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ONE-DIMENSIONAL ASYMPTOTICALLY HOMOGENEOUS MARKOV CHAINS: CRAMÉR TRANSFORM AND LARGE DEVIATION PROBABILITIES

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Abstract

We consider a time-homogeneous ergodic Markov chain $\{X_n\}$ that takes values on the real line and has asymptotically homogeneous increments at infinity. We assume that the "limit jump" ξ of $\{X_n\}$ has negative mean and satisfies the Cramér condition, i.e., the equation $\mathbb{E}e^{\beta\xi} = 1$ has positive solution β. The asymptotic behavior of the probability $\mathbb{P}\{X_n > x\}$ is studied as $n \to \infty$ and $x \to \infty$. In particular, we distinguish the ranges of time n where this probability is asymptotically equivalent to the tail of a stationary distribution.

Key words and phrases: real-valued Markov chain, large deviation probabilities, transition phenomena, Cramér transform, invariant distribution.

1. Introduction

Let $P(x, B)$, $x \in \mathbb{R}$, $B \in \mathcal{B}(\mathbb{R})$, be some time-homogeneous transition probability in $\mathbb{R}$; here and in the sequel, $\mathcal{B}(\mathbb{R})$ denotes the σ-algebra of Borel sets in $\mathbb{R}$. In the present article the parameter n (time) ranges over the set $\{0, 1, 2, \dots\}$. Consider a Markov chain $\{X_n\}$ with values in $\mathbb{R}$ and the transition probabilities

$$P(x, B) = \mathbb{P}\{X_{n+1} \in B \mid X_n = x\}.$$

Let π_n be the distribution of X_n, i.e. $\pi_n(B) = \mathbb{P}\{X_n \in B\}$. Denote by $\xi(x)$ the random variable whose distribution corresponds to the distribution of the jump of $\{X_n\}$ from the state x, namely,

$$\mathbb{P}\big\{x + \xi(x) \in B\big\} = P(x, B), \quad B \in \mathcal{B}(\mathbb{R}).$$

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In this article we study *an asymptotically homogeneous* (*in space*) *Markov chain* X_n, i.e., a chain such that the distribution of the jump $\xi(x)$ has a weak limit as $x \to \infty$. Let the distribution of $\xi(x)$ converges weakly to the distribution F of a random variable ξ. We assume that $\mathbb{E}\xi < 0$ and $\mathbb{P}\{\xi > 0\} > 0$.

The Laplace transform $\varphi(\lambda) \equiv \mathbb{E}\, e^{\lambda\xi}$ of the random variable ξ is a convex function, $\varphi(0) = 1$, and $\varphi'(0) = \mathbb{E}\xi < 0$. Thus, the set $\{\lambda : \varphi(\lambda) \le 1\}$ is an interval of the form $[0, \beta]$, where $\beta = \sup\{\lambda : \varphi(\lambda) \le 1\}$. Since $\mathbb{P}\{\xi > 0\} > 0$, it follows that β is a finite number. In the article we consider the *Cramér case* corresponding to the situation when $\beta > 0$ and $\varphi(\beta) = 1$.

We assume that X_n is a Harris ergodic chain with the unique invariant distribution π. Then the distribution of X_n converges in the total variation distance to π as $n \to \infty$, i.e.,

$$|\pi_n - \pi|(\mathbb{R}) \equiv 2 \sup_{B \in \mathcal{B}(\mathbb{R})} \big|\pi_n(B) - \pi(B)\big| \to 0 \tag{1}$$

(here and in the sequel, for every signed measure μ and every set B, we denote the total variation of μ on B by $|\mu|(B)$). For a countable chain X_n, (1) takes place automatically provided the chain is irreducible, nonperiodic, and positive recurrent; for real-valued chains the corresponding ergodicity conditions can be found, for example, in [2, 10].

In view of (1), the family of distributions X_n is tight, i.e., $\sup_{n\ge 0} \mathbb{P}\{X_n > x\} \to 0$ as $x \to \infty$. In the present article we study the asymptotic behavior of the probability $\mathbb{P}\{X_n > x\}$ as $n \to \infty$ and $x \to \infty$.

The simplest and, at the same time, very impotent example of such Markov chain is provided by the random walk W_n with delay at the origin (which is called a space-homogeneous Markov chain in [4, 5]) defined by the recursion

$$W_{n+1} = (W_n + \xi_n)^+,$$

where $\xi_0, \xi_1, \dots$ are independent copies of the random variable ξ. Put $S_0 = 0$, $S_k = \xi_1 + \dots + \xi_k$, and $M_n = \max_{0 \le k \le n} S_k$. It is well known (see, for example, [7, Chapter VI, Section 9]) that the distribution of the chain W_n with zero initial state $W_0 = 0$ coincides with the distribution of M_n, i.e.,

$$\mathbb{P}\{W_n > x\} = \mathbb{P}\{M_n > x\}. \tag{2}$$

In particular, if $W_0 = 0$ then the sequence W_n is stochastically growing and, hence, has a weak limit. Denote by W_∞ the random variable with this limit distribution. The following Cramér estimate is well known:

$$\mathbb{P}\{W_\infty > x\} \le e^{-\beta x}, \quad x \ge 0.$$

In addition, if $\alpha \equiv \varphi'(\beta) = \mathbb{E}\xi e^{\beta\xi}$ is finite then

$$\mathbb{P}\{W_\infty > x\} \sim \frac{1-p}{\beta \widetilde{a}} e^{-\beta x} \quad \text{as} \quad x \to \infty \tag{3}$$

(see [7, Chapter XII, Section 5; 6]), where $p = \mathbb{P}\{M_\infty > 0\}$ and $\widetilde{a} = \mathbb{E}\{S_\tau e^{\beta S_\tau}; \tau < \infty\}$ with $\tau = \min\{n \ge 1 : S_n > 0\}$. Since $\mathbb{E}\xi < 0$, we have $p < 1$, and both τ and S_τ are defective random variables.

In the case when the distribution of ξ has a nonzero absolutely continuous component, the prestationary distributions of W_n were studied in detail by A. A. Borovkov in [1]. Theorems 7–11 of [1] provide the asymptotic expansion for the probability $\mathbb{P}\{\max_{0\le k\le n} S_k \ge x,\, S_n < x - y\}$ within the broad ranges of the parameters n, x, and y.

In [5] A. A. Borovkov and D. A. Korshunov generalized the results on the asymptotic behavior of the probability $\mathbb{P}\{W_n > x\}$ as $n \to \infty$ and $x \to \infty$ for the Markov chains with values on the real line. Namely, the case of the so-called *U-partially homogeneous* (*in space*) *Markov chain* was considered. We call a chain *U-partially homogeneous*, if, for every Borel set $B \subseteq (U, \infty)$, the transition probability $P(y, B)$ coincides with the probability $\mathbb{P}\{y + \xi \in B\}$ when y ranges over (U, ∞). In other words, in the domain (U, ∞) the stochastic behavior of X coincides with the process of summation of independent random variables with common distribution F.

Earlier the author found the conditions on the asymptotically space-homogeneous Markov chain under which the tail of the stationary distribution π is decreasing exponentially like in the estimate (3) for the distribution tail of the supremum of partial sums (see [2, Section 27, Theorems 3–5]). We reproduce the corresponding statements in view of their fundamental role in the subsequent exposition. We start with the large deviation principle (rough asymptotic behavior) which is valid under rather broad conditions. Following traditions, henceforth the values of a measure μ at the sets (x, y) and $(x, y]$ are denoted by $\mu(x, y)$ and $\mu(x, y]$.

Theorem 1. *Let the jumps of an asymptotically space-homogeneous Markov chain X_n satisfy the condition* $\sup_y \mathbb{E}e^{\beta\xi(y)} < \infty$. *If $\pi(y, \infty) > 0$ for every y then $\log \pi(x, \infty) \sim -\beta x$ as $x \to \infty$.*

In [5, Theorem 1] this result was generalized on the prestationary distributions; the large deviation principle was proved for the asymptotically homogeneous Markov chain.

In order to find the sharp asymptotic behavior of the probability $\pi(x, \infty)$, it is not sufficient to know only that the chain is asymptotically homogeneous. We need some additional information about the convergence rate of the jump distribution to the limit distribution F.

Theorem 2. *Let the distribution F be nonlattice and let the jumps of X_n satisfy the condition*

$$\int_{-\infty}^{\infty} e^{\beta t} \big|\mathbb{P}\{\xi(y) < t\} - \mathbb{P}\{\xi < t\}\big|\, dt \le \delta(y), \tag{4}$$

where $\delta(y)$ is a function regularly varying at infinity with index $-\alpha$, i.e., $\delta(uy) \sim u^{-\alpha}\delta(y)$ as $y \to \infty$ for every fixed $u > 0$. If

$$\int_0^\infty \delta(y)\,dy < \infty \tag{5}$$

(so that $\alpha \geq 1$) then the tail of the invariant distribution π satisfies the asymptotic equivalence

$$\pi(x,\infty) = ce^{-\beta x} + o\big(e^{-\beta x}\big) \quad \text{as} \quad x \to \infty, \tag{6}$$

where

$$c = \frac{1}{\beta \mathbb{E}\,\xi e^{\beta\xi}} \int_{-\infty}^{\infty} \Big(\mathbb{E}\, e^{\beta\xi(y)} - 1\Big) e^{\beta y}\,\pi(dy) \in [0,\infty). \tag{7}$$

If F is a lattice distribution with span $\Delta > 0$ and the chain X_n takes values on the lattice $\{n\Delta,\, n \in \mathbb{Z}\}$ then

$$\pi(n\Delta) = c_\Delta e^{-\beta n\Delta} + o\big(e^{-\beta n\Delta}\big) \quad \text{as} \quad n \to \infty, \tag{8}$$

where

$$c_\Delta = \frac{\Delta}{\mathbb{E}\,\xi e^{\beta\xi}} \sum_{k\in\mathbb{Z}} \Big(\mathbb{E}\, e^{\beta\xi(k\Delta)} - 1\Big) e^{\beta k\Delta} \pi(k\Delta) \in [0,\infty). \tag{9}$$

In [2] we gave sufficient conditions for the positivity of the constant c. Namely, if $\pi(y,\infty) > 0$ and $\mathbb{E}\, e^{\beta\xi(y)} \geq 1 - \gamma(y)$ for every y, where $\gamma(y) \geq 0$ and $\int_1^\infty \gamma(y) y (\log y)\,dy < \infty$, then $c > 0$.

The condition (5) says that, roughly speaking, the convergence rate of the distribution of $\xi(x)$ to that of ξ should be integrable. In Section 10 we give an example showing that, in some sense, the conditions (4) and (5) are necessary for (6).

In present article we obtain the exact asymptotics for the probability $\mathbb{P}\{X_n > x\}$ as $n \to \infty$ and $x \to \infty$ provided that the Markov chain X_n is asymptotically homogeneous in the space and satisfies the conditions like (4) and (5). We distinguish the ranges of time n where the probability $\mathbb{P}\{X_n > x\}$ is asymptotically equivalent to the tail $\pi(x,\infty)$ of the invariant distribution.

A few words about the technique of proving. In [5] the study of U-partially homogeneous chain is based on the total probability formula with respect to the last entry into the set $(-\infty, U]$; namely, on the formula

$$\mathbb{P}\{X_n > x\} = \sum_{k=0}^{n-1} \mathbb{P}\big\{X_k \leq U,\, X_j > U \text{ for every } j \in [k+1,n],\, X_n > x\big\}.$$

According to the Markov property the summand in the last sum equals

$$\begin{aligned}&\mathbb{P}\Big\{X_k \le U,\, X_j > U \text{ for every } j \in [k+1, n],\, X_n > x\Big\}\\ &\quad= \int_{-\infty}^{U} \mathbb{P}\{X_k \in du\} \int_{U}^{\infty} P(u, dv)\\ &\qquad\times \mathbb{P}\Big\{X_j > U \text{ for every } j \in [k+2, n],\, X_n > x \mid X_{k+1} = v\Big\}.\end{aligned}$$

Since the chain is U-partially homogeneous, above the level U the chain stochastically behaves like a partial sum process with the common step distribution F. This property allows us to calculate the last probability via the well-known theorem on the taboo probabilities of large deviations for sums of independent identically distributed random variables. For the asymptotically homogeneous chain this approach cannot be used since, in general, for *every high* level U, the stochastic behavior of the chain above this level cannot be described in terms of the partial sum process based on independent variables. Therefore we propose a new technique of proving.

The sketch of the proof follows: First, we apply the Cramér transform with corresponding parameter to the Markov chain under consideration. As a result, we obtain some object called the Markov evolution of masses. The main difference between the Markov evolution of masses and the usual Markov chain is that the jump of the Markov evolution of masses can have the total mass ("probability") other than 1; in particular, it can be greater than 1. Then some limit theory is developed for the Markov evolution of masses and for the Markov chains. In particular, we prove the analogs of the central limit theorem. After that, we apply the inverse Cramér transform to the Markov evolution of masses what allows us to compute the asymptotic behavior of the probability of the event $\{X_n > x\}$.

The article is organized as follows: We obtain the main results in Theorems 6 and 7 (in Sections 6 and 8 respectively) describing the asymptotic behavior of the large deviation probabilities of asymptotically homogeneous Markov chain. In Sections 2–5, we develop the preliminary theory. In particular, in Section 2, we discuss the notion and some properties of the Markov evolution of masses. In Section 3, we prove the local limit theorem and the local renewal theorem for the asymptotically homogeneous Markov chain. Sections 4 and 5 are devoted to more delicate asymptotic properties of the distribution of the Markov evolution of masses. The article is concluded with Section 10 in which a simple example of asymptotically homogeneous Markov chain demonstrates that the principal conditions of Theorem 6 on the integrability of the convergence rate of the distribution of $\xi(x)$ to that of ξ cannot be weakened.

2. The Markov evolution of masses

A Markov chain X_n with the distribution π_n may be considered as the *Markov evolution of unit mass* in the space $\mathbb{R}$. Specifically, at time $n = 0$ the unit mass is distributed on the space $\mathbb{R}$ according to the law π_0. At the next moment of time $n = 1$ the mass is redistributed according to the transition function $P(\cdot\,,\cdot)$, i.e., from every point $u \in \mathbb{R}$ the element of mass is redistributed on $\mathbb{R}$ according to the law $P(u,\cdot)$. Hence, at time $n = 1$ the total unit mass is distributed according to π_1, i.e., the mass of every measurable set $B \subseteq \mathbb{R}$ is equal to $\pi_1(B)$. And so on, at any time n.

