Lower limits for distributions of randomly stopped sums [1]

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Abstract

We study lower limits for the ratio $\frac{\overline{F^{*\tau}}(x)}{\overline{F}(x)}$ of tail distributions where $F^{*\tau}$ is a distribution of a sum of a random size τ of i.i.d. random variables having a common distribution F, and a random variable τ does not depend on summands.

AMS classification: Primary 60E05; secondary 60F10

Keywords: Convolution tail; Random sums of random variables; Lower limit; Heavy- and light-tailed distributions

1. Introduction. Let $\xi, \xi_1, \xi_2, \dots$ be independent identically distributed random variables. We assume that their common distribution F is unbounded from the right, that is, $\overline{F}(x) \equiv F(x,\infty) > 0$ for all x. Put $S_0 = 0$ and $S_n = \xi_1 + \ldots + \xi_n$, $n = 1, 2, \ldots$.

Let τ be a counting random variable which does not depend on $\{\xi_n\}_{n\ge1}$. Denote by $F^{*\tau}$ the distribution of a random sum $S_\tau = \xi_1 + \ldots + \xi_\tau$. In this paper we study lower limits (as $x \to \infty$) for the ratio $\frac{\overline{F^{*\tau}}(x)}{\overline{F}(x)}$.

We distinguish two types of distributions, heavy- and light-tailed. A random variable η has a *heavy-tailed* distribution if $\mathbf{E}e^{\varepsilon\eta} = \infty$ for all $\varepsilon > 0$, and *light-tailed* otherwise.

We consider only non-negative random variables and, in the case of heavy-tailed F, study conditions for

$$\liminf_{x\to\infty} \frac{\overline{F^{*\tau}}(x)}{\overline{F}(x)} \quad = \quad \mathbf{E}\tau \tag{1}$$

to hold. This problem has been given a complete solution in [5] for $\tau = 2$, and then in [3] for τ with a light-tailed distribution and for heavy-tailed summands. In the present work, we generalise results of [3] onto classes of distributions of τ which include all light-tailed distributions and also some heavy-tailed distributions. With each heavy-tailed distribution F, we associate a corresponding class of distributions of τ. For earlier studies on lower limits and on a related problem of justifying a constant K in the equivalence $\overline{F^{*2}}(x) \sim K\overline{F}(x)$, see e.g. [1, 2, 4, 7, 8] and further references therein.

Since the inequality "$\ge$" in (1) is valid for non-negative $\{\xi_n\}$ without any further assumptions (see, e.g., [9] or [3]), we immediately get the equality if $\mathbf{E}\tau = \infty$. Therefore, in the rest of the paper, we consider the case $\mathbf{E}\tau < \infty$ only. Our first result is

[1] The research of Denisov and Foss was partially supported by the EPSRC Grant EP/E033717/1. The research of Foss and Korshunov was partially supported by the Royal Society International Joint Project Grant 2005/R2-JP. The research of Korshunov was partially supported by Russian Science Support Foundation.

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Theorem 1. *Assume that* $\xi \geq 0$ *is heavy-tailed and* $\mathbf{E}\xi < \infty$*. Let, for some* $c > \mathbf{E}\xi$*,*

$$\mathbf{P}\{c\tau > x\} \quad = \quad o(\overline{F}(x)) \quad \text{ as } x \to \infty. \tag{2}$$

Then (1) *holds.*

The proof of Theorem 1 is based on a study of moments $\mathbf{E}e^{f(\xi)}$ for appropriately chosen concave function f. More precisely, we deduce Theorem 1 from the following general result which explores some ideas from [9, 5, 3].

Theorem 2. *Assume that* $\xi \geq 0$ *is heavy-tailed and* $\mathbf{E}\xi < \infty$*. Let there exists a function* $f : \mathbf{R}^+ \to \mathbf{R}$ *such that*

$$\mathbf{E}e^{f(\xi)} = \infty, \tag{3}$$

and, for some $c > \mathbf{E}\xi$*,*

$$\mathbf{E}e^{f(c\tau)} < \infty. \tag{4}$$

If $f(x) \geq \ln x$ *for all sufficiently large* x *and if the difference* $f(x) - \ln x$ *is an eventually concave function, then* (1) *holds.*

In particular, the equality (1) is valid provided $\mathbf{E}\xi^k = \infty$ and $\mathbf{E}\tau^k < \infty$ for some $k \geq 1$; it is sufficient to consider the function $f(x) = k \ln x$. Earlier this was proved in [3, Theorem 1] by a more simple method.

If we consider instead the function $f(x) = \gamma x$, $\gamma > 0$, then we obtain the equality (1) provided ξ is heavy-tailed but τ is light-tailed. This is Theorem 2 from [3].

Finally, the equality (1) is valid if F is a Weibull distribution with parameter $\beta \in (0,1)$, $\overline{F}(x) = e^{-x^\beta}$ and $f(x) = x^\beta$ or, more generally, $f(x) = x^\beta - c\ln x$ for $x \geq 1$ where $c \leq \beta$ is any fixed constant.

