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Theory and Methodology

The general routing polyhedron: A unifying framework

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Abstract

It is shown how to transform the General Routing Problem (GRP) into a variant of the Graphical Travelling Salesman Problem (GTSP). This transformation yields a projective characterisation of the GRP polyhedron. Using this characterisation, it is shown how to convert facets of the GTSP polyhedron into valid inequalities for the GRP polyhedron. The resulting classes of inequalities subsume several previously published classes.

Keywords: Polyhedra; Valid inequalities; General routing problem

1. Introduction

The *General Routing Problem* (GRP) is the NP-Hard problem of routing a single vehicle at minimum cost subject to the requirement that certain vertices must be visited at least once and certain edges must be traversed at least once (Orloff, 1974; Lenstra and Rinnooy-Kan, 1976). Formally, a GRP instance is given by a loopless, connected, undirected graph G with vertex set V and edge set E, a set $V_R \subseteq V$ of *required vertices*, a set $E_R \subseteq E$ of *required edges* and a cost c_e for each $e \in E$. The task is to find a minimum cost *tour*; i.e., a family of edges, possibly allowing repetitions, which represents a valid vehicle route.

The GRP has many practical applications (see Eiselt et al., 1995) and includes several well-known routing problems as special cases:

- the *Rural Postman Problem* (RPP), obtained when $V_R = \emptyset$ (Orloff, 1974). If, in addition, $E_R = E$, the RPP becomes the *Chinese Postman Problem* (Edmonds and Johnson, 1973).
- the *Steiner Graphical Travelling Salesman Problem* (SGTSP), obtained when $E_R = \emptyset$ (Cornuéjols et al., 1985; it was also called the *Road Travelling Salesman Problem* by Fleischmann, 1985). If, in addition, $V_R = V$, the SGTSP becomes the *Graphical Travelling Salesman Problem* or GTSP (Cornuéjols et al., 1985; Naddef and Rinaldi, 1993).

When a GTSP instance is defined on a complete graph, it is a relaxation of the well-known *Travelling Salesman Problem* or TSP (see Jünger et al., 1995). In the TSP, each vertex must be visited exactly once, which in turn implies that each edge

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may be traversed at most once. No such restriction applies to the GTSP.

The CPP is polynomially solvable (Edmonds and Johnson, 1973), but the GTSP and RPP are NP-Hard (Cornuéjols et al., 1985; Lenstra and Rinnooy-Kan, 1976, respectively). The GRP is therefore also NP-Hard. In fact, the GRP can be transformed into the TSP (Jünger et al., 1995), but only at the cost of introducing a large number of redundant variables.

Corberán and Sanchis (1996) give an integer programming formulation of the GRP. Assume, without loss of generality, that the end-vertices of each $e \in E_R$ are in V_R (clearly, the vehicle must pass through all such vertices). Associate a general integer variable x_e with each $e \in E$, representing the number of times e is traversed (if $e \notin E_R$), or one less than this number (if $e \in E_R$). For any $S \subset V$ let $\delta(S)$ denote the set of edges in G, known as the *cutset*, connecting S to $V \setminus S$, S_R denote $S \cap V_R$ and $\delta_R(S)$ denote $\delta(S) \cap E_R$. Finally, for any $F \subset E$, let $x(F)$ denote $\sum_{e \in F} x_e$. The formulation is:

$$\text{minimise} \sum_{e \in E} c_e x_e$$

subject to

$$x(\delta(S)) \geqslant 2 \quad (\forall S \subset V\colon\ \delta_R(S) = \emptyset, S_R \neq \emptyset, S_R \neq V_R), \qquad (1)$$

$$x(\delta(\{i\})) \equiv |\delta_R(\{i\})| \bmod 2 \quad (\forall i \in V), \qquad (2)$$

$$x_e \geqslant 0 \text{ and integer} \quad (\forall e \in E). \qquad (3)$$

Constraints (1), called *connectivity inequalities*, ensure that the tour is connected. Constraints (2), called *degree conditions*, ensure that the tour is Eulerian. They can be linearised through the addition of extra integer variables. The non-negativity conditions in Eq. (3) are sometimes called *trivial inequalities*.