Introduce the notion of *generalized transition kernel* $Q(u, B)$, $u \in \mathbb{R}$, $B \in \mathcal{B}(\mathbb{R})$, possessing all properties of ordinary Markov transition kernel except for the fact that the nonnegative function $Q(u,\mathbb{R})$ of the argument u is equal to one. Thus, the values of $Q(u,\mathbb{R})$ can be less or greater than one. Clearly, the function

$$Q^*(u, B) = \frac{Q(u, B)}{Q(u,\mathbb{R})}$$

represents a traditional Markov transition kernel.

Let Q_0 be some nonnegative measure on $\mathbb{R}$. Then the generalized transition kernel $Q(u, B)$ generates the family of nonnegative measures $\{Q_n\}$ defined by the recurrent equality

$$Q_{n+1}(B) = (Q_nQ)(B) \equiv \int_{\mathbb{R}} Q(u, B)Q_n(du), \quad n \geq 0.$$

Define the *Markov evolution of masses* (or simply the *Markov mass*) Y_n corresponding to the generalized transition kernel $Q(\cdot\,,\cdot)$ as follows: at time $n = 0$ the mass $Q_0(\mathbb{R})$ is distributed on $\mathbb{R}$ according to the law Q_0. During the time step $n \to n+1$ the element of mass Y_n at state $u \in \mathbb{R}$ changes $Q(u,\mathbb{R})$ times and the new element of mass Y_{n+1} is distributed on the space according to the measure $Q(u, B)/Q(u,\mathbb{R})$. Therefore, at each moment of time n the mass is distributed according to the law $Q_n(\cdot)$, i.e., the mass of every measurable set $B \in \mathcal{B}(\mathbb{R})$ equals $Q_n(B)$. While speaking about an ordinary Markov chain we use the term "the value of X_n at time n;" for the Markov evolution of masses we use the term "element of mass Y_n at time n" and denote the mass of B at time n by $\mathrm{Mes}\{Y_n \in B\}$.

Observe that, generally speaking, the finite-dimensional distributions of masses $(Y_0, \dots, Y_n)$ are not consistent. For example, if $Q(u,\mathbb{R}) \equiv 2$ then the mass of $B_0 \times \cdots \times B_n \times \mathbb{R}$ is twice greater than the mass of $B_0 \times \cdots \times B_n$. Thus, in general, the analog of the total probability formula does not hold. Nevertheless, in order to calculate the mass of the measurable set B at time n, we can "trace" all trajectories of the element of mass leading to B and "sum" the masses that are carrying along these trajectories according to the generalized transition kernel. For example, given two disjoint nonempty Borel sets B

and B_1, the measure of B at time n can be calculated by the following formula with respect to the last entry of the element of mass into B_1:

$$\begin{aligned}\mathrm{Mes}\{Y_n\in B\} = \mathrm{Mes}\big\{&Y_0\notin B_1,\dots,Y_{n-1}\notin B_1,\, Y_n\in B\big\}\\ &+\sum_{k=0}^{n-2}\mathrm{Mes}\big\{Y_k\in B_1,\, Y_{k+1}\notin B_1,\dots,Y_{n-1}\notin B_1,\, Y_n\in B\big\}\\ &+\mathrm{Mes}\{Y_{n-1}\in B_1,\, Y_n\in B\}.\end{aligned} \tag{10}$$

Note in addition that here

$$\mathrm{Mes}\Big\{Y_k\in B_1,\, Y_{k+1}\notin B_1,\,\dots,\, Y_{n-1}\notin B_1,\, Y_n\in B\Big\}$$
$$=\int\limits_{\mathbb{R}\setminus B_1} Q(u_{n-1},B)\int\limits_{\mathbb{R}\setminus B_1} Q(u_{n-2},du_{n-1})\ \cdots\int\limits_{\mathbb{R}\setminus B_1} Q(u_k,du_{k+1})\int\limits_{B_1} Q_k(du_k).$$

Denote by $\eta(u)$ the jump of the Markov evolution of masses Y_n from state u. By definition, $\eta(u)$ is a function on some measurable space with the total mass $Q(u,\mathbb{R})$ and the generalized distribution $Q(u,u+\cdot)$. Thus, $\mathrm{Mes}\big\{u+\eta(u)\in B\big\}=Q(u,B)$.

2.1. *Numerical characteristics.* By the *mean value* of a function Y on some measurable space with finite total mass we mean the integral $\mathcal{E}Y = \int_{\mathbb{R}} yQ(dy)$, where Q is the generalized distribution of Y. So,

$$\mathcal{E}Y_n=\int_{\mathbb{R}} yQ_n(dy),\quad \mathcal{E}\eta(u)=\int_{\mathbb{R}} yQ(u,u+dy).$$

Note that the mean value is a linear functional if we consider the functions on a fixed space with a fixed measure. But the equality

$$\mathcal{E}Y_{n+1}=\mathcal{E}Y_n+\int_{\mathbb{R}}\mathcal{E}\,\eta(u)Q_n(dy),$$

in general, is not valid. For example, if the distribution of the jump $\eta(u)$ does not depend on u and equals μ then $\mathcal{E}Y_1=\mathcal{E}Y_0\cdot\mu(\mathbb{R})+Q_0(\mathbb{R})\cdot\mathcal{E}\,\eta$, but not $\mathcal{E}Y_1=\mathcal{E}Y_0+\mathcal{E}\,\eta$.

Nevertheless, the time behavior of the exponential moments of the Markov evolution of masses, namely the Laplace transform and the characteristic function, is completely the same as one of the exponential moments of ordinary Markov chain. Since

$$\begin{aligned}\mathcal{E}e^{\lambda Y_{n+1}} &= \int_{\mathbb{R}} e^{\lambda y}Q_{n+1}(dy)\\ &=\int_{\mathbb{R}} e^{\lambda y}\int_{\mathbb{R}} Q(u,dy)Q_n(du)\\ &=\int_{\mathbb{R}} e^{\lambda(u+z)}\int_{\mathbb{R}} Q(u,u+dz)Q_n(du),\end{aligned}$$

the following equality holds:

$$\mathcal{E} e^{\lambda Y_{n+1}} = \int_{\mathbb{R}} e^{\lambda u} \mathcal{E} e^{\lambda \eta(u)} Q_n(du). \tag{11}$$

2.2. *The analog of Chebyshev's inequality.* For every positive increasing function $f(u)$, the inequality

$$\mathrm{Mes}\{Y \geq y\} \leq \frac{\mathcal{E}\, f(Y)}{f(y)}$$

is valid. In particular, for every $\lambda > 0$, we have

$$\mathrm{Mes}\{Y \leq y\} = \mathrm{Mes}\{-Y \geq -y\} \leq e^{\lambda y} \mathcal{E} e^{-\lambda Y}. \tag{12}$$

2.3. *The Cramér transform over a Markov chain: the inversion formula.* Let X_n be a real-valued Markov chain with transition probabilities $P(u, B)$ and distribution π_n. Given $\lambda > 0$, define a generalized transition kernel $P^{(\lambda)}(\cdot\,,\cdot)$ by the equality

$$P^{(\lambda)}(u, dv) = e^{\lambda(v-u)} P(u, dv);$$

the measure $P^{(\lambda)}(u, u + \cdot)$ represents the Cramér transform over the distribution $P(u, u + \cdot)$ with parameter λ. In addition, for every n, define the measure $\pi_n^{(\lambda)}$ as

$$\pi_n^{(\lambda)}(du) = e^{\lambda u} \mathbb{P}\{X_n \in du\}.$$

The following recurrent equality is true:

$$\begin{aligned}
\pi_{n+1}^{(\lambda)}(B) &= \int_B e^{\lambda v} \mathbb{P}\{X_{n+1} \in dv\} \\
&= \int_B e^{\lambda v} \int_{\mathbb{R}} P(u, dv) \mathbb{P}\{X_n \in du\} \\
&= \int_B \int_{\mathbb{R}} e^{\lambda(v-u)} P(u, dv) e^{\lambda u} \mathbb{P}\{X_n \in du\} \\
&= \int_B \int_{\mathbb{R}} P^{(\lambda)}(u, dv) \pi_n^{(\lambda)}(du) \\
&= \int_{\mathbb{R}} P^{(\lambda)}(u, B) \pi_n^{(\lambda)}(du).
\end{aligned}$$

Thus, the Markov evolution of masses $X_n^{(\lambda)}$ with generalized transition kernel $P^{(\lambda)}(\cdot\,,\cdot)$ is distributed according to $\pi_n^{(\lambda)}$, i.e.,

$$\pi_n^{(\lambda)}(B) = \mathrm{Mes}\big\{X_n^{(\lambda)} \in B\big\}$$

for all $n \geq 0$ and $B \in \mathcal{B}(\mathbb{R})$.

By the construction of $\pi_n^{(\lambda)}$, the following inversion formula is valid:

$$\pi_n(B) = \int_B e^{-\lambda u} \pi_n^{(\lambda)}(du). \tag{13}$$

In general, the following holds:

Lemma 1. *For every $n \in \mathbb{Z}^+$ and $u_0, \ldots, u_n \in \mathbb{R}$,*

$$\mathbb{P}\{X_0 \in du_0, \ldots, X_n \in du_n\} = e^{-\lambda u_n} \mathrm{Mes}\big\{X_0^{(\lambda)} \in du_0, \ldots, X_n^{(\lambda)} \in du_n\big\}.$$

Proof follows from the equalities

$$\begin{aligned}
&\mathbb{P}\{X_0 \in du_0, \ldots, X_n \in du_n\} \\
&\quad = \mathbb{P}\{X_0 \in du_0\} P(u_0, du_1) \cdots P(u_{n-1}, du_n) \\
&\quad = e^{-\lambda u_0} \mathrm{Mes}\big\{X_0^{(\lambda)} \in du_0\big\} e^{-\lambda(u_1 - u_0)} P^{(\lambda)}(u_0, du_1) \\
&\qquad \cdots e^{-\lambda(u_n - u_{n-1})} P^{(\lambda)}(u_{n-1}, du_n) \\
&\quad = e^{-\lambda u_n} \mathrm{Mes}\big\{X_0^{(\lambda)} \in du_0, \ldots, X_n^{(\lambda)} \in du_n\big\}.
\end{aligned}$$

3. The local renewal theorem for transient Markov chains

We start with some modifications of (the local limit) Theorems 7 (the lattice case) and 8 (the nonlattice case) of the article [9] which are essential for our subsequent study.

Let X_n^* be a real-valued Markov chain. Denote the jump of this chain at the state x by $\xi^*(x)$.

Theorem 3. *Let the jumps of X^* possess a minorant $\underline{\zeta}$ with $\mathbb{E}\underline{\zeta} > 0$ and $\mathrm{Var}\,\underline{\zeta} < \infty$, i.e., for every $x \in \mathbb{R}$, the following stochastic inequality holds*:

$$\xi^*(x) \geq_{\mathrm{st}} \underline{\zeta}. \tag{14}$$

Let $\xi^*(x) \Rightarrow \xi^*$ as $x \to \infty$, let the relations

$$\mathbb{E}\xi^*(x) = \alpha + o\big(1/\sqrt{x}\,\big),$$
$$\mathrm{Var}\ \xi^*(x) \to \sigma^2 > 0$$

hold, and let the family $\big\{\big(\xi^*(x)\big)^2,\ x \in \mathbb{R}\big\}$ be uniformly integrable. In addition, assume that the initial distribution of the chain satisfies the condition $\mathbb{P}\big\{X_0^* \leq -x\big\} = o\big(1/\sqrt{x}\,\big)$ as $x \to \infty$.

If ξ^* is a nonlattice random variable and, for every $A > 0$,

$$\sup_{|\lambda| \leq A} \big|\mathbb{E}\, e^{i\lambda\xi^*(x)} - \mathbb{E}\, e^{i\lambda\xi^*}\big| = o(1/x) \quad \text{as} \quad x \to \infty \tag{15}$$

then, for each fixed $\Delta > 0$, *the following relation holds as* $n \to \infty$:

$$\sup_{x\in\mathbb{R}}\left|\mathbb{P}\big\{X_n^* \in (x, x+\Delta]\big\} - \frac{\Delta}{\sqrt{2\pi n\sigma^2}}e^{-(x-n\alpha)^2/2n\sigma^2}\right| = o\big(1/\sqrt{n}\,\big).$$

If the chain X_n^* *takes its values on the lattice* $\{\Delta k,\ k \in \mathbb{Z}\}$, $\Delta > 0$, *this lattice is minimal, and*

$$\sup_{|\lambda|\le\pi/\Delta}\left|\mathbb{E}\,e^{i\lambda\xi^*(k\Delta)} - \mathbb{E}\,e^{i\lambda\xi^*}\right| = o(1/k) \quad as \quad k \to \infty,$$

then, as $n \to \infty$, *we have*

$$\sup_{k\in\mathbb{Z}^+}\left|\mathbb{P}\{X_n^* = k\Delta\} - \frac{\Delta}{\sqrt{2\pi n\sigma^2}}e^{-(k\Delta-n\alpha)^2/2n\sigma^2}\right| = o\big(1/\sqrt{n}\,\big).$$

Proof. As was observed in [9, Section 4.2], the condition (14) of the existence of minorant $\underline{\zeta}$ with positive mean and finite variance together with the condition on the left tail of the initial distribution provide the following estimate:

$$\mathbb{P}\big\{X_k^* \le k\mathbb{E}\underline{\zeta}/2 \ \text{ for some } k \ge n\big\} = o\big(1/\sqrt{n}\big) \quad \text{as} \quad n \to \infty.$$

Thus, all conditions of Theorems 7 and 8 of [9] are satisfied, which completes the proof.

Define the renewal measure generated by the Markov chain X_n^*:

$$H(B) \equiv \sum_{n=0}^{\infty}\mathbb{P}\{X_n^* \in B\}$$

and the renewal process

$$\widetilde{H}(B) \equiv \sum_{n=0}^{\infty}\mathbf{I}\{X_n^* \in B\}.$$

The equality $H(B) = \mathbb{E}\widetilde{H}(B)$ holds.

Lemma 2. *Let the minorization condition* (14) *hold with* $\mathbb{E}\underline{\zeta} > 0$ *and* $\mathbb{E}\underline{\zeta}^2 < \infty$. *Then there exists a random variable* θ *with finite mean such that* $\widetilde{H}(x, x+1] \le_{\rm st} \theta$ *for every* $x \in \mathbb{R}$.

