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AN ANALOG OF WALD'S IDENTITY FOR RANDOM WALKS WITH INFINITE MEAN

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Abstract: We deduce an analog of the classical Wald's identity $\mathbf{E}S_\tau = \mathbf{E}\tau\mathbf{E}\xi$ in the case of the infinite mean of summands. We find the conditions on τ under which $\mathbf{E}\min(S_\tau, x) \sim \mathbf{E}\tau\mathbf{E}\min(\xi, x)$ as $x \to \infty$.

Keywords: sums of random variables, stopping time, independence on the future, Wald's identity

Let $\xi, \xi_1, \xi_2, \dots$ be independent identically distributed random variables valued in $\mathbf{R}$. Put $S_n = \xi_1 + \dots + \xi_n$. Let τ be a random variable valued in $\{1, 2, \dots\}$. The following equality, well known as Wald's identity,

$$\mathbf{E}S_\tau = \mathbf{E}\tau\mathbf{E}\xi, \tag{1}$$

holds if both $\mathbf{E}\tau$ and $\mathbf{E}|\xi|$ are finite and τ is such that, for every n,

$$\text{the event } \{\tau \le n\} \text{ does not depend on } \xi_{n+1}. \tag{2}$$

For hitting times and, more generally, for some stopping times, the corresponding theorems were first proved by Wald in [1–3] and then they were improved by Wolfowitz [4] (also see [5, Chapters XII and XVIII]). The general result was formulated and proved by A. N. Kolmogorov and Yu. V. Prokhorov in [6] under a condition similar to (2) (also see [7, Chapter 4]).

In the present article we consider the case of the infinite mean value, $\mathbf{E}|\xi| = \infty$. In this case both left and right sides of (1) may be either nondefined or infinite (positive or negative). Thus we are interested in the asymptotic behavior of $\mathbf{E}\{S_\tau; -y \le S_\tau \le x\}$ as $x, y \to \infty$.

Given an arbitrary random variable X, we denote the positive part $\max(0, X)$ of X by X^+ and the negative part $\max(0, -X)$ of X by X^-, so that $X = X^+ - X^-$. Denote the truncated mean value of the positive part of ξ by

$$m^+(x) = \mathbf{E}\min(\xi^+, x) = \int_0^x \mathbf{P}\{\xi > y\}\, dy,$$

$x > 0$, and the truncated mean value of the negative part of ξ by

$$m^-(x) = \mathbf{E}\min(\xi^-, x) = \int_{-x}^0 \mathbf{P}\{\xi < y\} dy.$$

We observe first that the positive part of the sum may be estimated by the sum of positive parts of summands,

$$S_\tau^+ \le \xi_1^+ + \dots + \xi_\tau^+.$$

Together with the concavity of the function $y \to \min(y, x)$ this implies the inequality

$$\min(S_\tau^+, x) \le \min(\xi_1^+, x) + \dots + \min(\xi_\tau^+, x).$$

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Since $\min(\xi^+, x)$ has a finite mean, the last inequality and (1) yield the upper bound

$$\mathbf{E}\min(S_\tau^+, x) \le \mathbf{E}\tau m^+(x). \tag{3}$$

Symmetrically, $\mathbf{E}\min(S_\tau^-, x) \le \mathbf{E}\tau m^-(x)$. These bounds prompt us a possible version of the asymptotics of $\mathbf{E}\min(S_\tau^+, x)$. Moreover, we assume that $S_n \to \infty$ as $n \to \infty$ with probability 1; in the case $\mathbf{E}|\xi| = \infty$ it holds if and only if (see Corollary 1 in [8])

$$\int_{-\infty}^{0} \frac{|y|}{m^+(-y)} \mathbf{P}\{\xi \in dy\} < \infty. \tag{4}$$

Roughly speaking, this condition means that the right tail of the distribution of ξ is heavier than the left tail. In particular, $m^+(x) \to \infty$ as $x \to \infty$, that is, $\mathbf{E}\xi^+ = \infty$. The following theorem holds.

Theorem 1. *Let τ do not depend on the future in the sense that, for every n,*

$$\sigma(\xi_1, \dots, \xi_n, \mathbf{I}\{\tau \le n\}) \text{ does not depend on } \sigma(\xi_{n+1}, \xi_{n+2}, \dots). \tag{5}$$

If (4) holds then, as $x \to \infty$,

$$\mathbf{E}\min(S_\tau^+, x) \sim \mathbf{E}\tau m^+(x), \quad \mathbf{E}\min(S_\tau^-, x) \le \mathbf{E}\tau m^-(x) = o(m^+(x)).$$

Condition (5) yields, in particular, (2). If the random variable τ does not depend on the sequence $\{\xi_n\}$ then (5) holds. Also, (5) is valid if τ is a stopping time for the sequence $\{\xi_n\}$, that is, if the event $\tau \le n$ is $\sigma(\xi_1, \dots, \xi_n)$-measurable for every n.