The counterpart of Theorem 1 in the light-tailed case is stated next. But first we need some notations. By the Laplace transform of F at the point $\gamma \in \mathbf{R}$ we mean

$$\varphi(\gamma) = \int_0^\infty e^{\gamma x} F(dx) \in (0, \infty].$$

Put

$$\widehat{\gamma} = \sup\{\gamma : \varphi(\gamma) < \infty\} \in [0, \infty].$$

Note that the function $\varphi(\gamma)$ is monotone continuous in the interval $(-\infty, \widehat{\gamma})$, and $\varphi(\widehat{\gamma}) = \lim_{\gamma \uparrow \widehat{\gamma}} \varphi(\gamma) \in [1, \infty]$.

Theorem 3. *Let* $\widehat{\gamma} \in (0, \infty]$*, so that* $\varphi(\widehat{\gamma}) \in (1, \infty]$*. If* (2) *holds and, for any fixed* $y > 0$*,*

$$\liminf_{x \to \infty} \frac{\overline{F}(x-y)}{\overline{F}(x)} \quad \geq \quad e^{\widehat{\gamma} y}, \tag{5}$$

then

$$\liminf_{x \to \infty} \frac{\overline{F^{*\tau}}(x)}{\overline{F}(x)} \quad = \quad \mathbf{E}\tau\varphi^{\tau-1}(\widehat{\gamma}).$$

The paper is organised as follows. In Section 2, we formulate and prove a general result on characterisation of heavy-tailed distributions on the positive half-line. Section 3 is devoted to the estimation of the functional $\mathbf{E}e^{h(S_n)}$ for a concave function h. Sections 4 and 5 contain proofs of Theorems 2 and 1 respectively. Section 6 is devoted to the proof in light-tailed case.

2. Characterisation of heavy-tailed distributions. It was proved in [3, Lemma 2] that, for any heavy-tailed random variable $\xi \ge 0$ and for any real $\delta > 0$, there exists an increasing concave function $h : \mathbf{R}^+ \to \mathbf{R}^+$ such that $\mathbf{E}e^{h(\xi)} \le 1+\delta$ and $\mathbf{E}\xi e^{h(\xi)} = \infty$. In the present section, we obtain some generalisation of it.

Lemma 1. *Let $\xi \ge 0$ be a random variable with a heavy-tailed distribution. Let $f : \mathbf{R}^+ \to \mathbf{R}$ be a concave function such that*

$$\mathbf{E}e^{f(\xi)} = \infty. \tag{6}$$

Let a function $g : \mathbf{R}^+ \to \mathbf{R}$ be such that $g(x) \to \infty$ as $x \to \infty$. Then there exists a concave function $h : \mathbf{R}^+ \to \mathbf{R}$ such that $h \le f$ and

$$\mathbf{E}e^{h(\xi)} < \infty, \qquad \mathbf{E}e^{h(\xi)+g(\xi)} = \infty.$$

Proof. Without loss of generality assume $f(0) = 0$. We will construct a function $h(x)$ on the successive intervals. For that we introduce two positive sequences, $x_n \uparrow \infty$ as $n \to \infty$ and $\varepsilon_n \in (0,1]$. We put $x_0 = 0$, $h(0) = f(0) = 0$, $h'(0) = f'(0)$, and

$$h(x) = h(x_{n-1}) + \varepsilon_n \min(h'(x_{n-1})(x - x_{n-1}), f(x) - f(x_{n-1})) \quad \text{for } x \in (x_{n-1}, x_n];$$

here h' is the left derivative of the function h. The function h is increasing, since $\varepsilon_n > 0$ and f is increasing. Moreover, this function is concave, due to $\varepsilon_n \le 1$ and concavity of f. Since $h(x) - h(x_{n-1}) \le f(x) - f(x_{n-1})$ for $x \in (x_{n-1}, x_n]$, we have $h \le f$.

Now proceed with the very construction of x_n and ε_n. By conditions $g(x) \to \infty$ and (6), we can choose x_1 so large that $e^{g(x)} \ge 2^1$ for all $x \ge x_1$ and

$$\mathbf{E}\{e^{\min(h'(0)\xi, f(\xi))}; \xi \in (x_0, x_1]\} + e^{\min(h'(0)x_1, f(x_1))}\overline{F}(x_1) \quad > \quad \overline{F}(x_0) + 1.$$

Choose $\varepsilon_1 \in (0,1]$ so that

$$\mathbf{E}\{e^{\varepsilon_1 \min(h'(0)\xi, f(\xi))}; \xi \in (x_0, x_1]\} + e^{\varepsilon_1 \min(h'(0)x_1, f(x_1))}\overline{F}(x_1) \quad = \quad \overline{F}(x_0) + 1.$$

Put $h(x) = \varepsilon_1 \min(x, f(x))$ for $x \in (0, x_1]$. Then the latter equality is equivalent to

$$\mathbf{E}\{e^{h(\xi)}; \xi \in (x_0, x_1]\} + e^{h(x_1)}\overline{F}(x_1) \quad = \quad e^{h(x_0)}\overline{F}(x_0) + 1/2,$$