The convex hull in $\mathbb{R}^{|E|}$ of solutions to Eqs. (1)–(3) is a full-dimensional, unbounded polyhedron and will be denoted by GRP(G, V_R, E_R). It is easily shown that all valid inequalities for GRP(G, V_R, E_R) are of greater-than-or-equal type with no negative coefficients. When $E_R = \emptyset$, we will also denote the associated polyhedron by SGTSP(G, V_R). When, in addition, $V_R = V$, we will write GTSP(G).

In order to produce an algorithm capable of solving realistic GRP instances to optimality, it seems likely that an understanding of these integer polyhedra will be required (see Nemhauser and Wolsey, 1988). Results on the GRP polyhedron have been given by Corberán and Sanchis (1996) and Letchford (1997). The present paper generalises and unifies these results.

The outline of the paper is as follows. In Section 2, some of the known polyhedral results for the GTSP are reviewed and it is shown how to generalise these to the SGTSP. In Section 3, it is shown how to transform any instance of the GRP into an instance of the SGTSP. This yields a proof that each GRP polyhedron is equal to the projection of a face of an associated SGTSP polyhedron. Moreover, there is a simple mapping from valid inequalities for the latter polyhedron onto valid inequalities for the former. In Section 4, existing valid inequalities for the GRP are interpreted within the new framework. In Section 5, examples are given of how the results of Sections 2 and 3 can yield new, hitherto unknown facet-inducing inequalities for certain GRP instances. Finally, concluding comments are given in Section 6.

2. From the GTSP to the SGTSP

In this section, it is shown how to adapt valid inequalities and facets of the GTSP to the SGTSP. This is useful because there is a vast literature on GTSP polyhedra (see the excellent surveys, Jünger et al., 1995, 1997), but only two papers which present valid inequalities for the SGTSP (Cornuéjols et al., 1985 and Fleischmann, 1985).

Space does not permit a thorough review of the literature on GTSP polyhedra and the reader is referred to the above survey papers (see also Edmonds, 1965; Grötschel and Padberg, 1979; Cornuéjols et al., 1985; Grötschel and Pulleyblank, 1986; Fleischmann, 1987, 1988; Boyd and Cunningham, 1991; Naddef, 1990, 1992). However, certain results will be needed in this paper and are therefore presented now.

In Naddef and Rinaldi (1993), it is shown that all non-trivial facet-inducing inequalities for the GTSP on a complete graph have coefficients obeying the triangle inequality. Moreover, two important operations are defined for converting valid (or facet-inducing) inequalities for the GTSP on a complete graph into valid (respectively, facet-inducing) inequalities for the GTSP on a larger complete graph.

The first operation is called *zero-lifting*. Let n be a positive integer, ζ^- be a valid inequality for GTSP(K_n) and $u \in \{1, \ldots, n\}$ be an arbitrary vertex of K_n. Also let m be a positive integer and K_{n+m} be the complete graph obtained by adding m extra vertices and the appropriate extra edges. Then a valid inequality ζ^+ for GTSP(K_{n+m}), with the same RHS as ζ^-, can be formed as follows. Let $e = \{v, w\}$ be an edge of K_{n+m}. If $v, w \in \{1, \ldots, n\}$, x_e receives the same coefficient as before. If $v \in \{u, n+1, n+2, \ldots, n+m\}$ and $w \in \{n+1, n+2, \ldots, n+m\}$, x_e receives a coefficient of zero. If $v \in \{1, \ldots, n\} \backslash u$, and $w \in \{n+1, n+2, \ldots, n+m\}$, x_e receives the same coefficient as $x_{\{u,v\}}$.

The second operation is called 1-*node lifting*. It is a way of converting a valid inequality for GTSP(K_n) into a valid inequality for GTSP(K_{n+1}) with the same RHS, which is different from zero-lifting with $m = 1$. There is no need to formally define it here.

We will call an inequality *simple* if it is not a zero-lifting of any other inequality, and *primitive* if it is simple and is also not a 1-node lifting of any other inequality.