Proof. Consider the sums $\underline{Z}_n = \underline{\zeta}_1 + \cdots + \underline{\zeta}_n$, $\underline{Z}_0 = 0$, where $\underline{\zeta}_1, \underline{\zeta}_2, \ldots$ are independent copies of $\underline{\zeta}$. Since the mean $\underline{\zeta}$ is positive and the second moment is finite, we have (see, for example, [11, Corollary 2.5])

$$\theta \equiv \sum_{n=0}^{\infty}\mathbf{I}\big\{\underline{Z}_n \le 1\big\} < \infty, \quad \mathbb{E}\theta < \infty. \tag{16}$$

Let $\tau(x) = \min\{n \geq 0 : X^*_n > x\}$. Since $\underline{\zeta}$ minorizes the jumps, the Markov chain X^*_n and the sequence $\underline{Z}_n$ can be defined on a common probability space so that $X^*_{\tau(x)+n} \geq x + \underline{Z}_n$ with probability 1 for every $n \geq 0$. Therefore,

$$\begin{aligned}\widetilde{H}(x, x+1] &= \sum_{n=0}^{\infty} \mathbf{I}\big\{X^*_{\tau(x)+n} \in (x, x+1]\big\} \\ &\leq \sum_{n=0}^{\infty} \mathbf{I}\big\{X^*_{\tau(x)+n} \leq x+1\big\} \\ &\leq_{\text{st}} \sum_{n=0}^{\infty} \mathbf{I}\big\{x + \underline{Z}_n \leq x+1\big\},\end{aligned}$$

which, together with (16), implies the lemma conclusion.

From now on, we assume, in addition, that the chain X^*_n is asymptotically homogeneous. Let F^* be the limit distribution of the variable $\xi^*(x)$ and let ξ^* be a random variable with the distribution F^*.

Theorem 4. *Let F^* be a nonlattice distribution, let $\alpha = \mathbb{E}\xi^*$, and let the jumps of the chain X^* possess a minorant $\underline{\zeta}$ and a majorant $\bar{\zeta}$ with $\mathbb{E}\underline{\zeta} > 0$,* $\operatorname{Var}\underline{\zeta} < \infty$, *and* $\mathbb{E}\bar{\zeta} < \infty$, *i.e., for every x, the following stochastic inequalities are satisfied*:

$$\underline{\zeta} \leq_{\text{st}} \xi^*(x) \leq_{\text{st}} \bar{\zeta},$$

Then, for each fixed $\Delta > 0$, we have

$$\lim_{x\to\infty} H(x, x+\Delta] = \Delta/\alpha.$$

If, in addition, $\sigma^2 = \operatorname{Var}\xi^* < \infty$ and the local limit theorem holds, i.e., if the relation

$$\mathbb{P}\big\{X^*_n \in (x, x+\Delta]\big\} = \frac{\Delta}{\sqrt{2\pi n\sigma^2}} e^{-(x-n\alpha)^2/2n\sigma^2} + o\big(1/\sqrt{n}\,\big)$$

holds as $n \to \infty$ uniformly in x then, as $n \to \infty$ and $x \to \infty$,

$$\sum_{k=0}^{n} \mathbb{P}\big\{X^*_k \in (x, x+\Delta]\big\} = \frac{\Delta}{\alpha}\Phi_{\sigma^2}\left(\frac{n\alpha - x}{\sqrt{x/\alpha}}\right) + o(1).$$

Proof. Denote by $\tau(x)$ and $\chi(x)$ the time and the value of the first overshoot of the level x by the chain X^*_n:

$$\tau(x) = \min\big\{n \geq 0 : X^*_n > x\big\}, \quad \chi(x) = X^*_{\tau(x)} - x.$$

Let $\xi_1^*, \xi_2^*, \ldots$ be independent copies of the random variable ξ^*. Put $S_n^* = \xi_1^* + \cdots + \xi_n^*$. Denote by $\tau^*(x)$ and $\chi^*(x)$ the time and the value of the first overshoot of the level x by the sums S_n^*:

$$\tau^*(x) = \min\bigl\{n \ge 0 : S_n^* > x\bigr\}, \quad \chi^*(x) = S_{\tau(x)}^* - x;$$

and by

$$\widetilde{H}^*(B) = \sum_{n=0}^{\infty} \mathbf{I}\{S_n^* \in B\},$$

the renewal process. It is known (see, e.g., [11, Theorem 2.3]) that the distribution of $\chi^*(x)$ converges weakly as $x \to \infty$ to the distribution of the overshoot $\chi^*(\infty)$ of the so-called infinite level, and the distribution of $\chi^*(\infty)$ is absolutely continuous. In view of absolute continuity of the overshoot weak limit, the distribution of $\widetilde{H}^*(x, x+\Delta]$ converges weakly as $x \to \infty$ to some distribution, say G. By virtue of the local renewal theorem for sums of independent identically distributed random variables (see, for example, [7, Chapter XI; 11, Appendix]), the mean of the distribution G is equal to Δ/α.

The conditions of Theorem 4 allow us to apply Theorem 2.2 of [3] according to which the distribution of the overshoot $\chi(x)$ converges weakly to the distribution of $\chi^*(\infty)$. Therefore, the distribution of $\widetilde{H}(x, x+\Delta]$ converges weakly to the distribution G. Taking it into account that, by Lemma 3, the family $\bigl\{\widetilde{H}(x, x+\Delta], x \in \mathbb{R}\bigr\}$ admits an integrable majorant, we obtain the convergence of the mean value of $\widetilde{H}(x, x+\Delta]$ to that of G as $x \to \infty$, i.e.,

$$H(x, x+\Delta] \to \Delta/\alpha,$$

and the first assertion of the theorem is proven.

If the local limit theorem is valid then, for any fixed s and t, $s < t$, the following convergence holds:

$$\sum_{k=x/\alpha+s\sqrt{x}}^{x/\alpha+t\sqrt{x}} \mathbb{P}\bigl\{X_k^* \in (x, x+\Delta]\bigr\} - \sum_{k=x/\alpha+s\sqrt{x}}^{x/\alpha+t\sqrt{x}} \frac{\Delta}{\sqrt{2\pi k\sigma^2}} e^{-(x-k\alpha)^2/2k\sigma^2} \to 0$$

as $x \to \infty$. Thus,

$$\sum_{k=x/\alpha+s\sqrt{x}}^{x/\alpha+t\sqrt{x}} \mathbb{P}\bigl\{X_k^* \in (x, x+\Delta]\bigr\} \to \frac{\Delta}{\alpha}\Bigl(\Phi_{\sigma^2}\bigl(t\alpha^{3/2}\bigr) - \Phi_{\sigma^2}\bigl(s\alpha^{3/2}\bigr)\Bigr).$$

The last convergence, together with the first assertion of the theorem, implies the second assertion. The proof is complete.

In the lattice case assertions of Theorem 4 can be formulated as follows: *if F is a lattice distribution with span $\Delta > 0$ then*

$$\lim_{k\to\infty} H(k\Delta) = \Delta/\alpha,$$

$$\sum_{j=0}^{n} \mathbb{P}\bigl\{X_j^* = k\Delta\bigr\} = \frac{\Delta}{\alpha}\Phi_{\sigma^2}\left(\frac{n\alpha - k\Delta}{\sqrt{k\Delta/\alpha}}\right) + o(1)$$

uniformly in k as $n \to \infty$.

4. Some preliminary estimates for the distribution of a Markov evolution of masses

We consider a Markov evolution of masses $\{Y_n\}$ with jumps $\{\eta(x)\}$. Denote $Q_n(B) = \mathrm{Mes}\{Y_n \in B\}$ and $Q(x, B) = \mathrm{Mes}\bigl\{x + \eta(x) \in B\bigr\}$. Put

$$\widehat{Q}(x) \equiv \sup_{y>x} Q(y, \mathbb{R}), \qquad \widehat{Q} \equiv \sup_{y\in\mathbb{R}} Q(y, \mathbb{R}).$$

First of all we find conditions on the Markov evolution of masses which provide the boundedness of the sequence of the whole space masses.

Lemma 3. *Let $Q_0(\mathbb{R}) < \infty$ and $\widehat{Q} < \infty$. If, for some sequence of levels x_n, $n \geq 0$,*

$$\sum_{n=0}^{\infty} Q_n(-\infty, x_n] < \infty, \tag{17}$$

$$\sum_{n=0}^{\infty} \widehat{q}(x_n) < \infty, \tag{18}$$

where

$$\widehat{q}(x) \equiv \sup_{y>x} \bigl|Q(y, \mathbb{R}) - 1\bigr|,$$

then the sequence of total masses is bounded:

$$\sup_{n\geq 0} Q_n(\mathbb{R}) < \infty;$$

moreover, there exists a finite limit

$$\lim_{n\to\infty} Q_n(\mathbb{R}) = Q \in [0, \infty).$$

Proof. By induction on n, we check the inequality

$$Q_{n+1}(\mathbb{R}) \leq Q_0(\mathbb{R}) \prod_{k=0}^{n} \widehat{Q}(x_k) + \widehat{Q} \sum_{k=0}^{n} Q_k(-\infty, x_k] \prod_{j=k+1}^{n} \widehat{Q}(x_j). \tag{19}$$

We have

$$Q_{n+1}(\mathbb{R}) = \left(\int_{-\infty}^{x_n} + \int_{x_n}^{\infty}\right) Q(x,\mathbb{R}) Q_n(dx) \leq \widehat{Q} Q_n(-\infty, x_n] + \widehat{Q}(x_n) Q_n(\mathbb{R}).$$

For $n = 0$ this estimate implies the inequality

$$Q_1(\mathbb{R}) \leq \widehat{Q} Q_0(-\infty, x_0] + \widehat{Q}(x_0) Q_0(\mathbb{R}),$$

which justifies the basis of the induction. By the induction hypothesis,

$$\begin{aligned} Q_{n+1}(\mathbb{R}) &\leq \widehat{Q} Q_n(-\infty, x_n] + \widehat{Q}(x_n) Q_n(\mathbb{R}) \\ &\leq \widehat{Q} Q_n(-\infty, x_n] \\ &\qquad + \widehat{Q}(x_n) \left[Q_0(\mathbb{R}) \prod_{k=0}^{n-1} \widehat{Q}(x_k) + \widehat{Q} \sum_{k=0}^{n-1} Q_k(-\infty, x_k] \prod_{j=k+1}^{n-1} \widehat{Q}(x_j) \right] \\ &= Q_0(\mathbb{R}) \prod_{k=0}^{n} \widehat{Q}(x_k) + \widehat{Q} \sum_{k=0}^{n} Q_k(-\infty, x_k] \prod_{j=k+1}^{n} \widehat{Q}(x_j), \end{aligned}$$

which implies the induction step.

Since the series (18) converges, we obtain

$$Q_{\sup} \equiv \sup_{n \geq 0} \prod_{k=0}^{n} \widehat{Q}(x_k) \leq \sup_{n \geq 0} \prod_{k=0}^{n} \big(1 + \widehat{q}(x_k)\big) < \infty.$$

From here and (19) we derive the estimate

$$Q_{n+1}(\mathbb{R}) \leq Q_0(\mathbb{R}) Q_{\sup} + \widehat{Q} Q_{\sup} \sum_{k=0}^{n} Q_k(-\infty, x_k], \tag{20}$$

which proves the first assertion of the lemma. Further, for every $n \geq 0$, we have

$$\begin{aligned} \big|Q_{n+1}(\mathbb{R}) - Q_n(\mathbb{R})\big| &= \left| \left(\int_{-\infty}^{x_n} + \int_{x_n}^{\infty}\right) \big(Q(x,\mathbb{R}) - 1\big) Q_n(dx) \right| \\ &\leq (\widehat{Q} + 1) Q_n(-\infty, x_n] + Q_n(\mathbb{R})\, \widehat{q}(x_n). \end{aligned}$$

Since the sequence $Q_n(\mathbb{R})$ is bounded, from (17) and (18) it follows that the sequence $Q_n(\mathbb{R})$ is fundamental. The proof is complete.

Lemma 4. *Assume that the Markov evolution of masses takes positive values only. Let $\widehat{Q} < \infty$ and let, for each fixed $\varepsilon > 0$, the condition* (18) *hold for the sequence $x_k = k\varepsilon$. If, for some $\lambda > 0$,*

$$E_{\sup} \equiv \sup_{u>0} \mathcal{E} e^{-\lambda\eta(u)} < 1 \tag{21}$$

then there exists $\widehat{c} < \infty$ such that, for any $n \geq 0$ and initial distribution Q_0, the following estimate holds:

$$Q_n(\mathbb{R}) \leq \widehat{c}\big(Q_0(\mathbb{R}) + 1\big).$$

Proof. By (11), we have the estimate

$$\begin{aligned}\mathcal{E} e^{-\lambda Y_n} &= \int_0^\infty e^{-\lambda u} \mathcal{E} e^{-\lambda\eta(u)} \mathrm{Mes}\{Y_{n-1} \in dy\} \\ &\leq E_{\sup} \int_0^\infty e^{-\lambda u} \mathrm{Mes}\{Y_{n-1} \in dy\} \\ &= E_{\sup} \mathcal{E} e^{-\lambda Y_{n-1}}.\end{aligned}$$

Therefore,

$$\mathcal{E} e^{-\lambda Y_n} \leq \mathcal{E} e^{-\lambda Y_0} (E_{\sup})^n \leq Q_0[0,\infty)(E_{\sup})^n.$$

Using the analog of the exponential Chebyshev inequality (12) with $y = n\varepsilon$, we arrive at the inequality

$$\mathrm{Mes}\{Y_n \leq n\varepsilon\} \leq e^{\lambda n\varepsilon} \mathcal{E} e^{-\lambda Y_n} \leq Q_0[0,\infty)\big(e^{\lambda\varepsilon} E_{\sup}\big)^n.$$

Since $E_{\sup} < 1$, there exists a sufficiently small $\varepsilon > 0$ such that

$$\delta \equiv e^{\lambda\varepsilon} E_{\sup} < 1. \tag{22}$$

With such choice of ε, the condition (17) of Lemma 3 is satisfied. The lemma assertion follows from the estimate (20).

In the following lemma we consider the Markov evolution of masses that do not necessarily satisfy (17). Nonfulfillment of this condition leads to the possibility of unbounded growth of the sequence of the total space masses. In the case of ordinary Markov chain, it is impossible since, if the mass tends to infinity, then the mass disappears near the origin. In the case of the Markov evolution of masses the jumps can generate masses greater than 1 and, therefore, the states near the origin can serve as a permanent sources of new masses.