By induction we construct an increasing sequence x_n and a sequence $\varepsilon_n \in (0,1]$ such that $e^{g(x)} \ge 2^n$ for all $x \ge x_n$, and

$$\mathbf{E}\{e^{h(\xi)}; \xi \in (x_{n-1}, x_n]\} + e^{h(x_n)}\overline{F}(x_n) \quad = \quad e^{h(x_{n-1})}\overline{F}(x_{n-1}) + 1/2^n$$

for any $n \ge 1$. For $n = 1$ this is already done. Make the induction hypothesis for some $n \ge 2$. For any $x > x_n$, denote

$$\begin{aligned}\delta(x,\varepsilon) \quad \equiv \quad & e^{h(x_n)}\Big(\mathbf{E}\{e^{\varepsilon \min(h'(x_n)(\xi - x_n), f(\xi) - f(x_n))}; \xi \in (x_n, x]\} \\ & + e^{\varepsilon \min(h'(x_n)(x - x_n), f(x) - f(x_n))}\overline{F}(x)\Big).\end{aligned}$$

By the convergence $g(x) \to \infty$, by heavy-tailedness of ξ, and by the condition (6), there exists x_{n+1} so large that $e^{g(x)} \ge 2^{n+1}$ for all $x \ge x_{n+1}$ and

$$\delta(x_{n+1}, 1) \quad > \quad e^{h(x_n)}\overline{F}(x_n) + 1.$$

Note that the function $\delta(x_{n+1}, \varepsilon)$ is continuously decreasing to $e^{h(x_n)}\overline{F}(x_n)$ as $\varepsilon \downarrow 0$. Therefore, we can choose $\varepsilon_{n+1} \in (0, 1]$ so that

$$\delta(x_{n+1}, \varepsilon_{n+1}) \quad = \quad e^{h(x_n)}\overline{F}(x_n) + 1/2^{n+1}.$$

Then

$$\mathbf{E}\{e^{h(\xi)}; \xi \in (x_n, x_{n+1}]\} + e^{h(x_{n+1})}\overline{F}(x_{n+1}) \quad = \quad e^{h(x_n)}\overline{F}(x_n) + 1/2^{n+1}.$$

Our induction hypothesis now holds with $n + 1$ in place of n as required.

Next, for any N,

$$\begin{aligned}
\mathbf{E}\{e^{h(\xi)}; \xi \le x_{N+1}\} \quad &= \quad \sum_{n=0}^{N} \mathbf{E}\{e^{h(\xi)}; \xi \in (x_n, x_{n+1}]\} \\
&= \quad \sum_{n=0}^{N} \Big(e^{h(x_n)}\overline{F}(x_n) - e^{h(x_{n+1})}\overline{F}(x_{n+1}) + 1/2^{n+1}\Big) \\
&\le \quad e^{h(x_0)}\overline{F}(x_0) + 1,
\end{aligned}$$

so that $\mathbf{E}e^{h(\xi)}$ is finite. On the other hand, since $e^{g(x)} \ge 2^k$ for all $x \ge x_k$,

$$\begin{aligned}
\mathbf{E}\{e^{h(\xi)+g(\xi)}; \xi > x_n\} \quad &\ge \quad 2^n \Big(\mathbf{E}\{e^{h(\xi)}; \xi \in (x_n, x_{n+1}]\} + e^{h(x_{n+1})}\overline{F}(x_{n+1})\Big) \\
&= \quad 2^n (e^{h(x_n)}\overline{F}(x_n) + 1/2^{n+1}).
\end{aligned}$$

Then, for any n, $\mathbf{E}\{e^{h(\xi)+g(\xi)}; \xi > x_n\} \ge 1/2$, which implies $\mathbf{E}e^{h(\xi)+g(\xi)} = \infty$. The proof is complete.

Lemma 2. *Let $\xi \ge 0$ be a random variable with a heavy-tailed distribution. Let $f_1 : \mathbf{R}^+ \to \mathbf{R}$ be any measurable function and $f_2 : \mathbf{R}^+ \to \mathbf{R}$ a concave function such that*

$$\mathbf{E}e^{f_1(\xi)} < \infty \quad \textit{and} \quad \mathbf{E}e^{f_1(\xi)+f_2(\xi)} = \infty.$$

Let a function $g : \mathbf{R}^+ \to \mathbf{R}$ be such that $g(x) \to \infty$ as $x \to \infty$. Then there exists a concave function $h : \mathbf{R}^+ \to \mathbf{R}$ such that $h \le f_2$ and

$$\mathbf{E}e^{f_1(\xi)+h(\xi)} < \infty \quad \textit{and} \quad \mathbf{E}e^{f_1(\xi)+h(\xi)+g(\xi)} = \infty.$$

Proof. Consider a new governing probability measure $\mathbf{P}^*$ defined in the following way:

$$\mathbf{P}^*\{d\omega\} = \frac{e^{f_1(\xi(\omega))}\mathbf{P}\{d\omega\}}{\mathbf{E}e^{f_1(\xi)}}.$$

