ERGODICITY IN A SENSE OF WEAK CONVERGENCE, EQUILIBRIUM-TYPE IDENTITIES AND LARGE DEVIATIONS FOR MARKOV CHAINS*

A. A. BOROVKOV and D. KORSHUNOV[1]

Institute of Mathematics,
Siberian Branch of the Russian Academy of Sciences,
Universitetskii pr. 4, 630090 Novosibirsk, Russia

1. ERGODICITY IN A SENSE OF WEAK CONVERGENCE

Almost all papers on ergodicity of Markov chains are devoted to the study of the Harris-irreducable (Harris-type) Markov chains. Ergodicity conditions for these Markov chains are investigated very well. These conditions (see, for instance, (Nummelin, 1984; Borovkov, 1993) imply convergence of transition probabilities to the stationary distribution in total variation and this free one from necessity to study other forms of convergence. Visa versa, if there is ergodicity in a sence of convergence in total variation then Harris-type ergodicity conditions of Markov chain are fulfilled.

So the problem of ergodicity in a sence of weak convergence arises first of all in the study of ergodicity of non Harris-type Markov chain. We know only a few results in this area. For non Harris-type Markov chain convergence in total variation, generally speaking, can not takes place and one have to study conditions under which some other form of convergence of transition probabilities takes place. The most natural of them is the weak convergence what makes one to introduce topology on state space of Markov chain. So the main subjects of discussion in this section are ergodicity conditions (for Markov chain in topological state space) which are not connected with Harris irreducability.

One can pick out three following approaches.

[1]Supported in part by the Russian Foundation for Fundamental Research; Grant No. 93-01-1416.

Let $X(x,n)$, $n=0,1,2,\ldots$, be a Markov chain (initial position $X(x,0)=x\in\mathcal{X}$) in measurable metric space $(\mathcal{X},\mathcal{B})$ with metric ρ, where $\mathcal{B}$ — Borel σ-algebra. Denote transition probabilities $P(x,n,B)=\mathbf{P}(X(x,n)\in B)$.

1. *General approach.* Ergodicity conditions in this case have three components:

1) Compactness-type condition: *there exists compact set V such that for*

$$\tau(x)=\inf\{n: X(x,n)\in V\}$$

(a) $\mathbf{P}(\tau(x)<\infty)=1$ *for any* x,

(b) $\sup_{x\in V}\mathbf{P}(\tau(x)>k)\le r(k)$, $\sum r(k)<\infty$,

(c) *For any* $\epsilon>0$, $n\ge 1$ *there exists compact set* $K(\epsilon,n)$ *such that*

$$\inf_{x\in V}\mathbf{P}(X(x,n)\in K(\epsilon,n))>1-\epsilon$$

(without "inf" condition (c) is always fulfilled).

2) Positivity-type condition: *for any open set G and $s\ge 1$ there exist numbers $k=k(G,s)$ and $p=p(G,s)$ such that*

$$\inf_{x\in V}P(x,k+j,G)\ge p, j=0,...,s.$$

3) Continuity-type condition. *Let f be bounded continuous function on $\mathcal{X}$,*

$$f_n(x)=A^n f=\int f(y)P(x,n,dy).$$

We assume that the family $\{f_n\}$ is equicontinuous on any compact set K. (This condition can be relaxed and have a form: *there exist $\delta(t)\downarrow 0$, $\epsilon(n)\downarrow 0$ such that* $|f_n(x_1)-f_n(x_2)|\le\delta(\rho(x_1,x_2))$ *for any* $x_1,x_2\in K$, $\rho(x_1,x_2)\ge\epsilon(n)$).

Theorem 1. *If conditions* 1)–3) *are fulfilled then there exists a unique invariant measure π and $P(x,n,\cdot)\Rightarrow\pi(\cdot)$.*

This result was published in (Borovkov, 1991).

2. Another approach is connected with the property of "*contraction*" *of transition probabilities* (see (Dobrushin, 1970)). It means that Kantorovich — Wasserstein distance $R(P(x_1,\cdot),P(x_2,\cdot))$ between $P(x_1,\cdot)$ and $P(x_2,\cdot)$ has to be less than $\rho(x_1,x_2)$, $P(x,\cdot)=P(x,1,\cdot)$.

3. *The third approach* can be used in the case when Markov chain $X(n)$ admits the representation

$$X(n+1)=f(X(n),\xi_n), \tag{1.1}$$

where ξ_n are i.i.d. random variables, function f satisfies some regularity conditions. It can be either monotonicity property (see (Loynes, 1962)) or contraction property.

