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Theory and Methodology

New inequalities for the General Routing Problem

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Abstract

A large new class of valid inequalities is introduced for the General Routing Problem which properly contains the class of 'K-C constraints'. These are also valid for the Rural Postman Problem. A separation algorithm is given for a subset of these inequalities which runs in polynomial time.

Keywords: General Routing Problem; Valid inequalities; Separation algorithms

1. Introduction

The *General Routing Problem* or GRP is an NP-hard vehicle routing problem in which both vertices and edges may require service (Orloff, 1974). It includes the *Rural Postman Problem* (Lenstra and Rinnooy Kan, 1976) and the *Road Travelling Salesman Problem* (Fleischmann, 1985) as special cases. Although it has real-life application (see Eiselt, Gendreau and Laporte, 1995), there is at present no optimisation algorithm available which can solve large GRP instances. This paper extends the current state of knowledge in this area.

The organisation of the paper is as follows: in Section 2, definitions and previous work are reviewed. In Section 3, a new class of valid inequalities (named Path-Bridge inequalities) is introduced for the GRP. They are also valid for the Rural Postman Problem (RPP). It is shown that they are facet-inducing under mild conditions. A polynomial exact separation routine for some of these is given in Section 4. It is concluded that larger problems should now be solvable by a linear programming-based dual cutting-plane method.

2. Definitions and previous work

The GRP was first defined by Orloff (1974). The undirected version, with which we are concerned in this paper, is formally defined by a graph G with vertex set $V=\{1,\ldots,N\}$ and edge set E, a set of edge costs $c_e \geqslant 0$ for each $e \in E$, a set $V_R \subseteq V$ of *required vertices* and a set $E_R \subseteq E$ of *required edges*. The task is to find a minimum-cost route which passes through all of the required vertices and edges.

A branch-and-bound algorithm was proposed for the GRP by Orloff (1974). Unfortunately, the paper contained some errors as shown by Lenstra and Rinnooy Kan (1976), who also showed that the GRP is strongly NP-hard. In fact, the GRP also contains

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some other well-known strongly NP-hard problems as special cases:

(i) When $V_R = \emptyset$, the GRP reduces to the *Rural Postman Problem* or RPP (Lenstra and Rinooy-Kan, 1976; Corberan and Sanchis, 1994).

(ii) When $E_R = \emptyset$, it reduces to the *Road Travelling Salesman Problem* or R-TSP (Fleischmann, 1985). If, in addition, $V_R = V$, the *Graphical Travelling Salesman Problem* or GTSP is obtained (Cornuejols, Fonlupt and Naddef, 1985; Naddef and Rinaldi, 1991, 1993).

Now note that, if $e \in E_R$, any feasible route must pass through the end-vertices of e. Therefore no loss of generality occurs if we assume that all such end-vertices are also in V_R. This is assumed in all that follows.

Corberan and Sanchis (1995) propose an Integer Programming approach to the GRP. Associate a general integer variable x_e with each $e \in E$ representing the number of times e is traversed (if $e \notin E_R$), or one less than this number (if $e \in E_R$). Given a proper subset S of vertices, let $\delta(S)$ represent the set of edges, commonly called the edge-cutset, connecting vertices in S to vertices in $V - S$. When S contains a single vertex i, we write $\delta(i)$ rather than $\delta(\{i\})$ for brevity.

The GRP can then be formulated as:

Minimise $\sum_{e \in E} c_e x_e$

Subject to

$$\sum_{e \in \delta(S)} x_e \geqslant 2 \quad (\forall S \subset V: \; E_R \cap \delta(S) = \emptyset, \; S \cap V_R \neq \emptyset, \; (V - S) \cap V_R \neq \emptyset), \tag{1}$$

$$\sum_{e \in \delta(i)} x_e + |E_R \cap \delta(i)| \text{ even} \quad (\forall i \in V), \tag{2}$$

$$x_e \geqslant 0 \text{ and integer.} \tag{3}$$

Now, for a given graph G, let GRP(G) denote the GRP polyhedron, i.e. the convex hull in $\mathbb{R}^{|E|}$ of solutions to (1)–(3) (the reader is advised to consult Nemhauser and Wolsey, 1988, for the fundamentals of polyhedral theory). Corberan and Sanchis (1995) show that the connectivity constraints (1) and the non-negativity conditions (3) induce facets of GRP(G) under very mild conditions. However, because of the nonlinear constraints (2) and the integrality conditions, more inequalities are required to define GRP(G). In particular, the following 'R-Odd Cut' constraints are valid and normally facet-inducing:

$$\sum_{e \in \delta(S)} x_e \geqslant 1 \quad (\forall S \subset V: \; E_R \cap \delta(S) \text{odd}). \tag{4}$$