Then

$$\mathbf{E}^* e^{f_2(\xi)} = \frac{\mathbf{E}e^{f_1(\xi)+f_2(\xi)}}{\mathbf{E}e^{f_1(\xi)}} = \infty.$$

In particular, ξ is heavy-tailed against the measure $\mathbf{P}^*$. Now it follows from Lemma 1 that there exists a concave function $h:\mathbf{R}^+\to\mathbf{R}$ such that $h\le f_2$, $h(x)=o(x)$, $\mathbf{E}^*e^{h(\xi)}<\infty$, and $\mathbf{E}^*e^{h(\xi)+g(\xi)}=\infty$. Equivalently,

$$\mathbf{E}e^{f_1(\xi)+h(\xi)}=\mathbf{E}e^{f_1(\xi)}\mathbf{E}^*e^{h(\xi)}<\infty$$

and

$$\mathbf{E}e^{f_1(\xi)+h(\xi)+g(\xi)}=\mathbf{E}e^{f_1(\xi)}\mathbf{E}^*e^{h(\xi)+g(\xi)}=\infty.$$

The proof is complete.

3. Growth rate of sums in terms of generalised moments. According to the Law of Large Numbers, the sum S_n growths like $n\mathbf{E}\xi$. In the following lemma we provide conditions on a function $h(x)$, guaranteeing an appropriate rate of growth for the functional $\mathbf{E}e^{h(S_n)}$.

Lemma 3. *Let ξ be a non-negative random variable. Let $h:\mathbf{R}^+\to\mathbf{R}$ be a non-decreasing eventually concave function such that $h(x)=o(x)$ as $x\to\infty$ and $h(x)\ge\ln x$ for all sufficiently large x. If $\mathbf{E}e^{h(\xi)}<\infty$, then, for any $c>\mathbf{E}\xi$, there exists a constant $K(c)$ such that $\mathbf{E}e^{h(S_n)}\le K(c)e^{h(nc)}$, for all n.*

To prove this lemma, we need the following assertion, which generalises the corresponding estimate from [6]:

Lemma 4. *Let η be a random variable with $\mathbf{E}\eta<0$. Let $h:\mathbf{R}\to\mathbf{R}$ be a non-decreasing and eventually concave function such that $h(x)=o(x)$ as $x\to\infty$ and $h(x)\ge\ln x$ for all sufficiently large x. If $\mathbf{E}e^{h(\eta)}<\infty$, then there exists x_0 such that the inequality $\mathbf{E}e^{h(x+\eta)}\le e^{h(x)}$ holds for all $x>x_0$.*

Proof. Since h is increasing, without loss of generality we may assume that η is bounded from below, that is, $\eta\ge M$ for some M. Also, we may assume that h is non-negative and concave on the whole half-line $[0,\infty)$.

Since h is concave, $h'(x)$ is non-increasing function. With necessity $h'(x)\to 0$ as $x\to\infty$, otherwise the condition $h(x)=o(x)$ is violated. If ultimately $h'(x)=0$, then h is ultimately a constant function and the proof of the theorem is obvious.

Consider now the case $h'(x)\to 0$ as $x\to\infty$ but $h'(x)>0$ for all x. Put $g(x)\equiv 1/h'(x)$, then $g(x)\uparrow\infty$ as $x\to\infty$. Since $\mathbf{E}\eta<0$, we can choose sufficiently large A such that

$$\varepsilon\equiv\mathbf{E}\{\eta;\eta\in[M,A]\}+e\mathbf{E}\{\eta;\eta>A\}\quad<\quad 0. \tag{7}$$

By concavity of h, for any x and $y\in\mathbf{R}$ we have the inequality $h(x+y)-h(x)\le h'(x)y$. Hence,

$$\begin{aligned}\mathbf{E}e^{h(x+\eta)-h(x)} &\le \mathbf{E}\{e^{h'(x)\eta};\eta\in[M,A]\}+\mathbf{E}\{e^{h'(x)\eta};\eta\in(A,g(x)]\}\\ &\quad+\mathbf{E}\{e^{h(x+\eta)-h(x)};\eta>g(x)\}\\ &\equiv E_1+E_2+E_3.\end{aligned} \tag{8}$$

Since $h'(x)\to 0$, the Taylor's expansion for the exponent up to the linear term implies, as $x\to\infty$,

$$E_1 \quad=\quad \mathbf{P}\{\eta\in[M,A]\}+h'(x)\mathbf{E}\{\eta;\eta\in[M,A]\}+o(h'(x)). \tag{9}$$

On the event $\eta\in(A,g(x)]$ we have $h'(x)\eta\le 1$ and, thus, $e^{h'(x)\eta}\le 1+eh'(x)\eta$. Then

$$E_2\quad\le\quad \mathbf{P}\{\eta\in(A,g(x)]\}+eh'(x)\mathbf{E}\{\eta;\eta\in(A,g(x)]\}. \tag{10}$$

