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THE CRITICAL CASE OF THE CRAMÉR–LUNDBERG THEOREM ON THE ASYMPTOTIC TAIL BEHAVIOR OF THE MAXIMUM OF A NEGATIVE DRIFT RANDOM WALK

D. A. Korshunov
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Abstract: We study the asymptotic tail behavior of the maximum $M = \max\{0, S_n, n \geq 1\}$ of partial sums $S_n = \xi_1 + \dots + \xi_n$ of independent identically distributed random variables $\xi_1, \xi_2, \dots$ with negative mean. We consider the so-called Cramér case when there exists a $\beta > 0$ such that $\mathbf{E}e^{\beta\xi_1} = 1$. The celebrated Cramér–Lundberg approximation states the exponential decay of the large deviation probabilities of M provided that $\mathbf{E}\xi_1 e^{\beta\xi_1}$ is finite. In the present article we basically study the critical case $\mathbf{E}\xi_1 e^{\beta\xi_1} = \infty$.

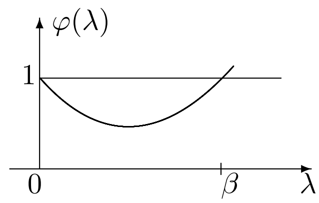
Keywords: maximum of a random walk, probabilities of large deviations, light tails, exponential change of measure, truncated mean value function

Let ξ, ξ_1, ξ_2, ... be a sequence of independent random variables with common distribution F and $S_n = \xi_1 + \dots + \xi_n$, $S_0 = 0$. We assume that $\mathbf{E}\xi < 0$; the case $\mathbf{E}\xi = -\infty$ is not excluded. Then $S_n \to -\infty$ with probability 1 and the maximum $M = \max_{n\geq 0} S_n$ is finite with probability 1. We further assume that $\mathbf{P}\{\xi > 0\} > 0$ and so $\mathbf{P}\{M > 0\} > 0$.

In the present article we consider a distribution F such that the random variable ξ possesses some positive exponential moment $\varphi(\lambda) \equiv \mathbf{E}e^{\lambda\xi}$; i.e., the value of

$$\lambda_+ = \sup\{\lambda \geq 0 : \varphi(\lambda) < \infty\}$$

is not equal to zero. Since $\varphi(0) = 1$, $\varphi'(0) = \mathbf{E}\xi < 0$, and the function $\varphi(\lambda)$ is convex, the typical shape of $\varphi(\lambda)$ is as follows:



Put $\beta = \sup\{\lambda > 0 : \varphi(\lambda) \leq 1\}$. Then $\beta > 0$ and β is finite since $\mathbf{P}\{\xi > 0\} > 0$.

By the results stemming from Lundberg and Cramér, the estimate $\mathbf{P}\{M > x\} \leq e^{-\beta x}$ holds for every $x \geq 0$. Moreover, if $\varphi(\beta) = 1$ and $\varphi'(\beta) < \infty$ then the following asymptotics holds:

$$\mathbf{P}\{M > x\} \sim ce^{-\beta x} \quad \text{as } x \to \infty, \tag{1}$$

where $c > 0$ is some constant (in the nonlattice case whereas in the lattice case x should be taken a multiple of the lattice span), e.g., see [1; 2, Chapter XIII, Section 5; 3, Section 21] or [4, Chapter XII, Section 5]. It was proved in [5] that the converse is true: if the exponential asymptotics (1) holds then $\varphi(\beta) = 1$ and $\varphi'(\beta) < \infty$.

The asymptotic behavior of the tail $\mathbf{P}\{M > x\}$ in the case $\varphi(\beta) = 1$, $\varphi'(\beta) = \infty$ (in this case $\beta = \lambda_+$ with necessity) remains poorly studied. It is well known (e.g., see [4, Chapter XII, Section 5]) that in this

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case $\mathbf{P}\{M > x\} = o(e^{-\beta x})$ as $x \to \infty$ with necessity. To the best of our knowledge the only result on the actual asymptotic behavior of $\mathbf{P}\{M > x\}$ in the case of the infinite value of $\varphi'(\beta)$ can be found in [3, Section 22, Subsection 5]. For the case of a regularly varying function $e^{\beta t}\mathbf{P}\{\xi > t\}$, the asymptotics (also regularly varying) was found in [3] for the integral $\int_0^x e^{\beta t}\mathbf{P}\{M > t\}\,dt$; together with the assumption of the monotonicity of the function under integration (which can be hardly checked in practice) its gives the asymptotic behavior of the function $e^{\beta x}\mathbf{P}\{M > x\}$. In the present article we obtain the asymptotics of $\mathbf{P}\{M > x\}$ as $x \to \infty$ using a novel approach.