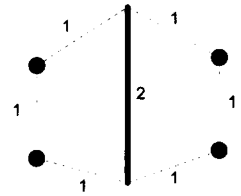

Fig. 1. GRP with c_e on edges; E_R and V_R in bold.

Corberan and Sanchis introduce two further classes of facet-inducing constraints, which they term 'K-C' and 'Honeycomb' constraints. However, the definitions of these are quite lengthy and so we do not give them here. They also show how to derive further GRP facets from facets of the GTSP. The Connectivity constraints (2) are a special case of this.

Now consider the GRP instance of Fig. 1. When this is formulated as above, the LP relaxation in Fig. 2 is obtained after violated connectivity and R-Odd constraints have been appended. Although it is integral, it violates constraints (2) (i.e., there are odd vertices), and therefore cannot represent a feasible tour. Furthermore, it can be shown that no K-C, Honeycomb or 'GTSP-type' constraint is violated. This observation led to the discovery of the Path-Bridge inequalities which are the subject of the following section.

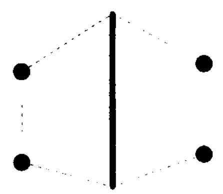
Fig. 2. LP relaxation after cuts added: Dashed lines represent $x_e = 1$.

Before proceeding, one more definition is presented which will later prove to be necessary. Suppose the set V of vertices of a graph G is partitioned into two sets T and $V-T$, with $|T|$ even. A *T-join* is a set of edges which meet each vertex in T an odd number of times, but meet each vertex in $V-T$ an even number of times (see, e.g., Nemhauser and Wolsey, 1988). It is easy to show that, if G is connected, there exists at least one T-join in G.

3. A new class of facets: The path-bridge inequalities

We now define a new class of inequalities. It will be shown that these include the K-C constraints and also some 'GTSP-type' constraints as special cases. Let P and B be integers with $P \geqslant 1$, $B \geqslant 0$, $P+B \geqslant 3$ and odd. Let n_i $(i=1,\dots,P)$ be integers greater than one. Consider a partition of V into the non-empty sets A, Z, V_j^i $(i=1,\dots,P;\ j=1,\dots,n_i)$. For convenience, for all i we identify V_j^i with A when $j=0$ and with Z when $j=n_i+1$. Let the partition be such that:

- each induced subgraph $G(V_j^i)$ is connected;
- each required edge either lies within some $G(V_j^i)$ or crosses from A to Z;
- $V_j^i \cap V_R \neq \emptyset$ $(i=1,\dots,P;\ j=1,\dots,n_i)$;
- there are some non-required edges in G between V_j^i and V_{j+1}^i for all i and $j=0,\dots,n_i$;
- there are B required edges with one endpoint in A and one endpoint in Z.

The (possibly empty) set of B required edges crossing from A to Z will be called the *bridge*. The resulting *Path-Bridge structure* is shown in Fig. 3. The circles represent the vertex sets, the bold edge represents the bridge and the dashed edges between the V_j^i represent non-required edges.

The structure is very similar to that of the Path inequalities for the TSP and GTSP given in Cornuejols, Fonlupt and Naddef (1985). Indeed, if $B=0$, the structure is identical. However, we allow the bridge to contain one or more required edges.

