (7.1).

Lemma 8. There exists $c < \infty$ such that, for any initial distribution of $(S(0), X(0))$, if

$$\frac{(w+b)\lambda_i}{\pi_1 q_i} \leqslant 1 - \Delta, \quad \Delta > 0,$$

then

$$\limsup_{l\to\infty} \mathbf{P}\{X_i(T(l)) \geqslant x\} \leqslant \frac{c\sigma_i}{\Delta q_i x}.$$

Proof. Let the process $\overline{X}$ and the Markov chain $G_i(l) \equiv \overline{X}_i(T(l))$ be the same as in the proof of lemma 5. Consider a quadratic test function. Let us estimate the mean value of

$$m_i(y) \equiv \mathbf{E}\{G_i^2(1) - y^2 \mid G_i(0) = y\}.$$

Denote by η, as in lemma 5, the period of time (in "real time") between two consecutive server's arrivals at the first station in the system $(S; \overline{X})$. In view of condition (1.10) we have that

$$\mathbf{E}\eta^2 < \infty. \tag{7.2}$$

Denote by $\xi_i \equiv \xi_i(\eta)$ the number of new customers at station i between two consecutive server's arrivals at the first station in the system $(S; \overline{X})$. By Wald's identity

$$\mathbf{E}\xi_i = \lambda_i \mathbf{E}\eta = \frac{\lambda_i(w+b)}{\pi_1}. \tag{7.3}$$

By the formula of total probability

$$\mathbf{E}\xi_i^2 = \int_0^\infty \big((\lambda_i t)^2 + \sigma_i t\big) \mathrm{d}\mathbf{P}\{\eta < t\} = \lambda_i^2 \mathbf{E}\eta^2 + \sigma_i \mathbf{E}\eta.$$

Since $\mathbf{E}\eta^2$ is finite and $\sigma_i \geqslant \lambda_i$, there exists $c_1 < \infty$ such that, for each i,

$$\mathbf{E}\xi_i^2 \leqslant c_1 \sigma_i. \tag{7.4}$$

Denote by ζ_i the indicator that at least one customer was served at station i between two consecutive server's arrivals at the first station in the system $(S; \overline{X})$. We have that $\mathbf{E}\zeta_i = \mathbf{E}\zeta_i^2 = q_i$.

Given $y \geqslant 1$ and $G_i(0) = y$, $G_i(1)$ is less than or equal to $y + \xi_i - \zeta_i$, in distribution. Hence, for $y \geqslant 1$,

$$\begin{aligned} m_i(y) &\leqslant \mathbf{E}(y + \xi_i - \zeta_i)^2 - y^2 = 2y\mathbf{E}(\xi_i - \zeta_i) + \mathbf{E}(\xi_i - \zeta_i)^2 \\ &\leqslant 2y\mathbf{E}(\xi_i - \zeta_i) + \mathbf{E}\xi_i^2 + \mathbf{E}\zeta_i^2. \end{aligned}$$

Therefore, by (7.3), (7.4) and the condition of the lemma, for $y \geqslant 1$

$$m_i(y) \leqslant 2y\left(\frac{\lambda_i(w+b)}{\pi_1} - q_i\right) + c_1\sigma_i + q_i \leqslant -2\Delta q_i y + c_1\sigma_i + q_i.$$

For $y = 0$, we have that $m_i(0) = \mathbf{E}\xi_i^2 \leqslant c_1\sigma_i$. By lemma 3 the following inequality is valid:

$$0 \leqslant \sum_{y=0}^{\infty} \pi_i(y) m_i(y),$$

where $\{\pi_i(y)\}_{y=0}^{\infty}$ are the stationary probabilities of the chain $G_i(l)$. Substituting the estimates for $m_i(y)$ gives the inequality

$$2\Delta q_i \sum_{y=0}^{\infty} y\pi_i(y) \leqslant c_1\sigma_i + q_i \sum_{y=1}^{\infty} \pi_i(y) \leqslant c_1\sigma_i + c_2\lambda_i,$$

using the relevant inequality in the proof of lemma 5. The last inequality, the relation $\sigma_i \geqslant \lambda_i$, and Chebyshev's inequality imply that

$$\sum_{y=x}^{\infty} \pi_i(y) \leqslant \frac{c\sigma_i}{\Delta q_i x}.$$

Now the assertion follows from (4.2). □

Lemma 9. Let $S(0)$ have distribution $\{\pi_i\}$, $X(0) = \mathbf{0}$, and let $\Delta > 0$. Then there exists $c < \infty$ such that, for all i with $\lambda_i(w + b)/\pi_1 q_i \leqslant 1 - \Delta$, and for all n,

$$\mathbf{P}\big\{S(n) = 1,\ X_i(n) \geqslant x\big\} \leqslant \frac{c\sigma_i}{q_i x}.$$