So, let $\varphi(\beta) = 1$ and $\varphi'(\beta) = \infty$. Applying the exponential change of measure (the Cramér transform) $G(dx) = e^{\beta x}F(dx)$ we again get a probability distribution. The distribution G has the heavy tail $\overline{G}(x) = G(x, \infty)$; that is, it does not possess finite positive exponential moments. Consider a sequence of independent random variables η, η_1, η_2, ... with common distribution G. Put $T_n = \eta_1 + \cdots + \eta_n$, $T_0 = 0$. We have $\mathbf{E}\eta = \varphi'(\beta) = \infty$. Thus, $T_n \to \infty$ a.s. as $n \to \infty$.

Denote the index of the first nonnegative sum among S_n (the first ascending ladder epoch) by $\sigma = \min\{n \ge 1 : S_n \ge 0\}$; denote the distribution of the corresponding first nonnegative sum (the first ascending ladder hight) by $F_+(B) = \mathbf{P}\{S_\sigma \in B\}$. Since $\mathbf{E}\xi < 0$; the distribution F_+ is defective, i.e., $F_+[0, \infty) < 1$.

Denote the index of the first nonnegative sum among T_n by $\tau = \min\{n \ge 1 : T_n \ge 0\}$ and denote the distribution of the corresponding first nonnegative sum by $G_+(B) = \mathbf{P}\{T_\tau \in B\}$. Since $\mathbf{E}\eta > 0$, the distribution G_+ is proper, i.e., $G_+[0, \infty) = 1$. Moreover (e.g., see [2, Chapter VIII, Theorem 2.3(c)]), $\mathbf{E}\tau < \infty$ and

$$p \equiv \mathbf{P}\{T_n \ge 0 \text{ for all } n \ge 1\} = \frac{1}{\mathbf{E}\tau}. \tag{2}$$

We have $0 < p < 1$. Denote the truncated mean value of the distribution G by

$$m(x) = \int\limits_0^x \overline{G}(y)dy. \tag{3}$$

The function $m(x)$ tends to infinity as $x \to \infty$ if and only if $\mathbf{E}\eta = \varphi'(\beta) = \infty$.

Theorem 1. *Let the distribution F be nonlattice and $\varphi'(\beta) = \infty$. In addition let the tail $\overline{G}(x)$ be regularly varying (as $x \to \infty$) with index $-\alpha$, $\alpha \in (0, 1]$. Then the following lower bound holds:*

$$\liminf_{x\to\infty} \mathbf{P}\{M > x\}m(x)e^{\beta x} \ge \frac{p}{\beta\Gamma(\alpha)\Gamma(2-\alpha)}.$$

If $\alpha \in (1/2, 1]$ then the following asymptotics holds as $x \to \infty$:

$$\mathbf{P}\{M > x\} \sim \frac{p}{\Gamma(\alpha)\Gamma(2-\alpha)}\frac{e^{-\beta x}}{\beta m(x)}.$$

Theorem 2. *Let the distribution F be concentrated on the lattice $\mathbf{Z}^+$ and let $\mathbf{Z}^+$ be the minimal lattice. If $\varphi'(\beta) = \infty$ and $\overline{G}(x)$ is regularly varying with index $-\alpha$, $\alpha \in (0, 1)$, then the following lower bound holds:*

$$\liminf_{k\to\infty} \mathbf{P}\{M = k\}k\overline{G}(k)e^{\beta k} \ge \frac{p \cdot \sin \pi\alpha}{\pi}.$$

If $\alpha \in (1/2, 1)$ then the following asymptotics holds as $k \to \infty$:

$$\mathbf{P}\{M = k\} \sim \frac{p \cdot \sin \pi\alpha}{\pi}\frac{e^{-\beta k}}{k\overline{G}(k)}.$$

The last equivalence holds in the case $\alpha \in (0, 1/2]$ as well, provided that the sequence $G\{k\}$ is ultimately decreasing.