Lemma 5. *Let $Q_0(\mathbb{R}) < \infty$, $\widehat{Q} < \infty$,*

$$\sup_{n\geq 0} Q_n(-\infty, 0] < \infty$$

and, for each fixed $\varepsilon > 0$, *the condition* (18) *hold for the sequence* $x_k = k\varepsilon$. *If, for some* $\lambda > 0$, *the condition* (21) *is satisfied then there exists* $\widehat{c} < \infty$ *such that, for any* $n \geq 0$ *and* $x > 0$, *the following estimate holds*:

$$Q_n(-\infty, x] \leq \widehat{c}\,(x+1).$$

Proof. In view of the condition $\sup_n Q_n(-\infty, 0] < \infty$, it is necessary and sufficient to prove that, for some c_1,

$$Q_n(0, x] \leq c_1(x+1). \tag{23}$$

We make use of the formula (10) on the last entrance into the set $(-\infty, 0]$:

$$Q_n(0,x] = \mathrm{Mes}\big\{Y_0 > 0, \ldots, Y_{n-1} > 0,\ Y_n \in (0,x]\big\}$$
$$+ \sum_{k=0}^{n-1} \mathrm{Mes}\big\{Y_k \leq 0,\ Y_{k+1} > 0,\ \ldots,\ Y_{n-1} > 0,\ Y_n \in (0,x]\big\}.$$

By Lemma 4, there exists $c_2 < \infty$ such that, for any $n \geq 0$ and $k < n$,

$$\mathrm{Mes}\big\{Y_0 > 0, \ldots,\ Y_n > 0\big\} \leq c_2,$$
$$\mathrm{Mes}\big\{Y_k \leq 0,\ Y_{k+1} > 0,\ \ldots,\ Y_n > 0\big\} \leq c_2.$$

Here the second estimate follows from the fact that the value

$$\mathrm{Mes}\big\{Y_k \leq 0,\ Y_{k+1} > 0\big\} \leq Q_k(-\infty, 0]\widehat{Q}$$

is bounded uniformly in $k \geq 0$.

Choose $\varepsilon > 0$ so that (22) is valid. If $n \leq 2x/\varepsilon$ then the lemma assertion follows from the inequalities

$$Q_n(0,x] \leq nc_2 \leq 2xc_2/\varepsilon.$$

If $n > 2x/\varepsilon$ then

$$Q_n(0,x] \leq \sum_{k=0}^{n-x/\varepsilon} \mathrm{Mes}\big\{Y_k \leq 0,\ Y_{k+1} > 0, \ldots, Y_{n-1} > 0,\ Y_n \in (0,x]\big\}$$
$$+ xc_2/\varepsilon. \tag{24}$$

We now estimate the kth term in the sum. Consider an auxiliary Markov evolution of masses Z_n taken values on the positive half-line with initial distribution

$$\mathrm{Mes}\{Y_k \leq 0,\ Y_{k+1} \in B\}, \quad B \in \mathcal{B}(0, \infty)$$

(thus, $\mathrm{Mes}\{Z_0 \in \mathbb{R}\} = \mathrm{Mes}\{Y_k \le 0,\, Y_{k+1} > 0\} \le c_2$), and with jumps on positive half-line possessing the distribution $\mathrm{Mes}\{\eta(u) \in B\}$, $B \in \mathcal{B}(0,\infty)$. We obtain the inequality

$$\mathrm{Mes}\{Y_k \le 0,\, Y_{k+1} > 0,\, \ldots,\, Y_{n-1} > 0,\, Y_n \in (0,x]\} \le \mathrm{Mes}\{Z_{n-k-1} \le x\}.$$

It follows from Lemma 4 that

$$\mathrm{Mes}\{Z_{n-k-1} \le (n-k-1)\varepsilon\} \le c_2 \delta^{n-k-1}$$

with $\delta < 1$. Therefore, for $n > 2x/\varepsilon$ and $k \le n - x/\varepsilon$ (thus, $x \le (n-k)\varepsilon$),

$$\begin{aligned}
&\mathrm{Mes}\{Y_k \le 0,\, Y_{k+1} > 0,\, \ldots,\, Y_{n-1} > 0,\, Y_n \in (0,x]\} \\
&\quad\le \mathrm{Mes}\{Z_{n-k-1} \le x\} \\
&\quad\le \mathrm{Mes}\{Z_{n-k-1} \le (n-k-1)\varepsilon\} \\
&\quad\le c_2 \delta^{n-k-1}.
\end{aligned}$$

Finally,

$$Q_n(0,x] \le c_2 \sum_{k=0}^{n-x/\varepsilon} \delta^{n-k-1} + xc_2/\varepsilon \le \frac{c_2}{1-\delta} + xc_2/\varepsilon.$$

Both the estimate (23) and the lemma are proven.

5. An analog of the central limit theorem for a Markov evolution of masses

The characteristic function of the sum of independent random variables is equal to the product of the characteristic functions of the summands. If we deal with a Markov chain or, moreover, a Markov evolution of masses, then the characteristic function is not a product of something in view of the non-homogeneity of jumps. In the following lemma we establish to what extent the time-behavior of the characteristic function of a Markov evolution of masses differs from the time-behavior of the characteristic function of a sequence of partial sums of independent variables.

Consider a Markov evolution of masses $\{Y_n\}$ with jumps $\{\eta_n(x)\}$. Denote $Q_n(B) = \mathrm{Mes}\{Y_n \in B\}$ and $Q(x,B) = \mathrm{Mes}\{x + \eta(x) \in B\}$. Let

$$\widehat{Q} \equiv \sup_{y \in \mathbb{R}} Q(y,\mathbb{R}) < \infty, \quad Q \equiv \sup_{n \ge 0} Q_n(\mathbb{R}) < \infty.$$

Lemma 6. *Let x_j be an arbitrary sequence of levels in $\mathbb{R}$. For all $\lambda \in \mathbb{R}$, $n \geq 1$, $k \leq n$ and a complex number $\varphi \in \mathbb{C}$, $|\varphi| \leq 1$, the following inequality holds*:

$$\left|\mathcal{E} e^{i\lambda Y_n} - \varphi^{n-k}\mathcal{E} e^{i\lambda Y_k}\right| \leq (\widehat{Q}+1)\sum_{j=k}^{n-1} Q_j(-\infty, x_j] + Q\sum_{j=k}^{n-1}\varepsilon_j,$$

where

$$\varepsilon_j = \sup_{x > x_j}\left|\mathcal{E} e^{i\lambda\eta(x)} - \varphi\right|. \tag{25}$$

Proof. Take $j \in [k+1, n]$. By (11), we have

$$\mathcal{E} e^{i\lambda Y_j} = \int_{\mathbb{R}} \big(\mathcal{E} e^{i\lambda\eta(x)}\big) e^{i\lambda x} Q_{j-1}(dx).$$

Therefore,

$$\begin{aligned}\left|\mathcal{E} e^{i\lambda Y_j} - \varphi\mathcal{E} e^{i\lambda Y_{j-1}}\right| &= \left|\int_{\mathbb{R}} \big(\mathcal{E} e^{i\lambda\eta(x)} - \varphi\big) e^{i\lambda x} Q_{j-1}(dx)\right| \\ &\leq \left|\int_{x_{j-1}}^{\infty} \big(\mathcal{E} e^{i\lambda\eta(x)} - \varphi\big) e^{i\lambda x} Q_{j-1}(dx)\right| \\ &\quad + \left|\int_{-\infty}^{x_{j-1}} \big(\mathcal{E} e^{i\lambda\eta(x)} - \varphi\big) e^{i\lambda x} Q_{j-1}(dx)\right| \\ &\leq \varepsilon_{j-1} Q_{j-1}(\mathbb{R}) + \big(\widehat{Q}+1\big) Q_{j-1}(-\infty, x_{j-1}]\end{aligned}$$

in view of (25). Combining the last estimate with the inequality

$$\begin{aligned}\left|\mathcal{E} e^{i\lambda Y_n} - \varphi^{n-k}\mathcal{E} e^{i\lambda Y_k}\right| &\leq \sum_{j=k+1}^{n}\left|\varphi^{n-j}\mathcal{E} e^{i\lambda Y_j} - \varphi^{n-(j-1)}\mathcal{E} e^{i\lambda Y_{j-1}}\right| \\ &= \sum_{j=k+1}^{n}\left|\mathcal{E} e^{i\lambda Y_j} - \varphi\mathcal{E} e^{i\lambda Y_{j-1}}\right|,\end{aligned}$$

we deduce the lemma assertion.

In the formula (25) the value ε_j is defined as the maximal difference between the jump characteristic function and some complex number $\varphi \in \mathbb{C}$, $|\varphi| \leq 1$, on some phase subspace rather than on the whole space. We are going to apply this lemma below in the case when the mass of the corresponding subspace is close to the mass of the whole real line.

In the following theorem we give sufficient conditions under which a Markov evolution of masses on real line $[0, \infty)$ satisfies the central limit theorem in some sense.

Theorem 5. *Let the conditions of Lemma 4 be satisfied and let $Q_0[0,\infty) < \infty$. Assume that the family of jump squares $\{\eta^2(x),\ x \ge 0\}$ is uniformly integrable. If, for some $\alpha > 0$ and $\sigma^2 > 0$, the relations*

$$\mathcal{E}\big(\eta(x)-\alpha\big) = o\big(1/\sqrt{x}\,\big), \tag{26}$$

$$\mathcal{E}\big(\eta(x)-\alpha\big)^2 \to \sigma^2 \tag{27}$$

hold as $x \to \infty$ then the distribution of the mass $(Y_n - n\alpha)/\sqrt{n}$ converges weakly as $n \to \infty$ to the normal law with zero mean and variance σ^2, i.e., for every $y \in \mathbb{R}$ the following convergence holds:

$$Q_n\big[0, n\alpha + y\sqrt{n}\,\big] \to Q\Phi_{\sigma^2}(y),$$

with $Q = \lim_{n\to\infty} Q_n(\mathbb{R})$.

Proof is carried out by the method of characteristic functions. Hereinafter $\lambda \in \mathbb{R}$. In view of the uniform integrability of the family of the jump squares, the following decomposition is valid:

$$\mathcal{E}e^{i\lambda(\eta(x)-\alpha)} = Q(x,\mathbb{R}) + i\lambda\mathcal{E}\big(\eta(x)-\alpha\big) - \frac{\lambda^2}{2}\mathcal{E}\big(\eta(x)-\alpha\big)^2 + o(\lambda^2)$$

as $\lambda \to 0$ uniformly in x. Taking into account the conditions (26) and (27), we obtain the inequality

$$\Big|\mathcal{E}e^{i\lambda(\eta(x)-\alpha)} - (1-\lambda^2\sigma^2/2)\Big| \le \varepsilon(x,\lambda)\big(\lambda/\sqrt{x} + \lambda^2\big) + \big|Q(x,\mathbb{R}) - 1\big|,$$

where $\varepsilon(x,\lambda) \to 0$ as $\lambda \to 0$ and $x \to \infty$. Fix arbitrary $\lambda \in \mathbb{R}$ and $\varepsilon > 0$. From the last inequality we have

$$\Big|\mathcal{E}e^{i\lambda(\eta(x)-\alpha)/\sqrt{n}} - (1-\lambda^2\sigma^2/2n)\Big| \le \widetilde{\varepsilon}(n,x)/\sqrt{nj} + \widehat{q}(j\varepsilon),$$

where $\widetilde{\varepsilon}(n,x) \to 0$ as $n \to \infty$ and $j \to \infty$ uniformly in the domain $x > j\varepsilon$. Applying now Lemma 6 with $\varphi = 1 - \lambda^2\sigma^2/2n$ and $x_j = j\varepsilon$ to the Markov evolution of masses $(Y_n - n\alpha)/\sqrt{n}$, we obtain the estimate

$$\left|\mathcal{E}e^{i\lambda\frac{Y_n-n\alpha}{\sqrt{n}}} - \left(1-\frac{\lambda^2\sigma^2}{2n}\right)^{n-k}\mathcal{E}e^{i\lambda\frac{Y_k-k\alpha}{\sqrt{n}}}\right|$$
$$\le \big(\widehat{Q}+1\big)\sum_{j=k}^{n-1} Q_j[0,j\varepsilon] + Q\sum_{j=k}^{n-1}\left(\frac{o(1)}{\sqrt{nj}} + \widehat{q}(j\varepsilon)\right).$$

Since all conditions of Lemma 4 are satisfied, the value Q is finite and $Q_j[0,j\varepsilon] \le Q_0[0,\infty)\delta^j$, $\delta < 1$. Thus, for each fixed $\lambda \in \mathbb{R}$, the difference

$$\left|\mathcal{E}e^{i\lambda\frac{Y_n-n\alpha}{\sqrt{n}}} - \left(1-\frac{\lambda^2\sigma^2}{2n}\right)^{n-k}\mathcal{E}e^{i\lambda\frac{Y_k-k\alpha}{\sqrt{n}}}\right|$$

can be made arbitrarily small uniformly in $n > k$ by choosing a sufficiently large k. For each fixed k,

$$\mathcal{E} e^{i\lambda \frac{Y_k - k\alpha}{\sqrt{n}}} \to Q_k(\mathbb{R})$$

as $n \to \infty$ and $Q_k(\mathbb{R}) \to Q$ as $k \to \infty$ by Lemma 3. Thus, in view of the convergence

$$\left(1 - \frac{\lambda^2\sigma^2}{2n}\right)^{n-k} \to e^{-\lambda^2\sigma^2/2} \quad \text{as} \quad n \to \infty$$

for each fixed k, the following holds:

$$\mathcal{E} e^{i\lambda \frac{Y_n - n\alpha}{\sqrt{n}}} \to Q e^{-\lambda^2\sigma^2/2} \quad \text{as} \quad n \to \infty,$$

which completes the proof of the theorem.

6. Large deviation probabilities for an asymptotically homogeneous Markov chain

In this section we consider an asymptotically space-homogeneous Markov chain, i.e., $\xi(u) \Rightarrow \xi$ as $u \to \infty$. We assume that F is a nonlattice distribution of the random variable ξ; the lattice case is discussed in Section 8.

As before, the parameter $\beta > 0$ is defined as the solution to the equation $\varphi(\beta) = \mathbb{E} e^{\beta\xi} = 1$. The measure, defined by the equality

$$F^{(\beta)}(du) = e^{\beta u} F(du), \tag{28}$$

is probabilistic. Let $\xi^{(\beta)}$ be a random variable with the distribution $F^{(\beta)}$. Assume that

$$\alpha \equiv \mathbb{E}\xi^{(\beta)} = \varphi'(\beta) \in (0, \infty),$$
$$\sigma^2 \equiv \operatorname{Var}\xi^{(\beta)} = \varphi''(\beta) - \big(\varphi'(\beta)\big)^2 < \infty.$$

Theorem 6. *Let $\mathbb{E} e^{\beta X_0}$ be finite and let the family of jumps $\{\xi(u), u \in \mathbb{R}\}$ possess a stochastic majorant $\bar{\xi}$ such that*

$$\mathbb{E}\,\bar{\xi}^2 e^{\beta\bar{\xi}} < \infty. \tag{29}$$

Assume that the chain jumps satisfy the following conditions:

$$\inf_{u \in \mathbb{R}} \mathbb{E} e^{\beta\xi(u)} > 0, \tag{30}$$

$$\mathbb{E}\xi(u) e^{\beta\xi(u)} = \alpha + o\big(1/\sqrt{u}\big) \quad \textit{as} \quad u \to \infty. \tag{31}$$

Moreover, suppose that, for each fixed $A > 0$, *there exists a bounded decreasing function* $\delta(u) = o(1/u)$ *integrable at infinity and such that*

$$\sup_{\lambda\in[-A,A]}\left|\mathbb{E}\,e^{(\beta+i\lambda)\xi(u)} - \mathbb{E}\,e^{(\beta+i\lambda)\xi}\right| \le \delta(u) \tag{32}$$

for every $u \in \mathbb{R}$. *Then the following relation holds*:

$$\mathbb{P}\{X_n > x\} = ce^{-\beta x}\Phi_{\sigma^2}\left(\frac{n\alpha - x}{\sqrt{x/\alpha}}\right) + o\big(e^{-\beta x}\big)$$

as $x \to \infty$ *uniformly in* $n \ge 0$, *where*

$$c = \frac{1}{\beta\alpha}\int_{-\infty}^{\infty}\Big(\mathbb{E}\,e^{\beta\xi(y)} - 1\Big)e^{\beta y}\,\pi(dy) \in [0,\infty). \tag{33}$$

The condition (30) is equivalent to the fact that there is no sequence of points $u_k \in \mathbb{R}$ such that $\xi(u_k) \Rightarrow -\infty$ as $k \to \infty$.