We have

$$E_3 \quad = \quad \mathbf{E}\{e^{h(\eta)}e^{h(x+\eta)-h(x)-h(\eta)}; \eta > g(x)\}. \tag{11}$$

By concavity of h, for $x > 0$, the difference $h(x+y) - h(y)$ is non-increasing in y. Therefore, for any $y > g(x)$,

$$\begin{aligned} h(x+y) - h(x) - h(y) &\leq h(x+g(x)) - h(x) - h(g(x)) \\ &\leq h'(x)g(x) - h(g(x)) \\ &= 1 - h(g(x)) \\ &\leq 1 - \ln g(x), \end{aligned}$$

due to the condition $h(x) \geq \ln x$ for all sufficiently large x. This estimate and (11) imply

$$\begin{aligned} E_3 &\leq \mathbf{E}\{e^{h(\eta)}; \eta > g(x)\}e^{1-\ln g(x)} \\ &= o(1)/g(x) = o(h'(x)) \quad \text{as } x \to \infty, \end{aligned} \tag{12}$$

by the condition $\mathbf{E}e^{h(\eta)} < \infty$. Substituting (9), (10) and (12) into (8) and taking into account the choice (7) of A, we get

$$\begin{aligned} \mathbf{E}e^{h(x+\eta)} &= e^{h(x)}\mathbf{E}e^{h(x+\eta)-h(x)} \\ &\leq e^{h(x)}(1 + h'(x)\varepsilon + o(h'(x))) \quad \text{as } x \to \infty. \end{aligned}$$

Since $\varepsilon < 0$, the latter estimate implies $\mathbf{E}e^{h(x+\eta)} < e^{h(x)}$ for all sufficiently large x. The proof is complete.

Proof of Lemma 3. Put $\eta_n = \xi_n - c$. We have $\mathbf{E}\eta_n < 0$ and $\mathbf{E}e^{h(\eta_n)} < \infty$. By Lemma 4, there exists $x_0 > 0$ such that $\mathbf{E}e^{h(x+\eta_n)} \leq \mathbf{E}e^{h(x)}$ for $x > x_0$. Then, by monotonicity of $h(x)$ and by non-negativity of S_{n-1},

$$\mathbf{E}e^{h(S_n)} \quad \leq \quad \mathbf{E}e^{h(S_n+x_0)} = \mathbf{E}e^{h(S_{n-1}+x_0+c+\eta_n)} \leq \mathbf{E}e^{h(S_{n-1}+x_0+c)}.$$

Now, by the induction arguments, $\mathbf{E}e^{h(S_n)} \leq e^{h(cn+x_0)} \leq e^{h(cn)}e^{h(x_0)}$. The proof is complete.

4. Proof of Theorem 2. Before starting the proof of Theorem 2, we formulate the following proposition from [3, Corollary 1]:

Proposition 1. *Let there exist a concave function $r : \mathbf{R}^+ \to \mathbf{R}$ such that $\mathbf{E}e^{r(\xi)} < \infty$ and $\mathbf{E}\xi e^{r(\xi)} = \infty$. If F is heavy-tailed and $\mathbf{E}\tau e^{r(S_{\tau-1})} < \infty$, then* (1) *holds.*

We also need two auxiliary technical results.

Lemma 5. *Let $\chi \geq 0$ be any random variable. Then there exists a differentiable concave function $g : \mathbf{R}^+ \to \mathbf{R}^+$, $g(0) = 0$, such that $g'(x) \leq 1$ for all x, $g(x) \to \infty$ as $x \to \infty$, and $\mathbf{E}e^{g(\chi)} < \infty$.*

Proof. Consider an increasing sequence $\{x_n\}$ such that $x_0 = 0$, $x_1 = 1$, $x_{n+1} - x_n > x_n - x_{n-1}$, and $\mathbf{P}\{\chi > x_n\} \leq e^{-n}$. Put $g_1(x_n) = n/2$ and continiously linear between these points. Then, for any $x \in (x_n, x_{n+1})$ and $y \in (x_{n+1}, x_{n+2})$ we have

$$g_1'(x) = \frac{1}{2(x_{n+1} - x_n)} > \frac{1}{2(x_{n+2} - x_{n+1})} = g_1'(y),$$

so that g_1 is concave. By the construction, $g_1(x) \uparrow \infty$ as $x \to \infty$ and $g_1'(x) \le 1$ where the derivative exists. Finally,

$$\mathbf{E}e^{g_1(\chi)} \le \sum_{n=0}^{\infty} e^{g_1(x_{n+1})}\mathbf{P}\{\chi > x_n\} \le \sum_{n=0}^{\infty} e^{(n+1)/2}e^{-n} < \infty.$$

A procedure of smoothing, say $g(x) = \int_x^{x+1} g_1(y)dy - \int_0^1 g_1(y)dy$, completes the proof.