Recall some relations between the functions $m(x)$, $\overline{G}(x)$, and $\overline{F}(x)$ in the nonlattice case. If $\alpha \in [0,1)$ then (see [6, Chapter XIII, Section 5])

$$m(x) \sim \frac{x\overline{G}(x)}{1-\alpha} \quad \text{as } x \to \infty.$$

If the function $e^{\beta x}\overline{F}(x)$ is regularly varying with index $-\gamma$, $\gamma \in (1,2]$, then

$$\overline{G}(x) = -\int_x^\infty e^{\beta y} d\overline{F}(y) = e^{\beta x}\overline{F}(x) + \beta \int_x^\infty \overline{F}(y) e^{\beta y} dy \sim \frac{\beta}{\gamma - 1} x\overline{F}(x) e^{\beta x}.$$

So, if the function $e^{\beta x}\overline{F}(x)$ is regularly varying with index $-\gamma$, $\gamma \in (1,2)$, then the function $\overline{G}(x)$ is regularly varying as well, with index $-\alpha = 1-\gamma$; in this case

$$m(x) \sim \frac{\beta}{(\gamma-1)(2-\gamma)} x^2 \overline{F}(x) e^{\beta x}.$$

On the contrary, the regular variation of the tail $\overline{G}(x)$ does not imply the regular variation of $e^{\beta x}\overline{F}(x)$ in general. In order to guarantee the regular variation of the last function it is necessary to know some information about the *local* behavior of G. That is the main reason why we impose the regular variation condition on $\overline{G}(x)$ rather than on $e^{\beta x}\overline{F}(x)$ in our theorems.

Proof of Theorem 1 follows from Lemmas 1 and 2 to be demonstrated below. Theorem 2 follows from Lemmas 1 and 4.

Lemma 1. *The following equality of measures holds:*

$$\mathbf{P}\{M \in B\} = \int_B e^{-\beta u} H_+(du)$$

for $B \in \mathscr{B}(0,\infty)$, where the renewal measure H_+ is governed by the measure G_+:

$$H_+(du) = \sum_{n=0}^\infty G_+^{*(n)}(du).$$

Proof. For all $n \in \mathbf{Z}^+$ and $u_0, \dots, u_n \in \mathbf{R}$, we have the identities

$$\begin{aligned}
&\mathbf{P}\{S_1 \in du_1, \dots, S_n \in du_n\} \\
&= \mathbf{P}\{\xi_1 \in du_1\}\mathbf{P}\{u_1 + \xi_2 \in du_2\} \dots \mathbf{P}\{u_{n-1} + \xi_n \in du_n\} \\
&= e^{-\beta u_1}\mathbf{P}\{\eta_1 \in du_1\} e^{-\beta(u_2-u_1)}\mathbf{P}\{u_1 + \eta_2 \in du_2\} \\
&\quad \times \dots \times e^{-\beta(u_n - u_{n-1})}\mathbf{P}\{u_{n-1} + \eta_n \in du_n\} \\
&= e^{-\beta u_n}\mathbf{P}\{\eta_1 \in du_1\}\mathbf{P}\{u_1 + \eta_2 \in du_2\} \dots \mathbf{P}\{u_{n-1} + \eta_n \in du_n\} \\
&= e^{-\beta u_n}\mathbf{P}\{T_1 \in du_1, \dots, T_n \in du_n\}.
\end{aligned}$$

Therefore (we can use also the equality (5.1) of [2, Chapter XIII, Section 5]),

$$\mathbf{P}\{S_1 < 0, \dots, S_{n-1} < 0, S_n \in du\} = e^{-\beta u}\mathbf{P}\{T_1 < 0, \dots, T_{n-1} < 0, T_n \in du\}.$$

Summing over $n \geq 1$ yields the equality

$$F_+(du) = e^{-\beta u} G_+(du),$$

which implies the equality of the lemma, since, for $u > 0$,

$$\mathbf{P}\{M \in du\} = \sum_{n=0}^\infty F_+^{*(n)}(du).$$

Lemma 2. *Let the distribution G be long-tailed, that is, $\lim_{x\to\infty} \frac{\overline{G}(x+t)}{\overline{G}(x)} = 1$ for every fixed t. Then $\overline{G}_+(x) \sim \overline{G}(x)/p$ as $x \to \infty$.*

Proof is a light modification of the proof of Lemma 1 in [7]. Define the taboo renewal measure, avoiding the nonnegative half-line:

$$H(B) = \mathbf{I}\{0 \in B\} + \sum_{n=1}^{\infty} \mathbf{P}\{T_1 < 0, \dots, T_n < 0, T_n \in B\}.$$

This measure is finite, since $H(0,\infty) = 0$ and (see (2))

$$H(-\infty, 0] = 1 + \sum_{n=1}^{\infty} \mathbf{P}\{T_1 < 0, \dots, T_n < 0\} = 1 + \sum_{n=1}^{\infty} \mathbf{P}\{\tau > n\} = \mathbf{E}\tau = 1/p < \infty.$$

By the law of total probability

$$\overline{G}_+(x) = \int_{-\infty}^{0} \overline{G}(x-t) H(dt). \tag{4}$$

Therefore,

$$\frac{\overline{G}_+(x)}{\overline{G}(x)} = \int_{-\infty}^{0} \frac{\overline{G}(x-t)}{\overline{G}(x)} H(dt).$$

The function under integration does not exceed 1 and tends to 1 for each fixed t, since G is long-tailed. Thus, the Lebesgue Dominated Convergence Theorem yields the convergence of the integral to $H(-\infty, 0] = 1/p$. This implies the claim of the lemma.