In this theorem we do not assume that the function $\delta(u)$ is regularly varying at infinity as it is assumed by the condition (4) in Theorem 2; so, the condition (32) is weaker than (4). Moreover, since

$$\begin{aligned}
&\left|\mathbb{E}\,e^{(\beta+i\lambda)\xi(u)} - \mathbb{E}\,e^{(\beta+i\lambda)\xi}\right| \\
&\quad= \left|\int_{-\infty}^{\infty} e^{(\beta+i\lambda)v}d_v\Big(\mathbb{P}\big\{\xi(u) < v\big\} - \mathbb{P}\{\xi < v\}\Big)\right| \\
&\quad= |\beta + i\lambda|\left|\int_{-\infty}^{\infty} e^{(\beta+i\lambda)v}\Big(\mathbb{P}\big\{\xi(u) < v\big\} - \mathbb{P}\{\xi < v\}\Big)\,dv\right| \\
&\quad\le |\beta + i\lambda|\int_{-\infty}^{\infty} e^{\beta v}\Big|\mathbb{P}\big\{\xi(u) < v\big\} - \mathbb{P}\{\xi < v\}\Big|\,dv,
\end{aligned}$$

we can propose the following condition sufficient for (32):

$$\int_{-\infty}^{\infty} e^{\beta v}\Big|\mathbb{P}\big\{\xi(u) < v\big\} - \mathbb{P}\{\xi < v\}\Big|\,dv \le \delta(u).$$

From Theorem 6 we deduce

Corollary 1. *Let* $\widehat{y}(x) \to \infty$ *as* $x \to \infty$. *Then we have the asymptotic*

$$\mathbb{P}\{X_n > x\} = e^{-\beta x}\big(c + o(1)\big)$$

as $x \to \infty$ *uniformly in* $n \ge x/\alpha_0 + \widehat{y}(x)\sqrt{x}$. *If* $c > 0$ *then the following equivalence holds in the above-indicated ranges of* n:

$$\mathbb{P}\{X_n > x\} \sim \pi(x,\infty) \;\text{ as }\; x \to \infty.$$

Proof of Theorem 6. First of all, note that the condition (32) with $\lambda = 0$ implies the relation

$$\mathbb{E}\, e^{\beta\xi(u)} = 1 + O\big(\delta(u)\big) = 1 + o(1/u) \quad \text{as} \quad u \to \infty. \tag{34}$$

Consider the Cramér transform with parameter β over the chain X_n, i.e., introduce the generalized transition kernel

$$P^{(\beta)}(u, dv) = e^{\beta(v-u)} P(u, dv).$$

Let $X_n^{(\beta)}$ be a Markov evolution of masses with the generalized transition kernel $P^{(\beta)}(\cdot\,,\cdot)$. The following equality is valid:

$$\pi_n^{(\beta)}(du) \equiv \mathrm{Mes}\big\{X_n^{(\beta)} \in du\big\} = e^{\beta u}\pi_n(du) \equiv e^{\beta u}\mathbb{P}\big\{X_n \in du\big\}.$$

Since $\mathbb{E}\, e^{\beta X_0}$ is finite, we have $\pi_0^{(\beta)}(\mathbb{R}) < \infty$. By (34), the total masses of the jumps of the Markov evolution of masses $X_n^{(\beta)}$ are uniformly bounded, i.e.

$$\widehat{Q} \equiv \sup_{u \in \mathbb{R}} P^{(\beta)}(u, \mathbb{R}) < \infty.$$

Represent the kernel $P^{(\beta)}$ as the sum of the transition probability P^* and the signed kernel P^{**} as follows:

$$P^*(x, \cdot) = \frac{P^{(\beta)}(x, \cdot)}{P^{(\beta)}(x, \mathbb{R})},$$
$$P^{**}(x, \cdot) = P^{(\beta)}(x, \cdot) - P^*(x, \cdot) = \frac{P^{(\beta)}(x, \mathbb{R}) - 1}{P^{(\beta)}(x, \mathbb{R})} P^{(\beta)}(x, \cdot).$$

The measure $P^{**}(x, \cdot)$ is negative in the case $P^{(\beta)}(x, \mathbb{R}) < 1$, positive in the case $P^{(\beta)}(x, \mathbb{R}) > 1$, and is equal to 0 in the case $P^{(\beta)}(x, \mathbb{R}) = 1$. Thus, the total variation $|P^{**}|(u, B)$ of the measure $P^{**}(u, \cdot)$ on the set B equals $\big|P^{**}(u, B)\big|$.

Applying the nth power of the kernel $P^{(\beta)}$ to the measure $\pi_0^{(\beta)}$, we obtain the measure $\pi_n^{(\beta)}$; so, we have $\pi_n^{(\beta)} = \pi_0^{(\beta)}(P^* + P^{**})^n$. Decomposing the power $(P^* + P^{**})^n$ into the sum with respect to the last application of the kernel P^{**}, we obtain the equality

$$(P^* + P^{**})^n = (P^*)^n + \sum_{k=0}^{n-1} (P^* + P^{**})^k P^{**} (P^*)^{n-1-k}.$$

From here we deduce the representation which is basic for our subsequent analysis:

$$\pi_n^{(\beta)} = \pi_0^{(\beta)}(P^*)^n + \sum_{k=0}^{n-1} \pi_k^{(\beta)} P^{**}(P^*)^{n-1-k}. \tag{35}$$

The main idea of the further considerations consists in the following: Since the sequence of measures $\pi_n^{(\beta)}(du)$ converges weakly to the measure $e^{\beta u}\pi(du)$ and the tail of the measure π behaves, as a rule, asymptotically as the exponential with parameter $-\beta$, the weak limit of the sequence of measures $\pi_n^{(\beta)}$ far away from the origin behaves like the Lebesgue measure up to some constant. In particular, (34) implies the tightness (in the same sense as that for probability measures) of the family of measures $\{\pi_n^{(\beta)}P^{**}\}$. In addition, the nth power of the transition kernel P^* satisfies the local limit theorem. All of these allows us to compute the local asymptotic of the measure $\pi_n^{(\beta)}$.

Lemma 7. *The family of measures* $\{\pi_k^{(\beta)}P^{**},\ k \ge 0\}$ *is tight in the sense that*

$$\sup_{k\ge 0} \left|\pi_k^{(\beta)}P^{**}\right|(-\infty,-x] = O\big(e^{-\beta x}\big),$$

$$\sup_{k\ge 0} \left|\pi_k^{(\beta)}P^{**}\right|(x,\infty) \to 0$$

as $x \to \infty$. *Moreover, the sequence of measures* $\pi_k^{(\beta)}P^{**}$ *converges in the total variation distance to the measure* $\pi^{(\beta)}P^{**}$ *as* $k \to \infty$, *where* $\pi^{(\beta)}(du) \equiv \pi(du)e^{\beta u}$.

Taking it into account that $\big(\pi^{(\beta)}P^{**}\big)(\mathbb{R})$ is equal to

$$\begin{aligned}\int_{-\infty}^{\infty} P^{**}(u,\mathbb{R})\pi^{(\beta)}(du) &= \int_{-\infty}^{\infty} \Big(P^{(\beta)}(u,\mathbb{R}) - P^*(u,\mathbb{R})\Big)\pi^{(\beta)}(du) \\ &= \int_{-\infty}^{\infty} \Big(\mathbb{E}e^{\beta\xi(u)} - 1\Big)e^{\beta u}\pi(du), \end{aligned} \tag{36}$$

we obtain

Corollary 2. *The constant* c *in* (33) *is finite.*

As far as the positivity of c is concerned, some sufficient conditions are given just after Theorem 2. Note that these conditions are satisfied automatically for the partially homogeneous chain with the uniformly bounded moments of order $2+\varepsilon$ of negative parts of jumps.

Proof of Lemma. We have the inequality

$$\big|\pi_k^{(\beta)} P^{**}\big|(B) \le \int_{-\infty}^{\infty} |P^{**}|(u,B)\pi_k^{(\beta)}(du). \tag{37}$$

From the definition of $P^{**}(u,\cdot)$ and the condition (32) with $\lambda = 0$ it follows that

$$\begin{aligned}
|P^{**}|(u,B) &= \frac{\big|P^{(\beta)}(u,\mathbb{R}) - 1\big|}{P^{(\beta)}(u,\mathbb{R})} P^{(\beta)}(u,B) \\
&= \frac{\big|\mathbb{E}\, e^{\beta\xi(u)} - 1\big|}{\mathbb{E}\, e^{\beta\xi(u)}} P^{(\beta)}(u,B) \\
&\le \frac{\delta(u)}{\mathbb{E}\, e^{\beta\xi(u)}} P^{(\beta)}(u,B);
\end{aligned} \tag{38}$$

thus, in view of (30), we deduce the estimate

$$|P^{**}|(u,B) \le c_1 P^{(\beta)}(u,B). \tag{39}$$

The measure $\pi_k^{(\beta)}$ is the Cramér transform with positive parameter β over a probability measure; therefore, its negative tail admits an exponential estimate like $\pi_k^{(\beta)}(-\infty,-x] \le e^{-\beta x}$, $x > 0$. Thus, from (37) and (39) with $B = (-\infty,-x]$ we can deduce the following estimates:

$$\begin{aligned}
\big|\pi_k^{(\beta)} P^{**}\big|(-\infty,-x] &\le c_1 \int_{-\infty}^{\infty} P^{(\beta)}\big(u,(-\infty,-x]\big)\pi_k^{(\beta)}(du) \\
&= c_1 \pi_{k+1}^{(\beta)}(-\infty,-x] \\
&\le c_1 e^{-\beta x}.
\end{aligned}$$

The proof of the first uniform estimate of the lemma is complete.

Now, check the second uniform convergence stated in the lemma. Fix arbitrary $\lambda \in (0,\beta)$; we have $\mathbb{E}\, e^{\lambda\xi} < 1$. Since $\xi(u) \Rightarrow \xi$ as $x \to \infty$ and the family of random variables $\big\{e^{\beta\xi(u)}\big\}$ is uniformly integrable, we obtain $\mathbb{E} e^{\lambda\xi(u)} \to \mathbb{E}\, e^{\lambda\xi} < 1$. Thus, there exists a sufficiently large U such that

$$\sup_{u>U} \mathbb{E}\, e^{\lambda\xi(u)} < 1. \tag{40}$$

Without loss of generality we may assume that $U = 0$. Then, by Lemma 5, we have

$$\sup_k \pi_k^{(\beta)}(-\infty,x] = O(x) \quad \text{as} \quad x \to \infty. \tag{41}$$

In view of (29), from (39) it follows that

$$\Big|P^{**}\big(u,(x,\infty)\big)\Big| \le c_1 \frac{\mathbb{E}\,\bar{\xi}^2 e^{\beta\bar{\xi}}}{(x-u)^2} = \frac{c_2}{(x-u)^2} \tag{42}$$

for $u \le x$. From (38) we infer that

$$\Big|P^{**}\big(u,(x,\infty)\big)\Big| \le \frac{\delta(u)}{\mathbb{E}\,e^{\beta\xi(u)}} P^{(\beta)}(u,\mathbb{R}) = \delta(u) \tag{43}$$

for all u and x. Inserting (42) and (43) into (37) with $B=(x,\infty)$, we obtain

$$\big|\pi_k^{(\beta)}P^{**}\big|(x,\infty) \le c_3\int_{-\infty}^{x/2}(x-u)^{-2}\pi_k^{(\beta)}(du) + \int_{x/2}^{\infty}\delta(u)\pi_k^{(\beta)}(du). \tag{44}$$

Here the first integral vanishes as $x\to\infty$. To calculate the second integral, we use the formula of integration by parts:

$$\int_{x/2}^{\infty}\delta(u)\pi_k^{(\beta)}(du) = \delta(u)\pi_k^{(\beta)}[0,u]\Big|_{x/2}^{\infty} + \int_{x/2}^{\infty}\pi_k^{(\beta)}[0,u]d\big(-\delta(u)\big).$$

By (41) and the relation $\delta(u)=o(1/u)$, the first term on the right-hand side of the last equality vanishes as $x\to\infty$ uniformly in k. By the same theorem and monotonicity of the function δ, we have the following estimate uniform in k:

$$\int_{x/2}^{\infty}\pi_k^{(\beta)}[0,u]d\big(-\delta(u)\big) \le c_4\int_{x/2}^{\infty}ud\big(-\delta(u)\big).$$

Successive integration by parts and integrability of the function δ at infinity imply that the second integral in (44) vanishes as $x\to\infty$ uniformly in k as well. Thus, the family of measures $\pi_k^{(\beta)}P^{**}$ is tight.

The weak convergence follows from the convergence in total variation of the sequence of measures $\pi_k^{(\beta)}$ as $k\to\infty$ to the measure $\pi^{(\beta)}$. The lemma is proven.

The end of the proof of Theorem 6 is carried out under three additional conditions: the condition of existence of minorant for the family of jumps of the chain with transition probability P^* $\big($see (45)$\big)$, a condition of sufficiently fast convergence rate of π_n to π $\big($see (46)$\big)$, and a condition of absolute continuity of the distribution π_n with respect to the invariant measure π $\big($see (47)$\big)$. The end of the proof in the general case is considered separately in Section 7.