Naddef (1990) reviewed a number of papers on the GTSP and related problems, and noted that many (but not all) of the known valid inequalities for GTSP(G) can be expressed in the form

$$\sum_{i=1}^{p} \alpha_i x(\delta(H_i)) + \sum_{j=1}^{q} \beta_j x(\delta(T_j)) \geqslant \gamma,$$

where the $H_i (i = 1, \ldots, p)$ are sets of vertices called *handles*, the $T_j (j = 1, \ldots, q)$ are sets of vertices called *teeth*, and the handles and teeth satisfy certain conditions. We will call such inequalities *handle–tooth-cutset* (HTC) inequalities. Fig. 1, adapted from Naddef (1990), displays every known class of HTC inequalities at the time of writing. An arrow from one class to another means that the former class is contained in the latter.

Given a set of handles and teeth, one can define an equivalence relation on the vertices, putting two vertices in the same class if they belong to the same handles and teeth. Naddef (1990, 1992) calls these classes *HT-classes*.

The following results are implicit in Naddef (1990) and Naddef and Rinaldi (1991, 1993). When an HTC inequality is primitive, all non-empty HT-classes have a cardinality of one. All such HT-classes are labelled *even* in Naddef and Rinaldi (1991). When an HTC inequality is simple but not primitive, it can be obtained from a primitive HTC inequality by applying 1-node lifting one or more times. With each application of 1-node lifting, an HT-class which was previously empty gains a single member. All such HT-classes are labelled *odd* in Naddef and Rinaldi (1991).

A non-simple HTC inequality can be obtained from a simple HTC inequality by applying zero-lifting one or more times. With each application of zero-lifting, an HT-class which previously had one member gains one or more new members. It follows that an HTC inequality is non-simple if and only if at least one HT-class has a cardinality of two or more.

We are now ready to present the main results of this section:

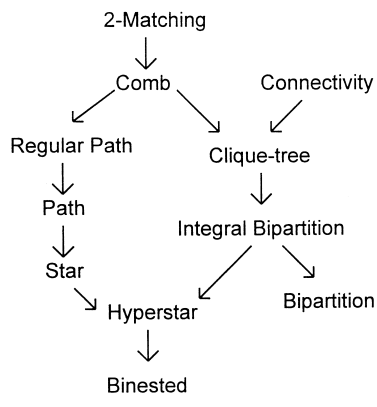

Fig. 1. The known classes of HTC inequalities.

Theorem 1. *Let $G(V, E)$ be a graph and $V_R \subseteq V$. Let $G^+(V, E^+)$ and $G^-(V_R, E^-)$ be complete graphs on V and V_R, respectively. We assume that the vertices and edges are indexed in a natural way so that G^- and G are subgraphs of G^+. Then if*

$$\zeta^+: \sum_{e \in E^+} \alpha_e x_e \geqslant \beta$$

is a valid inequality for GTSP(G^+) whose coefficients obey the triangle inequality, and

$$\zeta^-: \sum_{e \in E^-} \alpha_e x_e \geqslant \beta$$

is valid for GTSP(G^-), then

$$\zeta: \sum_{e \in E} \alpha_e x_e \geqslant \beta$$

is valid for SGTSP(G, V_R).

Proof. Note that, if the theorem is true for complete graphs G, then it is true for all graphs. This is because removing edges of G can only cause a shrinking of SGTSP(G, V_R). Thus we assume that G is complete. Suppose that an SGTSP tour exists that violates ζ. This tour can be written as a circular sequence of vertices, possibly with repetitions, representing the order in which the vertices are visited. Now consider the new SGTSP tour formed by taking "shortcuts"; that is, by observing the same sequence but omitting the visits to non-required vertices. The new tour uses only edges in E^- and, by the triangle inequality, also violates ζ. Yet this is impossible, since this would imply that ζ^- is not valid for GTSP(G^-). □

Theorem 1 can be applied whenever ζ^+ can be obtained from ζ^- by zero-lifting or 1-node lifting. In particular, we have the following corollary.