Let us consider the following contraction-type (Lipshitz-type) condition:

1) $\rho(f(x_1,\xi_1),f(x_2,\xi_1))\le c(\xi_1)\rho(x_1,x_2)$, $\mathbf{E}\ln c(\xi_1)\le-\beta<0$.

Besides we will need as well

2) Boundness-type condition: for some $x_0\in\mathcal{X}$ and any $\delta>0$ there exist n, m such that

$$\mathbf{P}(\rho(x_0,X(x_0,n))>N)<\delta$$

for all $n\ge m$.

Theorem 2. *Under conditions* 1), 2) *there exists stationary sequence* X^s, *satisfying* (1), *such that*

$$T^{-n}X(x,n+s) \xrightarrow{p} X^s, \tag{1.2}$$

where T *is measure conserving shift transformation, generated by stationary sequence* $\{\xi_n\}_{-\infty}^{\infty}$.

Condition 1) in this theorem can be relaxed (see (Borovkov, 1992, 1993)).

Of course it follows from Theorem 2 that distribution of $X(x,n)$ (i.e. $P(x,n,\cdot)$) converges weakly to stationary distribution (distribution of X^0). Similar results under stronger condition were obtained in (Bhattacharya and Lee, 1988; Dubins and Freedman, 1966).

Example 1. *Let* $\mathcal{X} = \mathbf{R}^d$, $\rho(x,y) = |x-y|$,

$$X(n+1) = G(\zeta_n F(X(n)) + \eta_n),$$

where G *and* F *are vector-functions,* ζ_n *is random matrix,* η_n *— random vector. Assume that* G *and* F *satisfy Lipshitz conditions:*

$$|G(x_1) - G(x_2)| < c_G|x_1 - x_2|, \quad |F(x_1) - F(x_2)| \le c_F|x_1 - x_2|.$$

Let us denote ζ^* *— conjugate matrix to* ζ, $\lambda(\zeta_1)$ *— maximal proper value of* $\zeta_1^*\zeta_1$.

Theorem 3. *If*

$$2\ln c_G c_F + \ln\lambda(\zeta_1) < 0,$$

$$\mathbf{E}\ln\lambda(\zeta_1) > -\infty, \quad \mathbf{E}(\ln|\eta_1|)^+ < \infty$$

then conditions of Theorem 2 *are fulfilled and hence sequence* $X(n)$ *converges to stationary one in a sence of* (2) ((Borovkov, 1993), see also (Brandt et.al., 1990)).

Next two sections of the talk stay apart from the first one.

2. EQUILIBRIUM-TYPE IDENTITIES

This section is devoted to the so-called equilibrium-type identities for stationary Markov chains. These identities are of interest by themselves and can be of use at least for the following three things:

- For the proof of necessity of ergodicity conditions for various Markov chains, and particularly for Markov chains arising in applications, such as polling systems, Jackson networks, etc.
- Equilibrium-type identities can be of use in obtaining estimates (sometimes, sharp) for probabilities of large deviations of stationary Markov chains (see Section 3).
- Equilibrium-type identities are very usefull in investigation of so-called transient phenomena for stationary Markov chains (see (Korshunov, 1993; Borovkov, 1993)).

The third part of the paper is devoted to the just mentioned problem of large deviations of stationary Markov chains. We will give an example showing how the equilibrium-type identities can be used for large deviations problem.

So, what are the equilibrium-type identities?

Let $X(x,n)$, $n = 0,1,2,\ldots$, be again a Markov chain in some measurable space $(\mathcal{X},\mathcal{B})$ with initial position $x : X(x,0) = x$, and with transition function $P(x,B) = \mathbf{P}(X(x,1) \in B)$. We assume that there exists stationary version X^n of this Markov chain with the same transition function $P(x,B)$ and with stationary distribution π.

Let us consider a measurable functional $g : \mathcal{X} \to \mathbf{R}$ and denote by $\gamma(x)$ the increment of $g(x)$ on Markov chain $X(n)$:

$$\gamma(x) = g(X(x,1)) - g(x)$$

and by $v(x)$ the mean of γ:

$$v(x) = \mathbf{E}\gamma(x) = \int (g(y) - g(x))P(x,dy).$$

If $\mathbf{E}|g(X^0)| < \infty$, then

$$\mathbf{E}v(X^0) = \int v(x)\pi(dx) = \mathbf{E}(g(X^1) - g(X^0)) = 0.$$

The equality

$$\mathbf{E}v(X^0) = 0 \tag{2.3}$$

will be referred to as equilibrium-type identity. So, if $\mathbf{E}|g(X^0)| < \infty$ it is obvious. But the problem is that

1. We almost never know whether $\mathbf{E}|g(X^0)| < \infty$ or not
2. If $\mathbf{E}g(X^0)$ does not exist then $\mathbf{E}v(X^0)$ can take any value.