Lemma 6. *Let $\chi \ge 0$ be a random variable such that, for some concave function $f : \mathbf{R}^+ \to \mathbf{R}^+$, $\mathbf{E}e^{f(\chi)} = \infty$. Then there exists a concave function $f_1 : \mathbf{R}^+ \to \mathbf{R}^+$ such that $f_1 \le f$, $f_1(x) = o(x)$ as $x \to \infty$, and $\mathbf{E}e^{f_1(\chi)} = \infty$.*

Proof. Take x_1 so large that $\mathbf{E}\{e^{\min(\chi, f(\chi))}; \chi \le x_1\} \ge 1$ and put $f_1(x) = \min(x, f(x))$ for $x \in [0, x_1]$. Then by induction, for any n, we can choose x_{n+1} such that

$$\mathbf{E}\{e^{f_1(x_n)+\min(n^{-1}f_1'(x_n)(\chi - x_n), f(\chi)-f(x_n))}; \chi \in (x_n, x_{n+1}]\} \ge 1.$$

Let $f_1(x) = f_1(x_n) + \min(n^{-1}f_1'(x_n)(x - x_n), f(x) - f(x_n))$ for $x \in (x_n, x_{n+1}]$. By construction, f_1 is concave, $f_1 \le f$, and $f_1'(x_{n+1}) \le f_1'(x_n)/n \to 0$ as $n \to \infty$.

Proof of Theorem 2. Without loss of generality, assume that $f(x) \ge \ln x$ for all x and that $f_2(x) \equiv f(x) - \ln x$ is concave on the whole positive half-line. By Lemma 6 and by measure change arguments like in the proof of Lemma 2 we may assume from the very beginning that

$$f(x) = o(x) \quad \text{as } x \to \infty.$$

Next we state the existence of a concave function $g : \mathbf{R}^+ \to \mathbf{R}$ such that $g(x) \to \infty$ as $x \to \infty$, $g(x) \le \ln x$ for all sufficiently large x, the difference $\ln x - g(x)$ is a non-decreasing function, and

$$\mathbf{E}e^{f(c\tau)+g(c\tau)} < \infty.$$

Indeed, by Lemma 5 and again measure change technique, there exists a differentiable concave function $g_1 : \mathbf{R}^+ \to \mathbf{R}^+$ such that $g_1(0) = 0$, $g_1(x) \uparrow \infty$, $g_1'(x) \le 1$, and $\mathbf{E}e^{f(c\tau)+g_1(c\tau)} < \infty$. Put $g(x) = g_1(\ln(x+1)) - 1$. Then g is a monotone function increasing to infinity and $g(x) \le \ln x$ for all sufficiently large x. In addition,

$$(\ln x - g(x))' = 1/x - g_1'(\ln(x+1))/(x+1) \ge 0,$$

so that the difference $\ln x - g(x)$ is a non-decreasing function as needed.

Since the function $f_2(x)$ is concave, by Lemma 2 with $f_1(x) = \ln x$, there exists a concave function h such that $h \le f_2$, $h(x) = o(x)$, $\mathbf{E}\xi e^{h(\xi)} < \infty$ and $\mathbf{E}\xi e^{h(\xi)+g(\xi)} = \infty$. Since $\ln x + h(x) + g(x) \le f(x) + g(x)$, by (4) and by the choice of g,

$$\mathbf{E}\tau e^{h(c\tau)+g(c\tau)} < \infty. \tag{13}$$

The concave function $r(x) = h(x) + g(x)$ satisfies all conditions of Proposition 1. Indeed, due to the inequality $g(x) \le \ln x$ for all sufficiently large x, we have $\mathbf{E}e^{r(\xi)} < \infty$ because $\mathbf{E}\xi e^{h(\xi)} < \infty$. It remains to check that $\mathbf{E}\tau e^{r(S_{\tau-1})} < \infty$. Since, by (13),

$$\mathbf{E}\{\tau e^{r(S_\tau)}; S_\tau \le c\tau\} \le \mathbf{E}\tau e^{r(c\tau)} < \infty,$$

it suffices to prove that

$$\mathbf{E}\{\tau e^{r(S_\tau)}; S_\tau > c\tau\} < \infty.$$

We proceed in the following way:

$$\begin{aligned}\mathbf{E}\{c\tau e^{r(S_\tau)}; S_\tau > c\tau\} &= \sum_{n=1}^{\infty} \mathbf{P}\{\tau = n\} cn \mathbf{E}\{e^{r(S_n)}; S_n > cn\} \\ &= \sum_{n=1}^{\infty} \mathbf{P}\{\tau = n\} e^{g(cn)+\ln(cn)-g(cn)} \mathbf{E}\{e^{h(S_n)+g(S_n)}; S_n > cn\}.\end{aligned}$$

By the monotonicity of the difference $\ln x - g(x)$, we obtain the following estimate

$$\mathbf{E}\{c\tau e^{r(S_\tau)}; S_\tau > c\tau\} \le \sum_{n=1}^{\infty} \mathbf{P}\{\tau = n\} e^{g(cn)} \mathbf{E}\{e^{\ln S_n + h(S_n)}; S_n > cn\},$$

Since the function $\ln x + h(x)$ is concave and $\ln x + h(x) \ge \ln x$, by Lemma 3,

$$\mathbf{E} e^{\ln S_n + h(S_n)} \le K(c) e^{\ln(nc)+h(cn)}$$

for some $K(c) < \infty$. Therefore,

$$\begin{aligned}\mathbf{E}\{c\tau e^{r(S_\tau)}; S_\tau > c\tau\} &\le K(c) \sum_{n=1}^{\infty} \mathbf{P}\{\tau = n\} e^{g(cn)} e^{\ln(cn)+h(nc)} \\ &= K(c) c \mathbf{E} \tau e^{h(c\tau)+g(c\tau)} < \infty,\end{aligned}$$

from (13). The proof of Theorem 2 is complete.