Let $\xi^*(x)$ be the jump at the state x of the Markov chain X_n^* with transition probabilities P^*, i.e., be a random variable such that $\mathbb{P}\big\{\xi^*(x)\in B\big\} =$

$P^*(x, x+B)$. Since $P^{(\beta)}(x, \mathbb{R}) \to 1$ as $x \to \infty$, by the definition of P^*, the following weak convergence holds:

$$\xi^*(x) \Rightarrow \xi^{(\beta)} \quad \text{as} \quad x \to \infty.$$

So, assume that the family of jumps $\{\xi^*(x),\, x \in \mathbb{R}\}$ possesses a minorant $\underline{\zeta}$ with positive mean and finite variance, i.e., the stochastic inequality

$$\xi^*(x) \geq_{\text{st}} \underline{\zeta} \tag{45}$$

takes place for every $x \in \mathbb{R}$. Let the convergence rate in (1) be sufficiently fast; namely,

$$\sum_{n=1}^{\infty} |\pi_n - \pi|(\mathbb{R}) < \infty. \tag{46}$$

Moreover, assume that, for every n, the measure π_n is absolutely continuous with respect to the measure π, i.e., the (nonnegative) Radon–Nikodým derivative is defined as

$$f_n(u) \equiv \frac{d\pi_n}{d\pi}(u), \tag{47}$$

and this derivative is bounded from above by some number $\rho < \infty$ uniformly in n and u. By the definition of the Cramér transform, we have

$$\frac{d\pi_n^{(\beta)}}{d\pi^{(\beta)}}(u) = \frac{d\pi_n}{d\pi}(u) = f_n(u).$$

Then the measure $\pi_n P^{**}$ is absolutely continuous with respect to the measure πP^{**}, and the corresponding signed density $f^{**}(u)$ can be estimated as follows:

$$\big|f_n^{**}(u)\big| \equiv \left|\frac{d\pi_n^{(\beta)} P^{**}}{d\pi^{(\beta)} P^{**}}(u)\right| \leq \rho. \tag{48}$$

This is possible due to the estimate

$$\begin{aligned}\pi_n^{(\beta)} P^{**}(B) &= \int_{\mathbb{R}} P^{**}(u, B) f_n(u) \pi^{(\beta)}(du) \\ &\leq \rho \int_{\mathbb{R}} P^{**}(u, B) \pi^{(\beta)}(du) = \rho \pi^{(\beta)} P^{**}(B)\end{aligned}$$

if $B \in \mathcal{B}(\mathbb{R})$.

Since

$$\mathbb{P}\big\{\xi^*(x) > u\big\} = \frac{\operatorname{Mes}\big\{\xi^{(\beta)} > u\big\}}{\mathbb{E}\, e^{\beta\xi(x)}} \leq \frac{\operatorname{Mes}\big\{\bar{\xi}^{(\beta)} > u\big\}}{\mathbb{E}\, e^{\beta\xi(x)}},$$

by (29) and (30), the family of random variables $\{\xi^*(x)\}$ possesses a square integrable majorant. In view of (31) and (34), we have

$$\mathbb{E}\,\xi^*(x) = \frac{\mathbb{E}\,\xi(x)e^{\beta\xi(x)}}{\mathbb{E}\,e^{\beta\xi(x)}} = \frac{\alpha + o\big(1/\sqrt{x}\,\big)}{1+o(1/x)} = \alpha + o\big(1/\sqrt{x}\,\big) \quad \text{as} \quad x \to \infty.$$

From the weak convergence $\xi(u) \Rightarrow \xi$ and the conditions (29) and (34) we infer that

$$\mathbb{E}\big(\xi^*(x)\big)^2 = \frac{\mathbb{E}\big(\xi(x)\big)^2 e^{\beta\xi(x)}}{\mathbb{E}\,e^{\beta\xi(x)}} \to \sigma^2 + \alpha^2 \quad \text{as} \quad x \to \infty.$$

For each fixed $A > 0$, from (32) and (34) it follows that

$$\sup_{\lambda\in[-A,A]} \Big|\mathbb{E}\,e^{i\lambda\xi^*(x)} - \mathbb{E}\,e^{i\lambda\xi^{(\beta)}}\Big| = \sup_{\lambda\in[-A,A]} \left|\frac{\mathbb{E}\,e^{(i\lambda+\beta)\xi(x)}}{\mathbb{E}\,e^{\beta\xi(x)}} - \mathbb{E}\,e^{(i\lambda+\beta)\xi}\right| = O\big(\delta(x)\big)$$

as $x \to \infty$. So, all conditions of Theorem 3 (in particular, the existence of a proper minorant) are fulfilled. Thus, the chain X_n^* satisfies the local limit theorem. In particular, for each fixed $\Delta > 0$, we obtain

$$\sup_y \mathbb{P}\big\{X_n^* \in (y, y+\Delta]\big\} \to 0 \quad \text{as} \quad n \to \infty. \tag{49}$$

According to (35), we have

$$\pi_n^{(\beta)}(y, y+\Delta] = \pi_0^{(\beta)}(P^*)^n(y, y+\Delta] + \sum_{k=0}^{n-1} \pi_k^{(\beta)} P^{**}(P^*)^{n-1-k}(y, y+\Delta].$$

By (49), the contribution of the term $\pi_0^{(\beta)}(P^*)^n(y, y+\Delta]$, as well as of each (for a fixed finite set of k's) of the terms $\pi_k^{(\beta)}P^{**}(P^*)^{n-1-k}(y, y+\Delta]$, to the resultant sum is negligible (of order $o(1)$ as $n \to \infty$ uniformly in y). Thus, for each fixed K, we have the relation

$$\pi_n^{(\beta)}(y, y+\Delta] = \sum_{k=K}^{n-1} \pi_k^{(\beta)} P^{**}(P^*)^{n-1-k}(y, y+\Delta] + o(1) \tag{50}$$

as $n \to \infty$ uniformly in y. Recall that $\pi_k^{(\beta)}P^{**}$ converges in total variation to the measure $\pi^{(\beta)}P^{**}$ as $k \to \infty$ (see Lemma 7). Hence, our immediate goal is to make such a change of measures in (50) and prove the following relation as $n \to \infty$ uniformly in y:

$$\pi_n^{(\beta)}(y, y+\Delta] = \sum_{k=0}^{n-1} \pi^{(\beta)} P^{**}(P^*)^{n-1-k}(y, y+\Delta] + o(1). \tag{51}$$

Justify the passage from (50) to (51).

Given A, we have

$$\begin{aligned}
&\pi_k^{(\beta)} P^{**} (P^*)^{n-1-k}(y, y+\Delta] \\
&\quad = \left(\int_{-\infty}^{A} + \int_{A}^{\infty} \right) (P^*)^{n-1-k}\big(u, (y, y+\Delta]\big)\big(\pi_k^{(\beta)} P^{**}\big)(du) \\
&\quad \equiv I_1(k, A) + I_2(k, A).
\end{aligned}$$

Using (48), we can estimate the second integral as follows:

$$\begin{aligned}
\big|I_2(k, A)\big| &= \left| \int_A^{\infty} (P^*)^{n-1-k}\big(u, (y, y+\Delta]\big) f_k^{**}(u)\big(\pi^{(\beta)} P^{**}\big)(du) \right| \\
&\le \rho \int_A^{\infty} (P^*)^{n-1-k}\big(u, (y, y+\Delta]\big)\big|\pi^{(\beta)} P^{**}\big|(du).
\end{aligned}$$

Hence,

$$\begin{aligned}
\sup_n \left| \sum_{k=0}^{n-1} I_1(k, A) \right| &\le \rho \int_A^{\infty} \sum_{k=0}^{n-1} (P^*)^{n-1-k}\big(u, (y, y+\Delta]\big)\big|\pi^{(\beta)} P^{**}\big|(du) \\
&\to \rho \big|\pi^{(\beta)} P^{**}\big|(A, \infty)\Delta/\alpha
\end{aligned}$$

by Theorem 4. Thus,

$$\lim_{A \to \infty} \sup_n \left| \sum_{k=0}^{n-1} I_1(k, A) \right| \to 0. \tag{52}$$

Now, consider the integrals $I_1(k, A)$ for a *fixed* A. We have the estimates

$$\begin{aligned}
&\left| I_1(k, A) - \int_{-\infty}^{A} (P^*)^{n-1-k}\big(u, (y, y+\Delta]\big)\big(\pi^{(\beta)} P^{**}\big)(du) \right| \\
&\quad \le \int_{-\infty}^{A} \big|\pi_k^{(\beta)} P^{**} - \pi^{(\beta)} P^{**}\big|(du) \\
&\quad = \big|\pi_k^{(\beta)} P^{**} - \pi^{(\beta)} P^{**}\big|(-\infty, A) \\
&\quad \le \big|\pi_k^{(\beta)} - \pi^{(\beta)}\big|(-\infty, A) \sup_{u \in \mathbb{R}} |P^{**}|(u, \mathbb{R}) \\
&\quad \le e^{\beta A} |\pi_k - \pi|(-\infty, A) \sup_{u \in \mathbb{R}} |P^{**}|(u, \mathbb{R}).
\end{aligned}$$

From this and (46) it follows that, for a sufficiently large K, the sum

$$\sum_{k=K}^{n-1} I_1(k,A)$$

can be arbitrarily close to the sum

$$\sum_{k=K}^{n-1} \int_{-\infty}^{A} (P^*)^{n-1-k}\big(u,(y,y+\Delta]\big)\big(\pi^{(\beta)}P^{**}\big)(du)$$

uniformly in all n. Together with (52), this justifies the passage from (50) to (51).

From (51), using Theorem 4 and taking the equality (36) and Corollary 2 into account, we deduce the asymptotic equality

$$\begin{aligned}\pi_n^{(\beta)}(y,y+\Delta] &= \big(\pi^{(\beta)}P^{**}\big)(\mathbb{R})\frac{\Delta}{\alpha}\Phi_{\sigma^2}\left(\frac{n\alpha-y}{\sqrt{y/\alpha}}\right)+o(1)\\ &= c\beta\Delta\Phi_{\sigma^2}\left(\frac{n\alpha-y}{\sqrt{y/\alpha}}\right)+o(1) \quad \text{as } y\to\infty,\ n\to\infty. \end{aligned} \tag{53}$$

Applying the inverse Cramér transform (13) to the Markov evolution of masses $X_n^{(\beta)}$, we obtain

$$\mathbb{P}\{X_n>x\}=\int_x^\infty e^{-\beta y}\pi_n^{(\beta)}(dy).$$

Therefore, for every $\Delta>0$, the upper estimate

$$\mathbb{P}\{X_n>x\}\le\sum_{k=0}^{\infty} e^{-\beta(x+k\Delta)}\pi_n^{(\beta)}(x+k\Delta,x+k\Delta+\Delta]\equiv s_1(\Delta)$$

and the lower estimate

$$\mathbb{P}\{X_n>x\}\ge\sum_{k=0}^{\infty} e^{-\beta(x+k\Delta+\Delta)}\pi_n^{(\beta)}(x+k\Delta,x+k\Delta+\Delta]\equiv s_2(\Delta).$$

are valid. The ratio $s_1(\Delta)/s_2(\Delta)$ of the upper bound to the lower bound equals $e^{\beta\Delta}$ and tends to 1 as $\Delta\to0$. For each fixed $\Delta>0$, from (53) we deduce the relation

$$\begin{aligned} s_1(\Delta) &= o(1)\sum_{k=0}^{\infty} e^{-\beta(x+k\Delta)} + c\Delta\beta\sum_{k=0}^{\infty} e^{-\beta(x+k\Delta)}\Phi_{\sigma^2}\left(\frac{n\alpha-(x+k\Delta)}{\sqrt{(x+k\Delta)/\alpha}}\right)\\ &= o\big(e^{-\beta x}\big)+c\Delta\beta\Phi_{\sigma^2}\left(\frac{n\alpha-x}{\sqrt{x/\alpha}}\right)\sum_{k=0}^{\infty} e^{-\beta(x+k\Delta)}\\ &= o\big(e^{-\beta x}\big)+c\frac{\Delta\beta}{1-e^{-\beta\Delta}}e^{-\beta x}\Phi_{\sigma^2}\left(\frac{n\alpha-x}{\sqrt{x/\alpha}}\right), \end{aligned}$$

which implies the asymptotic

$$\mathbb{P}\{X_n > x\} = c\Phi_{\sigma^2}\left(\frac{n\alpha - x}{\sqrt{x/\alpha}}\right)e^{-\beta x} + o\big(e^{-\beta x}\big) \tag{54}$$

as $n \to \infty$ and $x \to \infty$. For each fixed n, we have $\mathbb{P}\{X_n > x\} = o\big(e^{-\beta x}\big)$. Hence, (54) holds as $x \to \infty$ uniformly in $n \geq 0$.

So, the theorem is proven only under the additional conditions: existence of a stochastic minorant for the family of jumps $\{\xi^*(x)\}$ and absolute continuity of π_n with respect to the invariant measure π.

7. Completion of the proof of Theorem 6

In this section we construct an auxiliary Markov chain $\widetilde{Z}_n$ that is equivalent to the original chain X_n from the point of view of the large deviation probabilities but at the same time satisfies the additional conditions imposed on the chain X_n during the proof in the preceding section.

By the above construction and (30), the mean

$$\begin{aligned}\mathbb{E}\Big\{e^{\beta|\xi^{(\beta)}(u)|};\ \xi^{(\beta)}(u) \leq 0\Big\} &= \frac{1}{P^{(\beta)}(u,\mathbb{R})}\int_{-\infty}^{0} e^{-\beta y}\mathbb{P}\big\{\xi^{(\beta)}(u) \in dy\big\} \\ &= \frac{1}{\mathbb{E}\,e^{\beta\xi(u)}}\int_{-\infty}^{0}\mathbb{P}\big\{\xi(u) \in dy\big\} \\ &\leq \frac{1}{\mathbb{E}\,e^{\beta\xi(u)}}\end{aligned}$$

is bounded uniformly in $u \in \mathbb{R}$. In particular, the squares of the negative parts of the random variables $\xi^{(\beta)}(u)$, $u \in \mathbb{R}$, are uniformly integrable. Together with the weak convergence $\xi^{(\beta)}(u) \Rightarrow \xi^{(\beta)}$, this implies the existence of level $U \in \mathbb{R}$ such that the family $\big\{\xi^{(\beta)}(u),\ u > U\big\}$ possesses a minorant with positive mean and finite second moment. Choose sufficiently large U such that (40) is satisfied and $\inf_{n\geq 0}\mathbb{P}\{X_n \leq U\} > 0$.