We now give conditions under which identity (3) is true.

Theorem 4. *Let $g(x) \geq 0$ be a functional such that*

$$\mathbf{E}\max(0, v(X^0)) < \infty, \tag{2.4}$$

$$\mathbf{E}|\gamma(x)| \leq c(1 + |v(x)|). \tag{2.5}$$

Then (3) holds.

It is not always easy to verify conditions (4), (5); so we give some examples.

Example 2. *$\mathcal{X} = [0,\infty)$, $g(x) = x^k$, $k \geq 1$ is an integer. Then conditions (4), (5) are fulfilled if*

$$\limsup_{x\to\infty} \mathbf{E}\xi(x) < 0,$$

$$\sup_x \mathbf{E}|\xi(x)|^k < \infty,$$

where $\xi(x) = X(x,1) - x$.

Example 3. $\mathcal{X} = [0, \infty)$, $g(x) = e^{\lambda x}$. *Then* (4), (5) *are fulfilled if*

$$\limsup_{x \to \infty} \mathbf{E} e^{\lambda \xi(x)} < 1, \tag{2.6}$$

$$\sup_x \mathbf{E} e^{\lambda \xi(x)} < \infty. \tag{2.7}$$

Example 4. $\mathcal{X}$ *is a separable Banach space,* $\xi(x) = X(x, 1) - x$, $a(x) = \mathbf{E}\xi(x)$. *Let* $g(x) = l(x)$ *be a linear functional, such that*

$$\int \mathbf{E}|l(\xi(x))|\pi(dx) < \infty.$$

Then conditions (4), (5) *are fulfilled and*

$$\int l(a(x))\pi(dx) = 0.$$

If

$$\int \mathbf{E} \parallel \xi(x) \parallel \pi(dx) < \infty, \tag{2.8}$$

then

$$\int a(x)\pi(dx) = 0 \in \mathcal{X}. \tag{2.9}$$

In this last case we have an equilibrium-type identity in its most transparent form.

Of course, condition (8) is always fulfilled if $\sup_x \mathbf{E} \parallel \xi(x) \parallel < \infty$.

The assertion of Theorem 1 was used in form (9) to derive necessary conditions of ergodicity for Polling and Jackson Networks. It was done jointly with Prof. R. Shassberger. For some special cases and under more restrictive conditions, assertion (9) can be found in (Sennott et. al., 1983; Sennott, 1987; Borovkov et. al., 1992). Assertion (9) is very natural and we do not exclude that there are other papers with similar results (see also (Baccelli and Bremaud, 1987, p. 36)). Results mentioned in this section will be published in (Borovkov, 1993; Korshunov, 1993).

3. LARGE DEVIATIONS

Now we will speak about large deviations. We shall consider the problem mostly for a one-dimensional Markov chain on $[0, \infty)$. Besides, something will be said about a Markov chain in $\mathbf{R}^d_+$, $d > 1$.

So, let $\mathcal{X}$ be $[0, \infty)$. The subject of investigation is the asymptotic behavior of the probability

$$\mathbf{P}(X^0 > x) \quad \text{as} \quad x \to \infty.$$

Let us again denote by $\xi(x)$ the increment of Markov chain. We can now write down

$$X(n+1) = X(n) + \xi_n(X(n)),$$

where

$$\xi_n(x) \stackrel{D}{=} \xi(x).$$

Of course we need some restrictions on $X(n)$ to acquire a regular behaviour of $\mathbf{P}(X^0 > x)$. We assume that Markov chain is asymptotically homogeneous, that is, that

$$\xi(x) \Rightarrow \xi \quad \text{as} \quad x \to \infty \tag{3.10}$$

in the sense of weak convergence. We shall assume as well that

$$\mathbf{E}\xi < 0. \tag{3.11}$$

Let us introduce some notations

$$\varphi(\lambda) = \mathbf{E}e^{\lambda\xi},$$

$$\mu_+ = \sup\{\lambda : \varphi(\lambda) < \infty\} > 0,$$

$\beta > 0$ is a solution of the equation

$$\varphi(\lambda) = 1.$$

This solution is defined if $\varphi(\mu_+) \geq 1$.