Corollary 2. *Let G, V_R and G^+ be defined as in Theorem 1. If*

$$\zeta^+: \sum_{e \in E^+} \alpha_e x_e \geqslant \beta$$

is a valid HTC inequality for GTSP(G^+) such that every even HT-class contains at least one vertex in V_R, then

$$\zeta^+: \sum_{e \in E} \alpha_e x_e \geqslant \beta$$

is valid for SGTSP(G, V_R).

Proof. It is known (Naddef and Rinaldi, 1993) that the coefficients in an HTC inequality obey the triangle inequality. Moreover, if every even HT-class contains at least one vertex in V_R, then ζ^+ can be obtained from a suitable ζ^- by a combination of 1-node lifting and zero-lifting. □

This means that all of the classes of inequalities in Fig. 1 have SGTSP analogues. Cornuéjols et al. (1985) noted that this was true for the connectivity inequalities and Fleischmann (1985) proved it for a special case of the star inequalities.

The use of Corollary 2 can be made more clear by an example. Fig. 2 shows a primitive 2-matching inequality (Edmonds, 1965) valid for GTSP(K_6). The bold ellipse represents the handle, the plain ellipses represent the teeth, and the small black circles represent the vertices. For this inequality, all α_i and β_j are equal to one and the RHS is 10. Fig. 3 shows a simple 2-matching inequality, valid for GTSP(K_8), obtained from the primitive inequality by applying 1-node lifting twice. Fig. 4 shows a non-simple 2-matching inequality, valid for GTSP(K_9), obtained from the simple 2-matching inequality by applying zero-lifting once. Finally, Fig. 5 shows a graph G on which an SGTSP instance is defined, where the white circles represent non-required vertices. Theorem 1 enables us to derive the inequality

$$\sum_{e \in \hat{E}} x_e \geqslant 10 \tag{4}$$

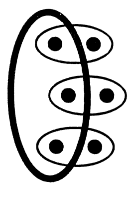

Fig. 2. A primitive 2-matching inequality, RHS = 10.

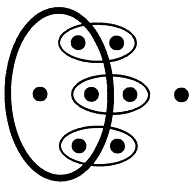

Fig. 3. A simple 2-matching inequality, RHS = 10.

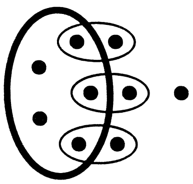

Fig. 4. A non-simple 2-matching inequality, RHS = 10.

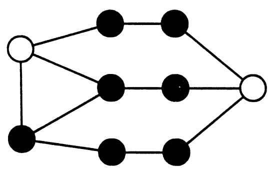

Fig. 5. Graph for an SGTSP instance.

for SGTSP(G, V_R), where $\hat{E}$ contains all edges apart from the vertical edge on the left of the diagram. Moreover, Eq. (4) can be shown to be facet-inducing by enumeration.

This raises the question of when inequalities resulting from Theorem 1 define facets of SGTSP(G). This question is likely to be hard to answer, since, to the knowledge of the author, necessary and sufficient conditions have not yet been found even for the star inequalities to induce facets of GTSP(G). A partial answer is however provided by the following.

Theorem 3. *A valid inequality for SGTSP(G, V_R) derived from an application of Theorem 1 induces a facet of SGTSP(G, V_R) if it induces a facet of GTSP(G).*

Proof. By assumption, the inequality induces a facet of GTSP(G). Therefore, since GTSP(G) is full-dimensional, there are $|E|$ affinely independent vectors satisfying the inequality as an equality. Since the SGTSP on G is a relaxation of the GTSP on G, these $|E|$ affinely independent vectors are also contained in SGTSP(G, V_R). The proof is completed by noting that SGTSP(G, V_R) is also $|E|$-dimensional. □

The author does not know whether this condition is also necessary.

3. From the SGTSP to the GRP

In this section, the results of the previous section are adapted to the GRP. To begin with, it is shown how to transform any GRP instance into an SGTSP instance, by an appropriate modification of G and the cost function. The transformation is of interest in its own right, since it preserves useful properties, such as sparsity or planarity, that G might have. However, its main use will be to provide new valid inequalities for GRP polyhedra.