Proof. Since $T(k) \geqslant k$, it follows that

$$\begin{aligned}\lim_{k\to\infty} \mathbf{P}\big\{S(k) = 1,\ X_i(k) \geqslant x\big\} &= \lim_{k\to\infty} \frac{1}{k} \sum_{m=0}^{k} \mathbf{P}\big\{S(m) = 1,\ X_i(m) \geqslant x\big\} \\ &\leqslant \limsup_{k\to\infty} \frac{1}{k} \sum_{l=0}^{k} \mathbf{P}\big\{X_i\big(T(l)\big) \geqslant x\big\}.\end{aligned}$$

In view of lemma 2, for each n,

$$\mathbf{P}\big\{S(n) = 1,\ X_i(n) \geqslant x\big\} \leqslant \lim_{k\to\infty} \mathbf{P}\big\{S(k) = 1,\ X_i(k) \geqslant x\big\}.$$

Now apply lemma 8. □

Turning now to the proof of (7.1) let $S(0)$ have distribution $\{\pi_i\}$, and let $X(0) = \mathbf{0}$. As derived in (4.5) we have, for all n,

$$\begin{aligned}&\mathbf{P}\big\{X_i(n) \geqslant \beta_i/\alpha_i \text{ for some } i \geqslant I\big\} \\ &\quad \leqslant \sum_{m=0}^{n} \mathbf{P}\big\{S(n-m) = 1,\ X_i(n-m) \geqslant \beta_i/\alpha_i \text{ for some } i \geqslant I\big\} \mathbf{P}\{\tau > m\}\end{aligned}$$

$$+\sum_{m=0}^{n}\sum_{l=0}^{\infty}\mathbf{P}\bigl\{T(l)=n-m,\ T(l+1)>n,\ B_I[n-m,n]\bigr\}+\mathbf{P}\bigl\{T(0)>n\bigr\}$$
$$=R_1(I,n)+R_2(I,n)+\mathbf{P}\bigl\{T(0)>n\bigr\}. \tag{7.5}$$

By the assumption that $\limsup_{i\to\infty}\lambda_i(w+b)/\pi_1 q_i<1$, and by lemma 9, we obtain that for some $c<\infty$, I large enough, and all n

$$R_1(I,n)\leqslant c\sum_{i\geqslant I}\frac{\alpha_i\sigma_i}{\beta_i q_i}\mathbf{E}\tau=\frac{c}{\pi_1}\sum_{i\geqslant I}\frac{\alpha_i\sigma_i}{\beta_i q_i}. \tag{7.6}$$

Substituting the last estimate into (7.5), we obtain

$$\mathbf{P}\bigl\{X_i(n)\geqslant\beta_i/\alpha_i\ \text{for some}\ i\geqslant I\bigr\}\leqslant c'\sum_{i\geqslant I}\frac{\alpha_i\sigma_i}{\beta_i q_i}+\sup_n R_2(I,n)+\mathbf{P}\bigl\{T(0)>n\bigr\}.$$

Now (7.1) follows from the last estimate, condition (1.11), and convergence (4.8). Note, that (7.1) is now proved for the special case when $S(0)$ has distribution $\{\pi_i\}$ and $X(0)=\mathbf{0}$. For arbitrary stations j, k, arbitrary m, and $\boldsymbol{x}\in\mathcal{X}^*$, put

$$p_{jk}(m,\boldsymbol{x})\equiv\mathbf{P}\bigl\{S(m)=k;\ X(m)=\boldsymbol{x}\mid S(0)=j,\ X(0)=\mathbf{0}\bigr\}.$$

Then it follows from what was said above that

$$\limsup_{n\to\infty}\sum_{j,k,\boldsymbol{x}}\pi_j p_{jk}(m,\boldsymbol{x})\mathbf{P}\bigl\{X_i(n)\geqslant\beta_i/\alpha_i\ \text{for some}\ i\geqslant I\mid S(m)=k;\ X(m)=\boldsymbol{x}\bigr\}\to 0$$

as $I\to\infty$. In particular, if $p_{jk}(m,\boldsymbol{x})>0$, then

$$\limsup_{n\to\infty}\mathbf{P}\bigl\{X_i(n)\geqslant\beta_i/\alpha_i\ \text{for some}\ i\geqslant I\mid S(m)=k;\ X(m)=\boldsymbol{x}\bigr\}\to 0,$$
$$I\to\infty. \tag{7.7}$$

Let now $S(0)$ have arbitrary distribution, and let $X(0)$ be distributed in $\mathcal{X}^*$. Then (7.1) holds if for each initial station j and initial queue lengths $\boldsymbol{x}\in\mathcal{X}^*$

$$\limsup_{n\to\infty}\mathbf{P}\bigl\{X_i(n)\geqslant\beta_i/\alpha_i\ \text{for some}\ i\geqslant I\mid S(0)=j,\ X(0)=\boldsymbol{x}\bigr\}\to 0,\quad I\to\infty.$$

This convergence follows from (7.7), because $\boldsymbol{x}\in\mathcal{X}^*$ and because, by virtue of the irreducibility of the system $(S;X)$, there exist k and m such that $p_{jk}(m,\boldsymbol{x})>0$. The proof of theorem 4 is complete. □

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