Proof. In view of (3), in order to prove the first equivalence it suffices to check that

$$\liminf_{x\to\infty} \frac{\mathbf{E}\min(S_\tau^+, x)}{m^+(x)} \ge \mathbf{E}\tau. \tag{6}$$

The proof follows the idea of "a single big jump" which is known in the theory of subexponential distributions. Fix N and A; later we will let N, $A \to \infty$. We have the following inequality:

$$\mathbf{E}\min(S_\tau^+, x) \ge \sum_{n=1}^{\infty} \sum_{j=1}^{\min(n,N)} \mathbf{E}\{\min(S_n^+, x); \tau = n, \xi_j > A, |\xi_k| \le A \text{ for all } k < j\}.$$

Since the joint occurrence of the events $\{\xi_j > A,\ |\xi_k| \le A \text{ for all } k < j\}$ and $\{S_k - S_j \ge -A \text{ for all } k > j\}$ implies $S_n^+ \ge \xi_j^+ - NA$,

$$\begin{aligned}\mathbf{E}\min(S_\tau^+, x) \ge \sum_{n=1}^{\infty} \sum_{j=1}^{\min(n,N)} \mathbf{E}\big\{ &\min(\xi_j^+ - NA, x); \tau = n, \xi_j > A, \\ &|\xi_k| \le A \text{ for all } k < j, \inf_{k>j}(S_k - S_j) \ge -A\big\}.\end{aligned}$$

By the boundedness of $\min(\xi^+ - NA, x)$ from below and above, it is possible to change the order of summation and to get the following inequality:

$$\begin{aligned}\mathbf{E}\min(S_\tau, x) \ge \sum_{j=1}^{N} \mathbf{E}\big\{ &\min(\xi_j^+ - NA, x); \tau \ge j, \xi_j > A, \\ &|\xi_k| \le A \text{ for all } k < j, \inf_{k>j}(S_k - S_j) \ge -A\big\} \equiv \Sigma.\end{aligned}$$

Since $\{\tau \geq j\} = \overline{\{\tau \leq j-1\}}$, by (5) we have

$$\Sigma = \sum_{j=1}^{N} \mathbf{P}\big\{\tau \geq j, |\xi_k| \leq A \text{ for all } k < j\big\}\mathbf{E}\big\{\min(\xi_j, x); \xi_j > A\big\}\mathbf{P}\big\{\inf_{k>j}(S_k - S_j) \geq -A\big\}.$$

By (4), which is equivalent to the convergence $S_n \to \infty$, the infimum $\inf_k \geq 1 S_k$ is a proper random variable. Therefore, for every $\varepsilon > 0$, there exists A such that

$$\mathbf{P}\big\{\inf_{k\geq 1} S_k \geq -A\big\} \geq 1 - \varepsilon.$$

Then

$$\Sigma \geq (1-\varepsilon)\mathbf{E}\{\min(\xi^+, x); \xi > A\} \sum_{j=1}^{N} \mathbf{P}\{\tau \geq j, |\xi_k| \leq A \text{ for all } k < j\}.$$

Hence,

$$\liminf_{x\to\infty} \frac{\mathbf{E}\min(S_\tau^+, x)}{m^+(x)} \geq (1-\varepsilon) \sum_{j=1}^{N} \mathbf{P}\{\tau \geq j, |\xi_k| \leq A \text{ for all } k < j\}.$$

Letting A to ∞ we deduce the following estimate:

$$\liminf_{x\to\infty} \frac{\mathbf{E}\min(S_\tau^+, x)}{m^+(x)} \geq (1-\varepsilon) \sum_{j=1}^{N} \mathbf{P}\{\tau \geq j\}.$$

By the arbitrary choice of $\varepsilon > 0$ and N, it implies the lower bound (6). The proof of the first equivalence of the theorem is complete.