Corresponding to the Path-Bridge structure we have the following *Path-Bridge inequality* or PBI:

$$\sum_{e\in E} \alpha_e x_e \geqslant 1 + \sum_{i=1}^{P} \frac{n_i+1}{n_i-1},$$

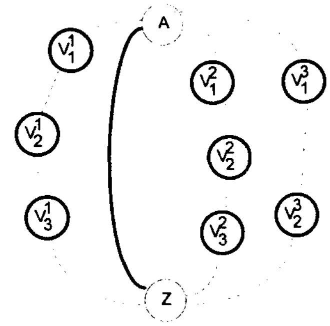

Fig. 3. Path-Bridge structure.

where:

$\alpha_e = 1$ if $i \in A$, $j \in Z$;

$\alpha_e = (q-p)/(n_r-1)$

if e connects V_p^r to V_q^r $(p<q)$, either $p \neq 0$ or $q \neq n_r+1$;

$$\alpha_e = 1/(n_r-1) + 1/(n_s-1) + \left| \frac{p-1}{n_r-1} - \frac{q-1}{n_s-1} \right|$$

if $i \in V_p^r$, $j \in V_q^s$, $r \neq s$, $1<p<n_r$, $1<q<n_s$; and

$\alpha_e = 0$, otherwise.

Just as in Cornuejols, Fonlupt and Naddef (1985), we can define the special *n-regular* PBIs in which n_i for each path is the same number n. In this case the constraint can be rewritten so that:

$\alpha_e = n-1$ if $i \in A$, $j \in Z$;

$\alpha_e = q-p$

if e connects V_p^r to V_q^r, $p<q$, either $p \neq 0$ or $q \neq n+1$;

$\alpha_e = 2 + |p-q|$

if $i \in V_p^r$, $j \in V_q^s$, $r \neq s$, $1<p<n$, $1<q<n$; and

$\alpha_e = 0$, otherwise,

and the right-hand side equals $P \cdot n + P + n - 1$.

Lemma. *PBIs are valid for* GRP(G).

Proof. If $P=1$, we have a K-C constraint (Corberan and Sanchis, 1994). If $B=0$, we have a Path constraint. If $P \geqslant 3$ and odd but $B \neq 0$, the argument in Corneujols, Fonlupt and Naddef (1985) for the validity of ordinary Path inequalities for the GTSP transfers directly, because the extra required edges in the bridge do nothing to resolve the parity problem presented by the odd number of paths. The argument in Cornuejols, Fonlupt and Naddef also shows the validity when $P \geqslant 2$ and even, because a parity problem is introduced by the odd number of edges in the bridge.

Now let T equal the set of all vertices in G which meet an odd number of required edges. Clearly, $|T \cap V_j^i|$ is even for $i = 1,\dots, P$; $j = 1,\dots, n_i$. Also, both $|T \cap A|$ and $|T \cap Z|$ will be even if and only if P is odd. It is now possible to prove:

Theorem. *The PBIs with P odd induce facets of* GRP(G).

Proof. We need only consider the case $P \geqslant 3$, since when $P = 1$ we have a K-C constraint. Each $G(V_j^i)$ is connected by assumption. Therefore there exists at least one T-join in G which uses only edges within the $G(V_j^i)$. Let J be one such T-join. Consider the GTSP instance obtained from the GRP instance by making all vertices required and all edges non-required. Then the corresponding Path inequality defines a facet of GTSP(G) (Cornuejols, Fonlupt and Naddef, 1985). But from every GTSP tour satisfying the Path inequality as an equality we can derive a GRP tour satisfying the PBI at equality by adding one copy of each edge in J. In this way we derive $|E|$ affinely independent tours. The proof is completed by noting that

$$\dim(\mathrm{GRP}(G)) = \dim(\mathrm{GTSP}(G)) = |E|.$$

A similar argument can be given to show that the inequality induces a facet when P is even. It is an inequality of this latter kind which cuts off the extreme point shown in Fig. 2. This inequality, having $P = 2$, $B = 1$ and $n_1 = n_2 = 2$, is the simplest possible PBI different from a K-C or Path constraint.

4. A separation routine

In order to actually use PBIs as cutting-planes, we require an algorithm which will generate a violated PBI in an LP relaxation, if one exists. Such an algorithm is called a *separation routine* (see Nemhauser and Wolsey, 1988). In this section we give a polynomial time exact separation routine for a simple subset of PBIs. It should be noted that polynomial exact separation routines are already known for the Connectivity and R-Odd constraints and a rapid heuristic routine is known for the K-C constraints (Corberan, 1995).