Our next assumption is

$$\sup_x \mathbf{E}e^{\mu\xi(x)} < \infty \tag{3.12}$$

for

$$\mu = \sup\{\lambda : \varphi(\lambda) \leq 1\} = \begin{cases} \beta, & \text{if} \quad \varphi(\mu_+) \geq 1, \\ \mu_+, & \text{if} \quad \varphi(\mu_+) < 1. \end{cases}$$

Theorem 5. *If* (10)–(12) *hold, then*

$$\ln \mathbf{P}(X^0 > x) \sim -\mu x \quad \text{as} \quad x \to \infty.$$

It is quite easily to obtain an upper estimate for $\ln \mathbf{P}(X^0 > x)$ using an equilibrium-type identity. To illustrate the use of these identities we prove here

Theorem 6. *If* (10)–(12) *hold and* $\lambda < \mu$, *then*

$$\mathbf{P}(X^0 > x) \leq ce^{-\lambda x}.$$

Proof. Put $g(x) = e^{\lambda x}$. According to Example 3, the conditions of Theorem 4 are fulfilled if (6)–(7) are true. We already have (7), since (12) is valid. And we have (6) because for $\lambda < \mu$ it follows from (10), (12) that

$$\mathbf{E}e^{\lambda\xi(x)} \to \varphi(\lambda) < 1 \quad \text{as} \quad x \to \infty. \tag{3.13}$$

In our case

$$v(x) = \mathbf{E}e^{\lambda(x+\xi(x))} - e^{\lambda x} = e^{\lambda x}(\mathbf{E}e^{\lambda\xi(x)} - 1).$$

Therefore, by Theorem 4

$$\int v(x)\pi(dx) = 0. \tag{3.14}$$

According to (13), we have

$$v(x) < -\delta e^{\lambda x}$$

for sufficiently large $x > N$ and so

$$\sup_x v(x) \le v_0 < \infty.$$

Further,

$$\int_N^\infty v(x)\pi(dx) \le -\delta \int_N^\infty e^{\lambda x}\pi(dx),$$

$$\int_N^\infty e^{\lambda x}\pi(dx) \le -\frac{1}{\delta}\int_N^\infty v(x)\pi(dx) \overset{(14)}{=} \frac{1}{\delta}\int_{-\infty}^N v(x)\pi(dx) \le \frac{v_0}{\delta}$$

It remains to use Chebyshev inequality: for $x \ge N$

$$\pi(x,\infty) \le \frac{v_0}{\delta}e^{-\lambda x}.$$

Since this inequality is true for any $\lambda < \mu$, it is easy to infer thence that

$$\limsup_{x\to\infty} \frac{1}{x}\ln \mathbf{P}(X^0 > x) \le -\mu.$$

To establish the reverse estimate for lim inf is more difficult.

If the rate of convergence (10) is known and sufficiently fast then we can obtain a more precise result about asymptotics of the probability $\mathbf{P}(X^0 > x)$ by itself.

Let us assume that β is defined and

$$\int_{-\infty}^\infty e^{\beta t}|\mathbf{P}(\xi(x) < t) - \mathbf{P}(\xi < t)|dt \le \epsilon(x) = x^{-\alpha}l(x) \tag{3.15}$$

where $l(x)$ is slowly varying at infinity function.

Theorem 7. *Let* (10)–(12) *hold and*

$$\mathbf{E}\xi e^{\beta\xi} < \infty, \quad \textstyle\int_0^\infty \epsilon(x)dx < \infty.$$

Then

$$\mathbf{P}(X^0 > x) = e^{-\beta x}(c + o(1)), \quad c < \infty.$$

If β is not defined or (15) is not true then asymptotic of $\mathbf{P}(X^0 > x)$ can be different.

A more advanced asymptotic analysis is also available, but although for less general Markov chains.

We consider here three types of Markov chains on $\mathcal{X} = (-\infty, \infty)$:

- The so-called *homogeneous Markov chain*

$$X(n+1) = (X(n) + \xi_n)^+$$

where $\xi_n \stackrel{D}{=} \xi$ are i.i.d. random variables.

- The so-called *almost homogeneous Markov chain*

$$X(n+1) = \begin{cases} (X(n) + \xi_n)^+, & \text{if} X(n) > 0, \\ \eta_n, & \text{if} X(n) = 0, \end{cases}$$

where $\eta_n \stackrel{D}{=} \eta \geq 0$ are also i.i.d. random variables independent of $\{\xi_n\}$.