Enlarge the chain X_n by merging the states on the half-line $(-\infty, U]$ into one state U, i.e., consider the Markov chain Z_n taken values on the half-line $[U,\infty)$ with initial state $Z_0 = \max\{U, X_0\}$ and with the following transition probabilities $P_{Z,n}$ *nonhomogeneous in time*:

$$P_{Z,n}(u,B) = P_Z(u,B) = P(u,B) \ \text{ if } \ u > U \text{ and } B \subseteq (U,\infty);$$

$$P_{Z,n}\big(u,\{U\}\big) = P_Z\big(u,\{U\}\big) = P\big(u,(-\infty,U]\big) \ \text{ if } \ u > U;$$

$$P_{Z,n}(U,B) = \frac{1}{\mathbb{P}\{X_n \leq U\}}\int_{-\infty}^{U} P(u,B)\mathbb{P}\{X_n \in du\} \ \text{ if } \ B \subseteq (U,\infty);$$

$$P_{Z,n}\big(U,\{U\}\big) = \frac{1}{\mathbb{P}\{X_n \leq U\}}\int_{-\infty}^{U} P\big(u,(-\infty,U]\big)\mathbb{P}\{X_n \in du\}.$$

Note that only the transition probabilities from the state U can be nonhomogeneous in time. Moreover, in view of the convergence in variation (1), we have the asymptotic time-homogeneity as $n \to \infty$:

$$P_{Z,n}(U,B) \to \frac{1}{\pi(-\infty,U]} \int_{-\infty}^{U} P(u,B)\pi(du) \quad \text{if} \quad B \subseteq (U,\infty);$$
$$P_{Z,n}\big(U,\{U\}\big) \to \frac{1}{\pi(-\infty,U]} \int_{-\infty}^{U} P\big(u,(-\infty,U]\big)\pi(du).$$

By the construction of the initial distribution of Z_0 and the transition probabilities of the chain Z_n, we have

$$\mathbb{P}\{Z_n = U\} = \mathbb{P}\{X_n \le U\},$$
$$\mathbb{P}\{Z_n > x\} = \mathbb{P}\{X_n > x\} \quad \text{for} \quad x > U.$$

Consider one more chain, say $\widetilde{Z}_n$, with the atom U and initial state $\widetilde{Z}_0 = U$. Its transition probabilities $\widetilde{P}(u,\cdot)$ are equal to $P_Z(u,\cdot)$ for $u > U$, and those for $u = U$ are equal to

$$\widetilde{P}(U,B) = \frac{1}{\pi(-\infty,U]} \int_{-\infty}^{U} P(u,B)\pi(du) \quad \text{if} \quad B \subseteq (U,\infty);$$
$$\widetilde{P}\big(U,\{U\}\big) = \frac{1}{\pi(-\infty,U]} \int_{-\infty}^{U} P\big(u,(-\infty,U]\big)\pi(du).$$

The transition probabilities of the chain $\widetilde{Z}_n$ are time-homogeneous. The invariant measure $\widetilde{\pi}$ of this chain coincides with the measure π on the set (U,∞), and $\widetilde{\pi}\big(\{U\}\big) = \pi(-\infty,U]$. The jumps $\widetilde{\xi}(u)$ of the chain $\widetilde{Z}_n$ possess a minorant with positive mean and finite variance. This chain with atom is geometrically ergodic (see, for example, [10, Section 15]). Thus, $|\pi_n - \pi|(\mathbb{R}) = o(r^n)$ for some $r < 1$, and the condition (46) is fulfilled. Moreover, for every $B \in \mathcal{B}(U,\infty)$, we have the equality

$$\mathbb{P}\big\{\widetilde{Z}_n \in B\big\} = \sum_{k=0}^{n-1} \mathbb{P}\big\{\widetilde{Z}_k = U\big\}\mathbb{P}\Big\{\widetilde{Z}_{k+1} > U, \dots, \widetilde{Z}_{n-1} > U,\, \widetilde{Z}_n \in B \;\Big|\; \widetilde{Z}_k = U\Big\}.$$

Hence,

$$\mathbb{P}\big\{\widetilde{Z}_n \in B\big\} \le \sum_{k=0}^{n-1} \mathbb{P}\Big\{\widetilde{Z}_{k+1} > U, \dots, \widetilde{Z}_{n-1} > U,\, \widetilde{Z}_n \in B \;\Big|\; \widetilde{Z}_k = U\Big\};$$

on the other hand, there exists $\varepsilon > 0$ such that

$$\mathbb{P}\big\{\widetilde{Z}_m \in B\big\} \ge \varepsilon \sum_{k=0}^{m-1} \mathbb{P}\Big\{\widetilde{Z}_{k+1} > U, \dots, \widetilde{Z}_{m-1} > U,\, \widetilde{Z}_m \in B \;\Big|\; \widetilde{Z}_k = U\Big\}.$$

It follows that, for every $n < m$, the measure $\mathbb{P}\{\widetilde{Z}_n \in \cdot\}$ is absolutely continuous with respect to the measure $\mathbb{P}\{\widetilde{Z}_m \in \cdot\}$ and the corresponding density is bounded by $1/\varepsilon$. Thus, for every n, the measure $\mathbb{P}\{\widetilde{Z}_n \in \cdot\}$ is absolutely continuous with respect to the measure $\widetilde{\pi}$ too; the density is also bounded by $1/\varepsilon$.

Thus, the chain $\widetilde{Z}_n$ is satisfied all additional conditions (45)–(47) of the preceding section. Therefore, the equivalence

$$\mathbb{P}\{\widetilde{Z}_n > x\} = \widetilde{c}\,\Phi_{\sigma^2}\Big(\frac{x - n\alpha}{\sqrt{x/\alpha}}\Big)e^{-\beta x} + o\big(e^{-\beta x}\big) \tag{55}$$

uniform in $n \geq 0$ holds as $x \to \infty$, with

$$\begin{aligned}\widetilde{c} &= \frac{1}{\alpha\beta}\int_U^\infty \Big(\mathbb{E}\,e^{\beta\widetilde{\xi}(u)} - 1\Big)e^{\beta u}\widetilde{\pi}(du) \\ &= \frac{1}{\alpha\beta}\Big(\int_{U+0}^\infty \Big(\mathbb{E}\,e^{\beta(u+\widetilde{\xi}(u))} - e^{\beta u}\Big)\pi(du) + \Big(\mathbb{E}\,e^{\beta(U+\widetilde{\xi}(U))} - e^{\beta U}\Big)\pi(-\infty, U]\Big) \\ &= \frac{1}{\alpha\beta}\int_{-\infty}^\infty \Big(\mathbb{E}\,f\big(u+\xi(u)\big) - f(u)\Big)\pi(du);\end{aligned} \tag{56}$$

here $f(u) = \max\{e^{\beta U}, e^{\beta u}\}$. The values of the function $g(u) = e^{\beta u} - f(u)$ lie between $-e^{\beta U}$ and 0. Hence, if the chain X_n is in the stationary regime, i.e., if X_n is distributed according to π, then $\mathbb{E}\,g(X_{n+1}) = \mathbb{E}\,g(X_n)$. Therefore, the *equilibrium-type identity*

$$\int_{-\infty}^\infty \Big(\mathbb{E}\,g\big(u+\xi(u)\big) - g(u)\Big)\pi(du) = 0$$

is valid. Dividing this identity by $\alpha\beta$ and summing with (56), we obtain the final representation for the constant $\widetilde{c}$:

$$\widetilde{c} = \frac{1}{\alpha\beta}\int_{-\infty}^\infty \Big(\mathbb{E}\,e^{\beta(u+\xi(u))} - e^{\beta u}\Big)\pi(du).$$

Let $Z_n^{(\beta)}$ and $\widetilde{Z}_n^{(\beta)}$ be the Markov evolution of masses obtained from the chains Z_n and $\widetilde{Z}_n$ respectively by the Cramér transform with parameter β. Analyze the measures $\mathrm{Mes}\{Z_n^{(\beta)} \in \cdot\}$ and $\mathrm{Mes}\{\widetilde{Z}_n^{(\beta)} \in \cdot\}$ from the point of view of the last entry into the point U. With regard to the first measure, we have

$$\begin{aligned}
&\mathrm{Mes}\Big\{Z_n^{(\beta)} \in [y, y+\Delta)\Big\}\\
&\quad = \mathrm{Mes}\Big\{Z_0^{(\beta)} > U, \dots, Z_{n-1}^{(\beta)} > U,\ Z_n^{(\beta)} \in [y, y+\Delta)\Big\} + \sum_{k=0}^{n-1} \mathrm{Mes}\Big\{Z_k^{(\beta)} = U\Big\}\\
&\qquad \times \mathrm{Mes}\Big\{Z_{k+1}^{(\beta)} > U, \dots, Z_{n-1}^{(\beta)} > U,\ Z_n^{(\beta)} \in [y, y+\Delta) \ \Big|\ Z_k^{(\beta)} = U\Big\}. \qquad (57)
\end{aligned}$$

Recall that (40) holds. In view of (the central limit) Theorem 5, the value of

$$\mathrm{Mes}\Big\{Z_0^{(\beta)} > U, \dots, Z_{n-1}^{(\beta)} > U,\ Z_n^{(\beta)} \in [y, y+\Delta)\Big\}$$

and, for each fixed k, the value of

$$\mathrm{Mes}\Big\{Z_{k+1}^{(\beta)} > U, \dots, Z_{n-1}^{(\beta)} > U,\ Z_n^{(\beta)} \in [y, y+\Delta) \ \Big|\ Z_k^{(\beta)} = U\Big\}$$

are of order $o(1)$ as $n \to \infty$ uniformly in y. Hence, the replacement of each finite (with respect to k) set of the transition probabilities among $P_{Z,k}^{(\beta)}(U, \cdot)$ with the probabilities $\widetilde{P}^{(\beta)}(U, \cdot)$ changes $\mathrm{Mes}\big\{Z_n^{(\beta)} \in (y, y+\Delta]\big\}$ by the value of order $o(1)$. Taking the relation

$$\mathrm{Mes}\big\{Z_k^{(\beta)} = U\big\} = e^{\beta U}\mathbb{P}\{Z_k = U\} \to e^{\beta U}\pi(-\infty, U]$$

as $k \to \infty$ into account, from (57) we deduce that

$$\begin{aligned}
&\mathrm{Mes}\Big\{Z_n^{(\beta)} \in [y, y+\Delta)\Big\}\\
&\quad = o(1) + e^{\beta U}\pi(-\infty, U] \sum_{k=0}^{n-1} \mathrm{Mes}\Big\{\widetilde{Z}_{k+1}^{(\beta)} > U, \dots, \widetilde{Z}_{n-1}^{(\beta)} > U,\\
&\qquad\qquad \widetilde{Z}_n^{(\beta)} \in [y, y+\Delta) \ \Big|\ \widetilde{Z}_k^{(\beta)} = U\Big\}
\end{aligned}$$

as $n \to \infty$ uniformly in y. By the same reasons, we can obtain the same relation for the time-homogeneous Markov evolution of masses $\widetilde{Z}_n^{(\beta)}$:

$$\begin{aligned}
&\mathrm{Mes}\Big\{\widetilde{Z}(\beta)_n \in [y, y+\Delta)\Big\}\\
&\quad = o(1) + e^{\beta U}\pi(-\infty, U] \sum_{k=0}^{n-1} \mathrm{Mes}\Big\{\widetilde{Z}_{k+1}^{(\beta)} > U, \dots, \widetilde{Z}_{n-1}^{(\beta)} > U,\\
&\qquad\qquad \widetilde{Z}_n^{(\beta)} \in [y, y+\Delta) \ \Big|\ \widetilde{Z}_k^{(\beta)} = U\Big\}.
\end{aligned}$$

Thus,

$$\mathrm{Mes}\Big\{Z_n^{(\beta)} \in [y, y+\Delta)\Big\} = \mathrm{Mes}\Big\{\widetilde{Z}_n^{(\beta)} \in [y, y+\Delta)\Big\} + o(1).$$

Applying the inverse Cramér transform (with parameter $-\beta$, see Lemma 1) to the measures $\mathrm{Mes}\{Z_n^{(\beta)} \in \cdot\}$ and $\mathrm{Mes}\{\widetilde{Z}_n^{(\beta)} \in \cdot\}$, we obtain the relation

$$\mathbb{P}\{Z_n > x\} = \mathbb{P}\{\widetilde{Z}_n > x\} + o(1)$$

uniformly in x as $n \to \infty$, which, together with (55), completes the proof.

8. The lattice case

Let X_n be a Markov chain taken values on the lattice $\{k\Delta,\ k \in \mathbb{Z}\}$ with span $\Delta > 0$, and this lattice is minimal. Consider an asymptotically homogeneous chain, i.e., $\xi(k\Delta) \Rightarrow \xi$ as $k \to \infty$; the values of the variable ξ are proportional to Δ. We formulate the corresponding theorem on large deviation probabilities.

Theorem 7. *Let* $\mathbb{E}\, e^{\beta X_0}$ *be finite. Assume that the family of jumps* $\big\{\xi(k\Delta),\ k \in \mathbb{Z}\big\}$ *possesses a stochastic majorant* $\bar{\xi}$ *such that* $\mathbb{E}\, \bar{\xi}^2 e^{\beta\bar{\xi}} < \infty$. *Let the jumps of the chain satisfy the following conditions*:

$$\inf_{k\in\mathbb{Z}} \mathbb{E}\, e^{\beta\xi(k\Delta)} > 0,$$

$$\mathbb{E}\xi(k\Delta)e^{\beta\xi(k\Delta)} = \alpha + o(1/\sqrt{k}) \quad \text{as} \quad k \to \infty.$$

In addition, assume that there exists a bounded decreasing sequence $\delta(k) = o(1/k)$ *summable at infinity and such that the inequality*

$$\sup_{\lambda\in[-\pi/\Delta, \pi/\Delta]} \Big| \mathbb{E}\, e^{(\beta+i\lambda)\xi(k\Delta)} - \mathbb{E}\, e^{(\beta+i\lambda)\xi} \Big| \le \delta(k)$$

holds for every $k \in \mathbb{Z}$. *Then*

$$\mathbb{P}\big\{X_n = m\Delta\big\} = c_\Delta e^{-\beta m\Delta} \Phi_{\sigma^2}\left(\frac{n\alpha - m\Delta}{\sqrt{m\Delta/\alpha}}\right) + o\big(e^{-\beta m\Delta}\big)$$

uniformly in $n \ge 0$ *as* $m \to \infty$, *where*

$$c_\Delta = \frac{\Delta}{\mathbb{E}\xi e^{\beta\xi}} \sum_{k\in\mathbb{Z}} \Big(\mathbb{E}\, e^{\beta\xi(k\Delta)} - 1\Big) e^{\beta k\Delta} \pi(k\Delta) \in [0, \infty).$$

Proof can be carried out in the same way as in the nonlattice case. The only difference is generated by the lattice variant of the local renewal theorem, which implies the different multiple in the final asymptotic of the probability $\mathbb{P}\{X_n \ge m\Delta\}$.

9. On the positivity of the multiplier c in Theorem 6

As was noted just after Theorem 2, the constant

$$c \equiv \frac{1}{\beta\alpha} \int_{-\infty}^{\infty} \Big(\mathbb{E}\, e^{\beta\xi(y)} - 1 \Big) e^{\beta y}\, \pi(dy) \tag{58}$$

is positive if $\mathbb{E}\, e^{\beta\xi(y)} \geq 1 - \gamma(y)$, where $\gamma(y) \geq 0$ for every y and

$$\int_0^\infty \gamma(y) y \log y\, dy < \infty.$$

In the following theorem this condition is somehow weakened. The proof is a substantially improved version of the proof of Theorem 5 in [2, Section 27].