The transformation involves doing the following for each required edge $e = \{u, v\}$ (that is, u and v are the end-vertices of e):

- add two new *required vertices* (u' and v', say);
- add three new *edges* $\{u, u'\}$, $\{u', v'\}$ and $\{v', v\}$;
- give these new edges a very large cost, M;
- make e non-required.

Fig. 6 shows a graph G for a GRP instance (bold lines represent required edges). Fig. 7 shows the graph G' obtained after the transformation.

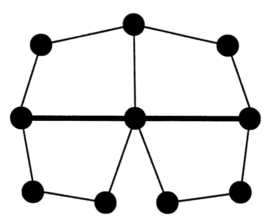

Fig. 6. Graph for a GRP instance.

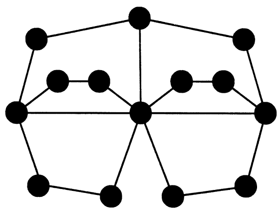

Fig. 7. Graph G' derived from the graph in Fig. 6.

We will call the new vertices (resp., edges) *fixing vertices (edges)*. The transformed network will be denoted by G', the fixing vertices by V^f and the fixing edges by E^f. Note that G' has $|V| + 2|E_R|$ vertices and $|E| + 3|E_R|$ edges. It is easily checked that an optimal solution to the SGTSP will be such that each $e \in E^f$ is crossed exactly once (this incurs an extra cost of 3 M $|E_R|$). There is a one-to-one correspondence between solutions to the original GRP and solutions to the SGTSP such that each $e \in E^f$ is crossed exactly once.

Now let SGTSP(G', $V_R \cup V^f$) denote the convex hull in $\mathbb{R}^{|E|+3|E_R|}$ of feasible solution vectors for the SGTSP tours in G'. For any required edge $e = \{u, v\}$ in G, consider once again the new vertices u' and v' in G'. The following three connectivity inequalities are valid for SGTSP(G', $V_R \cup V^f$):

$$x\{u,u'\} + x\{u',v'\} \geqslant 2,$$
$$x\{u,u'\} + x\{v',v\} \geqslant 2,$$
$$x\{u',v'\} + x\{v',v\} \geqslant 2.$$

Let Ψ denote the set of such connectivity inequalities, three for each required edge. That is, $|\Psi| = 3\ |E_R|$. Let $\boldsymbol{P^*}$ denote the face of SGTSP(G', $V_R \cup V^f$) containing all vectors satisfying the inequalities in Ψ at equality. Solving the simultaneous equations yields $x\{u, u'\} = x\{u', v'\} = x\{v', v\} = 1$. This means that the extreme points of $\boldsymbol{P^*}$ represent the SGTSP tours mentioned above, in which each fixing edge is crossed once. The one-to-one correspondence between these tours and tours of the GRP immediately implies the following theorem.

Theorem 4. *GRP(G, V_R, E_R) is equal to the projection of* $\boldsymbol{P^*}$ *onto the* $|E|$ *dimensional subspace defined by the original (i.e., non-fixing) edges.*

As a consequence of elementary properties of polyhedra, we have the following corollary.

Corollary 5. *If* $\sum_{e \in E \cup E^f} \alpha_e x_e \geqslant \beta$ *is a valid inequality for SGTSP(G', $V_R \cup V^f$), then* $\sum_{e \in E} \alpha_e x_e \geqslant \beta - \sum_{e \in E^f} \alpha_e$ *is valid for GRP(G, V_R, E_R). Moreover, any facet-inducing inequality for GRP(G, V_R, E_R) can be obtained in this way.*

In the light of Corollary 2, these results imply that each class of inequalities in Fig. 1 has a generalisation in terms of the GRP. The author calls the corresponding inequalities *projected clique-tree*, *projected path*, etc., inequalities.

4. Known inequalities interpreted

To the knowledge of the author, Corberán and Sanchis (1996) and Letchford (1997) are the only available references on GRP polyhedra. In this section, it will be shown how most of the inequalities presented in these papers can be obtained as projections of SGTSP inequalities which in turn are analogous to known GTSP inequalities.