- The so-called *partially homogeneous Markov chain on $\mathcal{X} = (-\infty, \infty)$*:

$$X(n+1) = X(n) + \xi_n(X(n))$$

where $\xi_n(x) \stackrel{D}{=} \xi(x)$ and $(x + \xi(x))^+ \stackrel{D}{=} (x + \xi)^+$, $x > 0$.

Homogeneous Markov chain. A homogeneous Markov chain is a well known and investigated model. Let us recall some basic results about it. We denote

$$S_n = \sum_{k=0}^{n} \xi_k, \quad S = \sup\nolimits_{k \geq 0} S_k.$$

Theorem 8. *For a homogeneous Markov chain $X^0 \stackrel{D}{=} S$,*
1) *If*

$$\begin{array}{ll} \mu_+ > 0, \quad \varphi(\mu_+) > 1 & \text{or} \\ \mu_+ > 0, \quad \varphi(\mu_+) = 1, \quad \varphi'(\mu_+) < \infty; \end{array}$$

then $\mathbf{P}(X^0 > x) = \mathbf{P}(S > x) = c_1 e^{-\beta x}(1 + o(1))$ *where c_1 is known* (Cramer, (Feller, 1971)).
2) *If*

$$\begin{array}{ll} \mu_+ > 0, \quad \varphi(\mu_+) < 1 & \text{or} \\ \mu_+ = 0, \quad \varphi'(\mu_+) > -\infty & \end{array}$$

then $\mathbf{P}(X^0 > x) = c_2 \int_x^\infty \mathbf{P}(\xi > t) dt (1 + o(1))$ (Borovkov, 1972).

Almost homogeneous Markov chain. What is the asymptotic behaviour of $\mathbf{P}(X^0 > x)$ for an almost homogeneous Markov chain? Let us denote by χ_- the first negative sum among $S_1, S_2, \ldots,$ by $H(t)$ a renewal function for a random variable χ_-, by $G(t) = \int_0^\infty \mathbf{P}(\eta > t + u) dH(u)$ (in regular cases $G(t) \sim c \int_t^\infty \mathbf{P}(\eta > u) du$), and by ζ a random variable with distribution function

$$\mathbf{P}(\zeta > x) = \frac{G(t)}{G(0)}.$$

Theorem 9. *For an almost homogeneous Markov chain*

$$\mathbf{P}(X^0 > x) = c_3 \mathbf{P}(S + \zeta > x), \quad x > 0,$$

where S and ζ are independent, c_3 is known.

This Theorem allows us to find the asymptotic behaviour of $\mathbf{P}(X^0 > x)$ like in Theorem 8, depending on the asymptotic behaviour of $\mathbf{P}(S > x)$ and $G(x)$.

Example 5. *If conditions of part* 1 *of Theorem* 8 *are fulfilled and* $\mathbf{E}e^{\beta\eta} < \infty$, *then*

$$\mathbf{P}(X^0 > x) = c_3 c_1 \mathbf{E}e^{\beta\zeta} e^{-\beta x}(1 + o(1)),$$

c_1, c_3 *are the same as in Theorems* 8, 9.

Partially homogeneous Markov chain. Let us consider now a partially homogeneous Markov chain, when $(x + \xi(x))^+ = (x + \xi)^+$, $x > 0$.

Theorem 10. *For a partially homogeneous Markov chain the asymptotic behaviour of* $\mathbf{P}(X^0 > x)$ *can be described as in Theorem* 9, *where we have to take as* η *the random variable with distribution*

$$\begin{aligned} \mathbf{P}(\eta \in B) &= \int_{-\infty}^{0} \frac{\pi(dy)}{\pi(-\infty, 0)} P(y, B) \\ &= \int_{-\infty}^{0} \frac{\pi(dy)}{\pi(-\infty, 0)} \mathbf{P}(y + \xi(y) \in B). \end{aligned}$$

So, if, for instance, conditions of part 1) *of Theorem* 8 *are fulfilled and* $\sup_{y \le 0} \mathbf{E}e^{\beta(y+\xi(y))} < \infty$ *then the behaviour of* $\mathbf{P}(X^0 > x)$ *will be the same as in Example* 5:

$$\mathbf{P}(X^0 > x) = c_4 e^{-\beta x}(1 + o(1)).$$

The results mentioned in this section will be published in (Borovkov, 1993).