A PBI will be termed *simple* if it is 2-regular and every V_j^i (for $i = 1,\dots, P$ and $j = 1, 2$) consists of a single required vertex. Then for any $i = 1,\dots, P$ we have that V_1^i and V_2^i are connected by a single edge, say e_i. Now let

$$S = A \cup \left(V_1^1 \cup \cdots \cup V_1^P\right),$$

$$\delta(e_i) = \delta\left(V_1^i \cup V_2^i\right)$$

and

$$F = e_1 \cup \cdots \cup e_P.$$

The simple PBI can now be rewritten as

$$\sum_{e \in \delta(S) - F} x_e + \sum_{e \in F} \left(x_e + x(\delta(e))\right) \geqslant 3|F| + 1, \quad (5)$$

which in turn is equivalent to

$$\sum_{e \in \delta(S) - F} x_e + \sum_{e \in F} \left(x_e + x(\delta(e)) - 3\right) \geqslant 1. \quad (6)$$

A violated constraint in this form can be identified in the same way as the *2-matching* constraints of the conventional TSP (Padberg and Rinaldi, 1990). In the present context we require $|F|$ to be odd and $|\delta(S) \cap E_{\mathrm{R}}|$ even, or vice-versa. We assume that all violated connectivity constraints have already been appended to the LP. The separation algorithm is then as follows:

Form a graph $G^*(V, E^*)$ with E^* containing the edges of E such that x_e is currently non-zero. Label all vertices *even* apart from those which are adjacent to an odd number of required edges in G; these should be labelled *odd*. Call those edges in E^* which correspond to non-required edges in G with both endpoints required *suitable*. Split each suitable edge in E^* into two halves by the creation of an extra even vertex in the middle. Give one half (the *normal* half) a weight of x_e and the other half (the

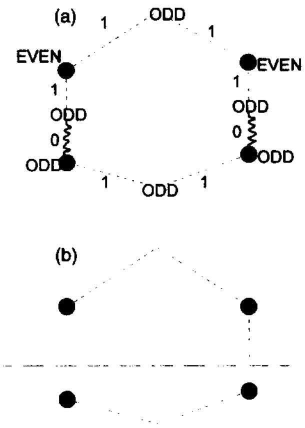

Fig. 4a. Graph derived from example in Fig. 2 (wavy lines represent altered halves).

Fig. 4b. An odd-cut with weight < 1.

altered half) a weight of $x_e + x(\delta(e)) - 3$. Reverse the label of any vertices in the resulting graph which are adjacent to an odd number of altered halves. Note that the resulting graph will always have non-negative edge-weights, because

$$2[x_e + x(\delta(e))] = x(\delta(e_i)) + x(\delta(V_1^i)) + x(\delta(V_2^i)),$$

which is $\geqslant 6$ if connectivity constraints are satisfied.

A minimum odd-cut in the resulting graph will have a weight less than 1 if a simple PBI is violated (the edges of G which are suitable candidates for the set F are precisely those for which the cut passes through their 'altered' half). Minimum odd-cuts can be obtained in polynomial time from a minimum cut-tree as explained in Padberg and Rao (1982). The process is illustrated in Figs. 4a and 4b.

The author has done some preliminary testing of this routine on some small GRPs. Typically, Connectivity and R-Odd constraints alone are not sufficient to arrive at the optimum, although the lower bound is often 95% of the optimal value. When Simple PB-inequalities are also used, the lower bound normally goes up to around 98.7%, though this depends on the number of non-required vertices.

5. Conclusion

The Path-Bridge inequalities are a wide generalisation of the K-C and Path constraints. It is already known that Path and K-C configurations can be linked together in a variety of ways to yield new facets (Naddef and Rinaldi, 1991; Corberan and Sanchis, 1995); the author has an example of an inequality derived from linking two PB configurations together. It appears therefore that the GRP polyhedron contains a bewildering variety of facet classes (and this is the case even for the less general RPP polyhedron).

It is the author's belief that future research should be focused primarily on devising new separation routines for some of the simpler classes of facets. In this way, one might hope to solve large-scale GRP and RPP instances to optimality. In particular, the simple PBI separation routine is of no use unless a number of required vertices are adjacent to eachother. It would be desirable to devise some 'shrinking' heuristics (see Fleischmann, 1985, and Padberg and Rinaldi, 1990), to be applied prior to the separation routine, in order to identify more general 2-regular PBIs.

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