4.1. GTSP-type inequalities

Suppose we are given a GRP instance and associated G, V_R and E_R. Let G' represent the transformed graph as defined in the previous section. If the subgraph of G obtained by deleting all non-required vertices and edges is disconnected, it will be possible to partition the vertices of G' into sets S^i ($i = 1,\ldots, m$) such that

1. $S^i_R \neq \emptyset$ for $i = 1,\ldots, m$.
2. No *crossing* edge (viz., edge with end-vertices in different sets) is a fixing edge.

Now let $p = |V| + 2|E_R|$, let the complete graph K_p have its vertices and edges indexed so that it contains G' as a subgraph, let the complete graph K_m have its vertices and edges indexed such that it is a subgraph of K_p with one vertex in each S^i, and let ζ^- be a valid inequality for GTSP(K_m). By zero-lifting, it will be possible to produce an inequality ζ^+, valid for GTSP(K_p), such that the coefficient of any non-crossing edge is zero. An application of Theorem 1 then yields an inequality ζ valid for SGTSP(G', $V_R \cup V^f$) such that the coefficient of any non-crossing edge is zero. This in turn implies that the coefficient of any fixing edge is zero. An application of Corollary 5 then allows ζ to be 'trivially' projected, yielding a valid inequality for GRP(G, V_R, E_R).

The resulting inequalities are called *GTSP-type inequalities* in Corberán and Sanchis (1996). In that paper, conditions are given for them to induce facets of GRP(G, V_R, E_R). The connectivity inequalities for the GRP are GTSP-type inequalities since they are obtained by trivially projecting connectivity inequalities for the GTSP.

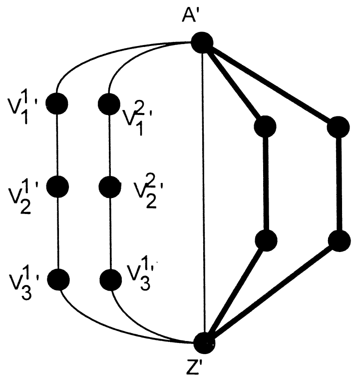

Fig. 8. Partition of transformed graph G' (crossing fixing edges in bold).

4.2. Path-bridge, K-C and R-odd cut inequalities

Let $p \geqslant 0$ and $b \geqslant 0$ be integers such that $p + b \geqslant 3$ and odd. If $p \geqslant 1$, let $n_i \geqslant 2$ $(i = 1, \ldots, p)$ be integers. Suppose there is a partition of V into sets A, Z, V^i_j $(i = 1, \ldots, p, j = 1, \ldots, n_i)$, such that:

(i) For all i and j, $V^i_j \cap V_R \neq \emptyset$ and $G(V^i_j)$ is connected.

(ii) There is a non-required edge connecting V^i_j to V^i_{j+1} for $i = 1, \ldots, p$ and $j = 1, \ldots, n_i - 1$.

(iii) There are b crossing required edges, all of which cross from A to Z.

If $b = 0$, then A and Z are permitted to be empty.

Given such a partition of V, a corresponding partition of $V_R \cup V^f$ (i.e., the vertices in G') can be defined as follows (see Fig. 8 for an illustration). We let A' equal the union of A and the fixing vertices arising from required edges having both end-vertices in A. We define Z' and $V^i_{j'}$ similarly. The remaining unassigned vertices in G' are each put into their own class. There are 2 b such vertices, since each of the b required edges in G is of the form $\{u, v\}$ with $\{u\} \in A$ and $\{v\} \in Z$ and there are two corresponding fixing vertices $\{u'\}$ and $\{v'\}$.