To prove the second assertion of the theorem, it suffices to prove only the relation $m^-(x) = o(m^+(x))$ as $x \to \infty$. Indeed,

$$\frac{m^-(x)}{m^+(x)} = \frac{x\mathbf{P}\{\xi < -x\}}{m^+(x)} + \int_{-x}^{0} \frac{|y|}{m^+(x)}\mathbf{P}\{\xi \in dy\}.$$

Since the function $x/m^+(x)$ is nondecreasing (e.g., see [9]), the first summand on the right does not exceed

$$\int_{-\infty}^{-x} \frac{|y|\mathbf{P}\{\xi \in dy\}}{m^+(-y)}.$$

The second summand on the right does not exceed

$$\int_{-x}^{-A} \frac{|y|}{m^+(-y)}\mathbf{P}\{\xi \in dy\} + \int_{-A}^{0} \frac{|y|}{m^+(x)}\mathbf{P}\{\xi \in dy\}$$

for any $A < x$. Therefore,

$$\limsup_{x\to\infty} \frac{m^-(x)}{m^+(x)} \leq \int_{-\infty}^{-A} \frac{|y|}{m^+(-y)}\mathbf{P}\{\xi \in dy\},$$

which completes the proof by (4) and the arbitrary choice of A.

Denote the first time when the random walk exceeds the nonlinear boundary $a(n) \geq 0$ by $\eta = \min\{n \geq 1 : S_n \geq a(n)\}$, and denote the overshot by $\chi = S_\eta - a(\eta)$.

Theorem 2. *Let* $\mathbf{E}\xi^+ = \infty$ *and* (4) *hold. If* $a(n) \le cn$ *for some* $c > 0$ *then* $\mathbf{E}\eta < \infty$ *and*

$$\mathbf{E}\min(S_\eta, x) \sim \mathbf{E}\min(\chi, x) \sim \mathbf{E}\eta m^+(x) \quad \text{as } x \to \infty.$$

The particular case of this theorem where $a(n) \equiv 0$ was proved in [9, Lemma 1] by a different method (also see [10, Lemma 2]).

Proof. By (4), we have the almost sure convergence $S_n - cn \to \infty$. Therefore, the minimum of the random walk $S_n - cn$ is finite with probability 1 which is equivalent to the finiteness of the mean of the first exit time from the negative half-line. Hence, $\mathbf{E}\eta < \infty$. Moreover, η satisfies (5) because η is a stopping time. Then it follows from Theorem 1 that

$$\mathbf{E}\min(S_\eta, x) \sim \mathbf{E}\eta m^+(x) \quad \text{as } x \to \infty.$$

Since $\chi \le S_\eta$,

$$\mathbf{E}\min(\chi, x) \le \mathbf{E}\eta m^+(x)$$

by virtue of (3). On the other hand, the inequalities $\min(\chi, x) \ge \min(S_\eta, x) - a(\eta)$, $a(\eta) \le b + c\eta$, and the finiteness of the mean of η imply the following lower bound:

$$\mathbf{E}\min(\chi, x) \ge \mathbf{E}\min(S_\eta, x) - b - c\mathbf{E}\eta \sim \mathbf{E}\eta m^+(x).$$

The proof is complete.

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References

1. *Wald A.*, "On cumulative sums of random variables," Ann. Math. Statist., **15**, No. 3, 283–296 (1944).
2. *Wald A.*, "Some generalizations of the theory of cumulative sums of random variables," Ann. Math. Statist., **16**, No. 3, 287–293 (1945).
3. *Wald A.*, "Differentiation under the expectation sign in the fundamental identity of sequential analysis," Ann. Math. Statist., **17**, No. 4, 493–497 (1946).
4. *Wolfowitz J.*, "The efficiency of sequential estimates and Wald's equation for sequential processes," Ann. Math. Statist., **18**, No. 2, 215–230 (1947).
5. *Feller W.*, An Introduction to Probability Theory and Its Applications. Vol. 2, John Wiley, New York (1971).
6. *Kolmogorov A. N. and Prokhorov Yu. V.*, "On sums of a random number of random terms," Uspekhi Mat. Nauk, **4**, No. 4, 168–172 (1949).
7. *Borovkov A. A.*, Probability Theory [in Russian], Nauka, Moscow (1986).
8. *Erickson K. B.*, "The strong law of large numbers when the mean is undefined," Trans. Amer. Math. Soc., **185**, 371–381 (1973).
9. *Denisov D., Foss S., and Korshunov D.*, "Tail asymptotics for the supremum of a random walk when the mean is not finite," Queueing Syst., **46**, No. 1–2, 15–33 (2004).
10. *Korshunov D. A.*, "The critical case of the Cramér–Lundberg theorem on the asymptotic tail behavior of the maximum of a negative drift random walk," Siberian Math. J., **46**, No. 6, 1077–1081 (2005).

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