In order to apply Lemma 1 for deriving the tail asymptotics of M, it is necessary to know the *local* behavior of the renewal measure H_+. The corresponding results for the renewal measure in the case of nonlattice distribution with infinite mean were proved in [8, Theorems 1 and 2]: if the function $\overline{G}_+(x)$ is regularly varying at infinity with index $-\alpha$, $\alpha \in (0, 1]$, then, for every fixed $h > 0$,

$$\liminf_{x\to\infty} H_+(x, x+h] \int_0^x \overline{G}_+(y)dy = \frac{h}{\Gamma(\alpha)\Gamma(2-\alpha)}; \tag{5}$$

if $\alpha \in (1/2, 1]$ then

$$\lim_{x\to\infty} H_+(x, x+h] \int_0^x \overline{G}_+(y)dy = \frac{h}{\Gamma(\alpha)\Gamma(2-\alpha)}. \tag{6}$$

To the best of our knowledge no other results are available in the literature on the local renewal theorem in the nonlattice case for $\alpha \in (0, 1/2]$ (except Theorem B in [9] which was announced but not proved later).

Lemma 3. *Let the distribution G be nonlattice and let the tail $\overline{G}(x)$ be regularly varying at infinity with index $-\alpha$, $\alpha \in (0, 1]$, and $m(x) \to \infty$. Then, for every fixed $h > 0$,*

$$\liminf_{x\to\infty} H_+(x, x+h] m(x) \ge \frac{hp}{\Gamma(\alpha)\Gamma(2-\alpha)};$$

if $\alpha \in (1/2, 1]$ then

$$\lim_{x\to\infty} H_+(x, x+h] m(x) = \frac{hp}{\Gamma(\alpha)\Gamma(2-\alpha)}.$$

Proof. Since $m(x) \to \infty$ as $x \to \infty$, it follows from Lemma 2 that

$$\int_0^x \overline{G}_+(y)dy \sim \frac{m(x)}{p}.$$

Using (5) and (6), we complete the proof.

Let us formulate some analogs and strengthenings of (5) and (6) for the lattice distributions. Let G_+ be the lattice distribution concentrated on $\mathbf{Z}^+$. It was proved in [10] that if $\overline{G}_+(x)$ is regularly varying at infinity with index $-\alpha$, $\alpha \in (0,1)$, then

$$\liminf_{k\to\infty} H_+\{k\}k\overline{G}_+(k) = \frac{\sin\pi\alpha}{\pi}; \tag{7}$$

if $\alpha \in (1/2, 1)$ then

$$\lim_{k\to\infty} H_+\{k\}k\overline{G}_+(k) = \frac{\sin\pi\alpha}{\pi}. \tag{8}$$

It was proved in [11, Corollary 3–B] (also see [12]) that (8) holds in the case $\alpha \in (0, 1/2]$, provided that $G_+\{k\}$ is ultimately decreasing.

Lemma 4. *Suppose that G is a lattice distribution concentrated on $\mathbf{Z}^+$ and $\mathbf{Z}^+$ is the minimal lattice. If the sequence $\overline{G}(k)$ is regularly varying at infinity with index $-\alpha$, $\alpha \in (0,1)$, then*

$$\liminf_{k\to\infty} H_+\{k\}k\overline{G}(k) \geq \frac{p\cdot\sin\pi\alpha}{\pi};$$

if $\alpha \in (1/2, 1)$ then

$$\lim_{x\to\infty} H_+\{k\}k\overline{G}(k) = \frac{p\cdot\sin\pi\alpha}{\pi}.$$

The last limit holds in the case $\alpha \in (0, 1/2]$ as well provided that the sequence $G\{k\}$ is ultimately decreasing.

Proof. The first two claims follow from Lemma 2, (7), and (8). The last claim of the lemma follows from the observation that the monotonicity of $G\{k\}$ implies the monotonicity of $G_+\{k\}$ due to the representation (compare with (4))

$$G_+\{k\} = \sum_{j=-\infty}^{0} G\{k-j\}H\{j\}.$$

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D. A. Korshunov
Sobolev Institute of Mathematics, Novosibirsk, Russia
E-mail address: korshunov@math.nsc.ru