There are now 3 b crossing fixing edges. Let C denote the union of these edges. Using Corollary 2, Theorem 3 and results in Cornuéjols et al. (1985), the following *path inequality* can be shown to induce a facet of SGTSP(G', $V_R \cup V^f$), where for convenience we identify A with V^i_0 and Z with $V^i_{n_i+1}$ for all i:

$$\sum_{\{i,j\} \in E \cup E^f} \alpha_{ij}\, x_{ij} \geqslant 1 + 3\,b + \sum_{i=1}^{p} \frac{n_i + 1}{n_i - 1},$$

where α_{ij} equals:

$1 \quad$ if $i \in A, \; j \in Z,$

$(r - q)/(n_s - 1) \quad$ if $i \in V^s_q$ and $j \in V^s_r (q < r),$

either $q \neq 0$ or $r \neq n_s + 1,$

$$1/(n_s - 1) + 1/(n_t - 1) + \left| \frac{q - 1}{n_s - 1} - \frac{r - 1}{n_t - 1} \right|$$

if $i \in V^s_q, \; j \in V^t_r, \; s \neq t, \; 1 < q < n_s, \; 1 < r < n_t,$

$1 \quad$ if $\{i, j\} \in C,$

$0 \quad$ otherwise.

An application of Corollary 5 then yields the *projected path inequality*

$$\sum_{\{i,j\} \in E} \alpha_{ij}\, x_{ij} \geqslant 1 + \sum_{i=1}^{p} \frac{n_i + 1}{n_i - 1}$$

valid for GRP(G, V_R, E_R).

When $p \geqslant 1$, the projected path inequality is a *path-bridge inequality* or PBI (Letchford, 1997). PBIs induce facets of GRP(G, V_R, E_R) under mild conditions. They also subsume the K–C inequalities of Corberán and Sanchis (1994, 1996): K–C inequalities are obtained when $p = 1$.

When $p = 0$, the path inequalities in G' reduce to 2-matching inequalities. The resulting *projected 2-matching inequalities* take the form $x(\delta(A)) \geqslant 1$ and are equivalent to the (generally facet-inducing) *R-odd cut inequalities* of Corberán and Sanchis (1996).

In fact, Corberán and Sanchis (1996) also define R-odd cut inequalities with $|\delta_R(A)| = b = 1$. These too can be viewed as projected 2-matching inequalities, since 2-matching inequalities remain valid for SGTSP(G', $V_R \cup V^f$), though not facet-inducing, when $b = 1$. It is also possible to derive R-odd cut inequalities with $b = 1$ as *projected connectivity inequalities*, but the details are omitted for brevity.

The above discussion is summarised in the following theorems.

Theorem 6. *Path-bridge inequalities (and therefore K–C inequalities) are projected path inequalities.*

Theorem 7. *R-odd cut inequalities are projected 2-matching inequalities.*

The author conjectures that projected path inequalities which are neither R-odd cut nor path-bridge inequalities are redundant for GRP(G, V_R, E_R).

4.3. Honeycomb inequalities

The only remaining class of inequalities known for the GRP is that of the *honeycomb inequalities* (Corberán and Sanchis, 1996). Like the path-bridge inequalities, these contain the K–C inequalities as a special case. Like GTSP-type inequalities, the honeycomb inequalities are defined in terms of a partition of V into sets S^i with $S^i_R \neq \emptyset$ for all i. However, some required edges must now be crossing. The number of crossing required edges leaving each S^i must be even or zero and the graph obtained by shrinking each S^i to a single vertex and deleting all required edges must be connected. Non-crossing edges have zero coefficient in the honeycomb inequality; the formula for computing the coefficients of the crossing edges, along with the RHS, is rather complex and is not discussed here.

Figs. 9–11 show some honeycomb configurations. The circles represent the S^i; a bold line represents a crossing required edge and a plain line represents one or more crossing non-required edges. All edges shown have coefficient 1 unless

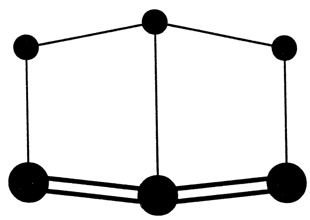

Fig. 9. First honeycomb, RHS = 6.