5. Proof of Theorem 1. Denote by G the distribution function of $c\tau$.

We will construct an increasing concave function $f : \mathbf{R}^+ \to \mathbf{R}$ such that

$$\mathbf{E} \xi e^{f(\xi)} = \infty \qquad \text{and} \qquad \mathbf{E} \tau e^{f(c\tau)} < \infty. \tag{14}$$

Then the desired relation 1) will follow by applying Theorem 2.

If G is light-tailed then one can take $f(x) = \lambda x$ for a sufficiently small $\lambda > 0$. ¿From now on we assume G to be heavy-tailed.

Consider new random variables ξ_* and τ_* with the following distributions:

$$\mathbf{P}\{\xi_* \in dx\} = \frac{xF(dx)}{\mathbf{E}\xi} \quad \text{and} \quad \mathbf{P}\{\tau_* = n\} = \frac{n\mathbf{P}\{\tau = n\}}{\mathbf{E}\tau}.$$

Denote by F_* and G_* the distributions of ξ_* and $c\tau_*$ respectively. Then both F_* and G_* are heavy-tailed and

$$\overline{G}_*(x) = o(\overline{F}_*(x)) \quad \text{as } x \to \infty. \tag{15}$$

The heavy-tailedness of G_* is equivalent to the following condition: for any $\varepsilon > 0$,

$$\int_1^\infty \overline{G}_*(\varepsilon^{-1} \ln x) dx \equiv \int_0^\infty e^x \overline{G}_*(x/\varepsilon) dx = \infty. \tag{16}$$

In terms of new distributions F_* and G_*, conditions (14) nay be reformulated as follows: we need to construct an increasing concave function f such that $\mathbf{E}e^{f(\xi_*)} = \infty$ and $\mathbf{E}e^{f(c\tau_*)} < \infty$, or, equivalently,

$$\int_1^\infty \overline{F}_*(f^{-1}(\ln x))dx = \infty \qquad \text{and} \qquad \int_1^\infty \overline{G}_*(f^{-1}(\ln x))dx < \infty. \tag{17}$$

The concavity of f is equivalent to the convexity of its inverse, $h = f^{-1}$. So, conditions (17) may be rewritten as: we have to present an increasing convex function h such that

$$\int_0^\infty e^x \overline{F}_*(h(x))dx = \infty \qquad \text{and} \qquad \int_0^\infty e^x \overline{G}_*(h(x))dx < \infty. \tag{18}$$

We will construct $h(x)$ as a piece-wise linear function. For this, we will introduce two increasing sequences, say $x_n \uparrow \infty$ and $a_n \uparrow \infty$, and let

$$h(x) \;=\; h(x_n) + a_n(x - x_n) \quad \text{for } x \in (x_n, x_{n+1}].$$

Then the convexity of f will follow from the increase of $\{a_n\}$.

Put $x_0 = 0$ and $f(x_0) = 0$. Due to (15) and (16), we can choose x_1 so large that

$$\frac{\overline{F}_*(y)}{\overline{G}_*(y)} \;\geq\; 2^1$$

for all $y > x_1$ and

$$\int_0^{x_1} e^x \overline{G}_*(h(x_0) + 1 \cdot (x - x_0))dx \;\geq\; 1.$$

Then there exists a sufficiently large $a_0 \geq 1$ such that

$$\int_0^{x_1} e^x \overline{G}_*(h(x_0) + a_0(x - x_0))dx \;=\; 1.$$

Now we use the induction argument to construct increasing sequences $\{x_n\}$ and $\{a_n\}$ such that

$$\frac{\overline{F}_*(y)}{\overline{G}_*(y)} \;\geq\; 2^{n+1} \tag{19}$$

for all $y > x_{n+1}$ and

$$\int_{x_n}^{x_{n+1}} e^x \overline{G}_*(h(x))dx \;=\; 2^{-n}.$$

For $n = 0$ this is already done. Make the induction hypothesis for some $n \geq 1$. For any $x > x_{n+1}$, denote

$$\delta(x, a) \;\equiv\; \int_{x_{n+1}}^{x} e^y \overline{G}_*(h(x_{n+1} + a(y - x_{n+1})))dy.$$

Due to (15) and (16), we can choose x_{n+2} so large that

$$\frac{\overline{F}_*(y)}{\overline{G}_*(y)} \geq 2^{n+2}$$

for all $y > x_{n+2}$ and

$$\delta(x_{n+2}, a_n) \geq 1.$$

Since the function $\delta(x_{n+2}, a)$ continuously decreases to 0 as $a \uparrow \infty$, we can choose $a_{n+1} > a_n$ such that

$$\delta(x_{n+2}, a_{n+1}) = 2^{-(n+1)}.$$

Then

$$\int_{x_{n+1}}^{x_{n+2}} e^x \overline{G}_*(h(x))dx = 2^{-(n+1)}.$$

Our induction hypothesis now holds with $n+1$ in place of n as required.