Multidimensional Markov chains. What can be said about *multidimensional Markov chains* in $\mathcal{X} = \mathbf{R}^d_+$? Let us assume that

$$\sup_x \mathbf{E}e^{(\lambda, \xi(x))} < \infty$$

for some $\lambda > 0$ and that the Markov chain in question is asymptotically homogeneous:

$$\xi(x_1, \ldots, x_{i_1}, \ldots, x_{i_k}, \ldots, x_d) \Rightarrow \xi(x_1, \ldots, \infty, \ldots, \infty, \ldots, x_d)$$

as $x_{i_1} \to \infty$, …, $x_{i_k} \to \infty$. In this case

$$\limsup_{t \to \infty} \frac{1}{t} \ln \mathbf{P}(\| X^0 \| > t) < 0,$$

i.e. the rate of the decreasing of $\mathbf{P}(\| X^0 \| > t)$ as $t \to \infty$ is exponential. This result was announced on conferences four years ago (see for instance (Borovkov, 1990)) and will soon appear in (Borovkov, 1993).

The only more precise result which we know in this area is that of (Malyshev, 1973). It deals with the simplest Markov chain in $\mathbf{R}^2_+$, for which $\xi(x)$ takes only the values $(0, \pm 1)$ and $(\pm 1, 0)$. The result looks too complicated to reproduce it here.

References

[1] Baccelli, F. and Bremaud, P. (1987). Palm Probabilities and Stationary Queues. Lecture Notes in Statistics 41. Springer-Verlag.

[2] Bhattacharya, R. N. and Lee, O. (1988). Ergodicity and central limit theorems for a class of Markov processes. *J. Multivariate Anal.* **27**, pp. 80–90.

[3] Borovkov, A. A. (1972). *Stochastic Processes in Queueing Theory*. Moscow, Nauka.

[4] Borovkov, A. A. (1990). Ergodicity and stability of Markov chains and of their generalizations. Multidimensional chains. *Probab. Theory and Math. Statistics: Proc. 5-th Vilnius conf.*, 1989, Vilnius, **1**, pp. 179–188.

[5] Borovkov, A. A. (1991). Ergodicity conditions of Markov chains not connected with Harris irreducability. *Siberian Math. J.* **32**, pp. 6–19.

[6] Borovkov, A. A. (1992). On the ergodicity of iterations of random Lipshitz transformations, *Dokl. Russian Acad. Nauk* **324**, pp. 249–251.

[7] Borovkov, A. A. (1993). *Ergodicity and Stability of Markov Processes.* Theory Probab. and Appl., Moscow. (To apper).

[8] Borovkov, A. A., Fayolle, G. and Korshunov, D.A. (1992). Transient phenomena for Markov chains and application. *Adv. in Appl. Probab.* **24**, pp. 322–342.

[9] Brandt, A., Franken, P. and Lisek, B. (1990). *Stationary stochastic models.* Akademie-Verlag/Wiley.

[10] Dobrushin, R. L. (1970). Definition of a system of random variables by means of conditional distributions. *Teor. Veroyatnost. i Primenen.* **25**, pp. 469–497.

[11] Dubins, L. E. and Freedman, D. A. (1966). Invariant probabilities for certain Markov processes. *Ann. Math. Statist.* **37**, pp. 837–847.

[12] Feller, W. (1971). *An Introduction to Probability Theory and Its Applications.* V. 2, J. Wiley, New York – London – Sydney – Toronto.

[13] Korshunov, D. A. (1993). Transition phenomena for real valued Markov chains. *Proc. of the Institute of Mathematics, Russian Academy of Sciences*, **20**, pp. 116–161.

[14] Loynes, R. (1962). The stability of a queue with non-independent inter-arrival and service times. *Proc. Cambridge Philos. Soc.* **58**, pp. 497–520.

[15] Malyshev, V. A. (1973). Asymptotic behaviour of stationary probabilities of two-dimensional positive random walks, *Siberian Math. J.* **14**, pp. 156–169.

[16] Nummelin, E. (1984). *General irreducible Markov chains and non-negative operators.* Cambridge Univ. Press, Cambridge.

[17] Sennott, L., Humblet, P. and Tweedie, R. (1983). Mean drifts and the non-ergodicity of Markov chains. *Operation Research* **31**, pp. 783–789.

[18] Sennott, L. (1987). Conditions for non-ergodicity of Markov chains. *J. Appl. Probab.* **24**, pp. 339–346.