Theorem 8. *Let the chain X_n be asymptotically space-homogeneous, i.e., $\xi(y) \Rightarrow \xi$ as $y \to \infty$. Assume that $\mathbb{E}\xi < 0$ and there exists $\beta > 0$ such that $\mathbb{E}\, e^{\beta\xi} = 1$. Let $\mathbb{E}\, e^{\beta\xi(y)} \geq 1 - \gamma(y)$, where $\gamma(y)$ is a nonnegative decreasing function, $\gamma(y) = o(1/y)$ as $y \to \infty$, and*

$$\int_0^\infty \gamma(y) y\, dy < \infty. \tag{59}$$

Let π be an arbitrary probability invariant distribution of the chain X_n such that $\pi(y, \infty) > 0$ for every y and $\pi(y, \infty) = O\big(e^{-\beta y}\big)$ as $y \to \infty$. Then the constant c defined by the equality (58) is positive.

Proof. Prove first that, under the conditions of the theorem, the equality

$$\int_{-\infty}^{\infty} e^{\beta u} \pi(du) = \infty \tag{60}$$

is valid.

Enlarge the chain X_n by averaging the states on the half-line $(-\infty, U]$ with respect to the measure π and by merging them into one state U; namely, consider the Markov chain $X_{U,n}$ taken values on the half-line $[U, \infty)$ with the following transition probabilities P_U:

$$P_U(y, B) = P(y, B) \;\text{ if }\; y > U \text{ and } B \subseteq (U, \infty);$$

$$P_U\big(y, \{U\}\big) = P\big(y, (-\infty, U]\big) \;\text{ if }\; y > U;$$

$$P_U(U, B) = \frac{1}{\pi(-\infty, U]} \int_{-\infty}^{U} P(u, B) \pi(du) \;\text{ if }\; B \subseteq (U, \infty);$$

$$P_U\big(U, \{U\}\big) = \frac{1}{\pi(-\infty, U]} \int_{-\infty}^{U} P\big(u, (-\infty, U]\big) \pi(du).$$

By the construction of the transition probabilities P_U, the invariant measure π_U of the chain $X_{U,n}$ coincides with the measure π on the set (U,∞), and $\pi_U\big(\{U\}\big)$ equals $\pi(-\infty,U]$. For every $y>U$, the jumps $\xi_U(y)$ of the chain $X_{U,n}$ satisfy the stochastic inequality

$$\xi_U(y) \geq_{\rm st} \xi(y). \tag{61}$$

Choose a level U so large that, for every $u>U$, the inequality

$$u\Big(\mathbb{E}\,e^{\beta\xi(u)}-1\Big)+\mathbb{E}\,\xi(u)e^{\beta\xi(u)}>0 \tag{62}$$

hold. This choice is possible due to $\gamma(u)=o(1/u)$ and

$$\liminf_{u\to\infty}\mathbb{E}\,\xi(u)e^{\beta\xi(u)} \geq \mathbb{E}\,\xi e^{\beta\xi}\in(0,\infty]$$

in view of the weak convergence $\xi(u)\Rightarrow\xi$.

Assume now that the integral in (60) is finite. Then, for every $\lambda\in[0,\beta]$, the mean drift of the exponent $e^{\lambda X_{U,n}}$ for one step in the stationary regime π_U is equal to zero, i.e., the following *equilibrium-type identity* holds:

$$\int_U^\infty e^{\lambda u}\Big(\mathbb{E}\,e^{\lambda\xi_U(u)}-1\Big)\pi_U(du)=0. \tag{63}$$

Differentiating this equality with respect to λ, we obtain

$$\int_U^\infty e^{\lambda u}u\Big(\mathbb{E}\,e^{\lambda\xi_U(u)}-1\Big)\pi_U(du)+\int_U^\infty e^{\lambda u}\mathbb{E}\,\xi_U(u)e^{\lambda\xi_U(u)}\pi_U(du)=0$$

for every $\lambda\in[0,\beta]$. Putting $\lambda=\beta$, we arrive at the equality

$$\int_U^\infty e^{\beta u}\Big[u\Big(\mathbb{E}\,e^{\beta\xi_U(u)}-1\Big)+\mathbb{E}\,\xi_U(u)e^{\beta\xi_U(u)}\Big]\pi_U(du)=0,$$

which cannot be true in view of (62), (61), and $\pi(U,\infty)>0$. Thus, (60) is proven.

Further, assume that $c=0$ in (58). Then, as it was demonstrated in the process of calculating the constant $\widetilde{c}$ in (56), *for each* level U, the equilibrium-type identity (63) is valid for $\lambda=\beta$. Therefore,

$$\Big(\mathbb{E}\,e^{\beta\xi_U(U)}-1\Big)e^{\beta U}\pi(-\infty,U]=\int_{U+0}^\infty\Big(1-\mathbb{E}\,e^{\beta\xi_U(u)}\Big)e^{\beta u}\pi(du). \tag{64}$$

By the definition of the transition probabilities P_U, the right-hand side of the last equality is equal to

$$e^{\beta U}\int_{-\infty}^U\mathbb{E}\,\max\Big\{0,e^{\beta\big(u+\xi(u)-U\big)}-1\Big\}\pi(du).$$

Since $\xi(u) \Rightarrow \xi$, there exist $\varepsilon > 0$ and $\delta > 0$ such that, for all sufficiently large U, the inequality $\mathbb{E}\max\{0, e^{\beta(u+\xi(u)-U)} - 1\} \ge \delta$ holds for every $u \in (U-\varepsilon, U]$ and, hence,

$$\Big(\mathbb{E}e^{\beta\xi_U(U)} - 1\Big)e^{\beta U}\pi(-\infty, U] \ge \delta e^{\beta U}\pi(U-\varepsilon, U]. \tag{65}$$

By (61) and the conditions of the theorem, the right-hand side of (64) does not exceed

$$\int_{U+0}^{\infty} \gamma(y)e^{\beta y}\pi(dy) \le \bar{c}\int_{U+1}^{\infty} \gamma(y)dy, \tag{66}$$

where $\bar{c} < \infty$. Inserting (65) and (66) into (64), we arrive at the inequality

$$e^{\beta U}\pi(U-\varepsilon, U] \le \frac{\bar{c}}{\delta}\int_{U+1}^{\infty} \gamma(y)dy. \tag{67}$$

Since this holds for all sufficiently large U, from the condition (59) it follows that the exponential moment of order β of the distribution π is finite, which contradicts (60). Thus, the assumption $c = 0$ leads to contradiction, and the theorem is proven.

The condition (59) can be considerably weakened by imposing stronger moment conditions on the jumps of X_n. If, for example, the chain has bounded jumps, i.e., if there exists a constant $A < \infty$ such that $|\xi(y)| \le A$ with probability 1 for every y, then it suffices to assume that

$$\int_0^{\infty} \gamma(y)y^{\mu}\,dy < \infty \tag{68}$$

for some $\mu > 0$. In order to prove this, observe that, in this case, we have

$$\int_1^{\infty} \frac{e^{\beta u}}{u^{1-\mu}}\pi(du) = \infty. \tag{69}$$

Indeed, since the mean drift $v_\lambda(y) \equiv \mathbb{E}V_\lambda\big(y+\xi(y)\big) - V_\lambda(y)$ of the test function $V_\lambda(y) = e^{\lambda y}y^{\mu-1}\mathbf{I}\{y > 1\}$ is equal to

$$\frac{e^{\lambda y}}{y^{1-\mu}}\left(\mathbb{E}e^{\lambda\xi(y)}\left(\frac{1}{1+\xi(y)/y}\right)^{1-\mu} - 1\right)$$

for $y > 1 + A$, under the jump boundedness condition, we have

$$\begin{aligned}\frac{dv_\lambda(y)}{d\lambda}\bigg|_{\lambda=\beta} &= \frac{ye^{\beta y}}{y^{1-\mu}}\Big(\mathbb{E}\,e^{\beta\xi(y)} - 1 - \frac{1-\mu}{y}\mathbb{E}\xi(y)e^{\beta\xi(y)} + O(1/y^2)\Big) \\ &\quad + \frac{e^{\beta y}}{y^{1-\mu}}\Big(\mathbb{E}\xi(y)e^{\beta\xi(y)} + O(1/y)\Big) \\ &= \frac{ye^{\beta y}}{y^{1-\mu}}\Big(o(1/y) - \frac{1-\mu}{y}\mathbb{E}\xi e^{\beta\xi} + O(1/y^2)\Big) \\ &\quad + \frac{e^{\beta y}}{y^{1-\mu}}\Big(\mathbb{E}\xi e^{\beta\xi} + O(1/y)\Big) \\ &= \frac{e^{\beta y}}{y^{1-\mu}}\Big(o(1) + \mu\mathbb{E}\xi e^{\beta\xi}\Big), \quad y \to \infty,\end{aligned}$$

which is a positive value for large y and implies (69) (as was observed in the preceding proof). Further, multiplying (67) by $U^{\mu-1}$ and summing up, we arrive at the inequalities

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{e^{\beta(U+k\varepsilon)}}{(U+k\varepsilon)^{1-\mu}}\pi\big(U+(k-1)\varepsilon, U+k\varepsilon\big] \\ &\le \frac{\bar{c}}{\delta}\sum_{k=0}^{\infty}\frac{1}{(U+k\varepsilon)^{1-\mu}}\int_{U+k\varepsilon+1}^{\infty}\gamma(y)dy \\ &\le c^*\int_U^\infty \gamma(y)y^\mu dy,\end{aligned}$$

where $c^* < \infty$, which imply the finiteness of the integral in (69) in view of the condition (68). This contradiction completes the proof.

10. On necessity of the condition for integrability of the convergence rate of the jump distribution to the limit distribution

In conclusion, we construct an example of Markov chain, which demonstrates that the condition (32) on the convergence rate of the jump distribution to the limit distribution F is so significant that it can be considered as almost necessary.

Consider a Markov chain X_n with values in $\mathbb{Z}^+$. Assume that the chain is continuous from above as well as from below, i.e., the chain changes its value at most by 1 in one step. Denote

$$p(k,k-1) \equiv \mathbb{P}\{X_{n+1}=k-1 \mid X_n=k\}, \;\; k \ge 1,$$
$$p(k,k+1) \equiv \mathbb{P}\{X_{n+1}=k+1 \mid X_n=k\}, \;\; k \ge 0,$$
$$p(0,0) \equiv \mathbb{P}\{X_{n+1}=0 \mid X_n=0\}.$$

Assume that

$$p(k,k-1)+p(k,k+1)=1, \quad p(0,0)+p(0,1)=1,$$

and

$$\sum_{k=1}^{\infty}\prod_{j=0}^{k-1}\frac{p(j,j+1)}{p(j+1,j)}<\infty.$$

It is known (see, for example, [8, Chapter 3, Section 7]) that the last condition is necessary and sufficient for the ergodicity of chain of this type. Denote the stationary probabilities of the chain by $\{\pi(k),\, k \in \mathbb{Z}^+\}$. The special simplicity of the system of equations

$$\pi(k+1)p(k+1,k)+\pi(k-1)p(k-1,k)=\pi(k), \quad k \ge 1,$$
$$\pi(1)p(1,0)+\pi(0)p(0,0)=\pi(0),$$
$$\sum_{k=0}^{\infty}\pi(k)=1$$

for the stationary probabilities $\{\pi(k)\}$ allows us to compute the stationary probabilities in explicit form (see again [8, Chapter 3, Section 7]):

$$\pi(k)=\pi(0)\prod_{j=0}^{k-1}\frac{p(j,j+1)}{p(j+1,j)}, \quad k \ge 1,$$

where

$$\pi(0)=\left(1+\sum_{k=1}^{\infty}\prod_{j=0}^{k-1}\frac{p(j,j+1)}{p(j+1,j)}\right)^{-1}.$$

Let $p(k,k+1) \to p$ and, therefore, $p(k,k-1) \to 1-p$ as $k \to \infty$, so that the limit distribution F is a Bernoulli distribution with parameter p and the Laplace transform

$$\varphi(\lambda)=pe^{\lambda}+(1-p)e^{-\lambda}.$$

In order to have an ergodic chain, we assume that $p < 1/2$. The unique nonzero solution β to the equation $\varphi(\lambda) = 1$ is equal to

$$\beta = \log \frac{1-p}{p} > 0.$$

Under the above conditions, the integrability condition (32) for the rate of convergence to the limit distribution is equivalent to the following:

$$\sum_{k=0}^{\infty} \bigl|p(k) - p\bigr| < \infty. \tag{70}$$

Theorem 9. *Let* $\varepsilon(k) \equiv p(k, k+1) - p \geq 0$ *for every* k. *Then the following two assertions are equivalent*:

(a) *there exists* $c > 0$ *such that*

$$\pi(k) \sim ce^{-\beta k} \quad \text{as } k \to \infty;$$

(b) *the condition* (70) *holds.*

Proof. We have

$$\pi(k) = \pi(0) \prod_{j=0}^{k-1} \frac{p + \varepsilon(j)}{1 - p - \varepsilon(j+1)} = \pi(0) \left(\frac{p}{1-p}\right)^k \prod_{j=0}^{k-1} \frac{1 + \bigl(\varepsilon(j)/(p)\bigr)}{1 - \bigl(\varepsilon(j+1)/(1-p)\bigr)}.$$

Taking the definition of β into account, we obtain

$$\pi(k) = \pi(0) e^{-\beta k} \prod_{j=0}^{k-1} \frac{1 + \bigl(\varepsilon(j)/(p)\bigr)}{1 - \bigl(\varepsilon(j+1)/(1-p)\bigr)}.$$

Hence, (a) is equivalent to the following:

$$\prod_{j=0}^{\infty} \frac{1 + \bigl(\varepsilon(j)/(p)\bigr)}{1 - \bigl(\varepsilon(j+1)/(1-p)\bigr)} < \infty.$$

In turn, this is equivalent to the convergence of the series

$$\sum_{j=0}^{\infty} \left(\frac{1 + \bigl(\varepsilon(j)/(p)\bigr)}{1 - \bigl(\varepsilon(j+1)/(1-p)\bigr)} - 1 \right) = \sum_{j=0}^{\infty} \frac{\bigl(\varepsilon(j)/(p)\bigr) + \bigl(\varepsilon(j+1)/(1-p)\bigr)}{1 - \bigl(\varepsilon(j+1)/(1-p)\bigr)}.$$

Since $\varepsilon(j) \to 0$, this is equivalent to the inequality

$$\sum_{j=0}^{\infty} \left(\frac{\varepsilon(j+1)}{1-p} + \frac{\varepsilon(j)}{p} \right) < \infty,$$

which is equivalent to (70). The proof is complete.

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