Now the inequalities (18) follow since, from the construction of function h,

$$\begin{aligned}\int_0^\infty e^x \overline{G}_*(h(x))dx &= \sum_{n=0}^\infty \int_{x_n}^{x_{n+1}} e^x \overline{G}_*(h(x))dx \\ &= \sum_{n=0}^\infty 2^{-n} < \infty.\end{aligned}$$

and, by (19),

$$\begin{aligned}\int_0^\infty e^x \overline{F}_*(h(x))dx &= \sum_{n=0}^\infty \int_{x_n}^{x_{n+1}} e^x \overline{F}_*(h(x))dx \\ &\geq \sum_{n=0}^\infty 2^n \int_{x_n}^{x_{n+1}} e^x \overline{G}_*(h(x))dx \\ &= \sum_{n=0}^\infty 2^n 2^{-n} = \infty.\end{aligned}$$

The proof of Theorem 1 is complete.

6. Proof of Theorem 3. We apply the exponential change of measure with parameter $\widehat{\gamma}$ and consider the distribution $G(du) = e^{\widehat{\gamma}u}F(du)/\varphi(\widehat{\gamma})$ and the stopping time ν with the distribution $\mathbf{P}\{\nu = k\} = \varphi^k(\widehat{\gamma})\mathbf{P}\{\tau = k\}/\mathbf{E}\varphi^\tau(\widehat{\gamma})$. Then it was proved in [3, Lemma 3] that

$$\liminf_{x\to\infty} \frac{\overline{G^{*\nu}}(x)}{\overline{G}(x)} \geq \frac{1}{\mathbf{E}\varphi^{\tau-1}(\gamma)} \liminf_{x\to\infty} \frac{\overline{F^{*\tau}}(x)}{\overline{F}(x)}. \tag{20}$$

¿From the definition of $\widehat{\gamma}$, the distribution G is heavy-tailed. Let us prove that

$$\mathbf{P}\{c\nu > x\} = o(\overline{G}(x)) \quad \text{as } x \to \infty. \tag{21}$$

Indeed, put $\lambda \equiv \ln \varphi(\widehat{\gamma}) > 0$; then

$$\begin{aligned} \mathbf{P}\{c\nu > x\} &= \frac{1}{\mathbf{E}\varphi^\tau(\widehat{\gamma})} \sum_{k>x/c} e^{\lambda k} \mathbf{P}\{\tau = k\} \\ &\le \frac{1}{\mathbf{E}\varphi^\tau(\widehat{\gamma})} \int_{x/c}^{\infty} e^{\lambda y} \mathbf{P}\{\tau \in dy\}. \end{aligned} \tag{22}$$

Integration by parts implies

$$\begin{aligned} \int_{x/c}^{\infty} e^{\lambda y} \mathbf{P}\{\tau \in dy\} &= -e^{\lambda y} \mathbf{P}\{\tau > y\}\Big|_{x/c}^{\infty} + \lambda \int_{x/c}^{\infty} e^{\lambda y} \mathbf{P}\{\tau > y\} dy \\ &= e^{\lambda x/c} \mathbf{P}\{c\tau > x\} + \frac{\lambda}{c} \int_{x}^{\infty} e^{\lambda y/c} \mathbf{P}\{c\tau > y\} dy, \end{aligned}$$

because $\mathbf{E}\varphi^\tau(\widehat{\gamma}) < \infty$ and, thus, $e^{\lambda y}\mathbf{P}\{\tau > y\} \to 0$ as $y \to \infty$. Now applying the condition (2) we obtain that the latter sum is of order

$$o\Big(e^{\lambda x/c}\overline{F}(x) + \frac{\lambda}{c}\int_x^\infty e^{\lambda y/c}\overline{F}(y)dy\Big) = o\Big(\int_x^\infty e^{\lambda y/c}F(dy)\Big) \quad \text{as } x \to \infty.$$

Together with (22) it implies (21). Therefore, by Theorem 1 we have the equality

$$\liminf_{x\to\infty} \frac{\overline{G^{*\nu}}(x)}{\overline{G}(x)} = \mathbf{E}\nu = \frac{\mathbf{E}\tau\varphi^\tau(\widehat{\gamma})}{\mathbf{E}\varphi^\tau(\widehat{\gamma})},$$

and, due to (20),

$$\liminf_{x\to\infty} \frac{\overline{F^{*\tau}}(x)}{\overline{F}(x)} \le \mathbf{E}\tau\varphi^{\tau-1}(\widehat{\gamma}). \tag{23}$$

The result now follows from Lemma .

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