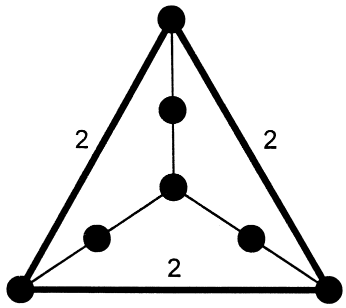

Fig. 10. Second honeycomb, RHS = 8.

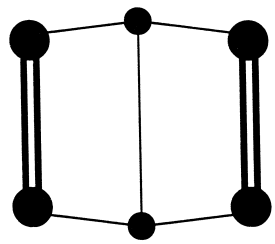

Fig. 11. Third honeycomb, RHS = 6.

otherwise indicated. For these examples, the coefficient of any other crossing non-required edge can be found by tracing a shortest path in the figure, from one end-vertex to the other, adding up coefficients along the way.

It turns out that these three honeycomb inequalities are projected HTC inequalities.

Fig. 12 shows the (multi)graph G' obtained by transforming the (multi)graph shown in Fig. 9, with the handles and teeth of a clique-tree inequality (Grötschel and Pulleyblank, 1986) superimposed (duplicate edges have been omitted for clarity). The clique-tree inequality has a RHS of 18 and is valid for SGTSP(G', $V_R \cup V^f$). The *projected clique-tree inequality* obtained by applying Corollary 5 is the first honeycomb inequality.

Fig. 13 shows the graph G' obtained by transforming the graph in Fig. 10, with the handles and teeth of an integral bipartition inequality (Boyd and Cunningham, 1991; Naddef, 1990) superimposed. The outer teeth have a coefficient of 2 and the inequality has a RHS of 26 and is valid for SGTSP(G', $V_R \cup V^f$). The resulting *projected integral bipartition inequality* is the second honeycomb inequality.

Fig. 14 shows the (multi)graph G' obtained from the (multi)graph in Fig. 11, with the handle and teeth of a binested inequality (Naddef, 1992) superimposed. The binested inequality has a RHS of 18 and is valid for SGTSP(G', $V_R \cup V^f$). The *projected binested inequality* is the third honeycomb inequality.

Thus, the three honeycomb inequalities are projected HTC inequalities as claimed. In fact, considering the relationships between inequalities displayed in Fig. 1, it is apparent that all three inequalities are *projected binested inequalities*. Many other facet-inducing honeycomb inequalities can also be shown to be projected binested inequalities.

The author conjectures, however, that there are honeycomb inequalities which are *not* projected binested inequalities. The reason for this belief is that Corberán and Sanchis (1996) showed that many honeycomb inequalities can be derived from two or more K–C inequalities using a so-called 2-*sum operation*. An analogous 2-sum operation was defined for GTSP inequalities in Naddef and Rinaldi (1991), where it was noted that many of the resulting inequalities cannot be expressed in HTC form.

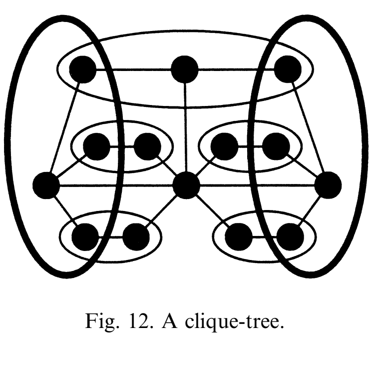

Fig. 12. A clique-tree.

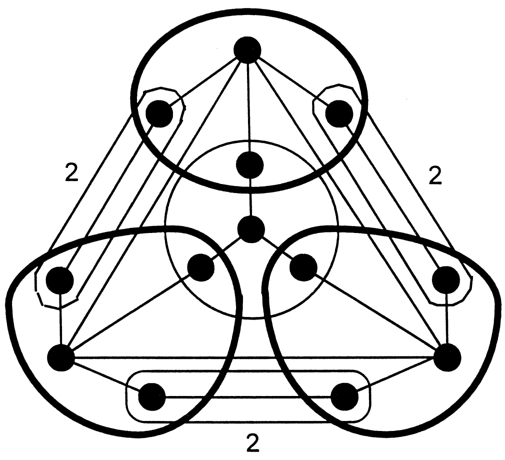

Fig. 13. A bipartition.

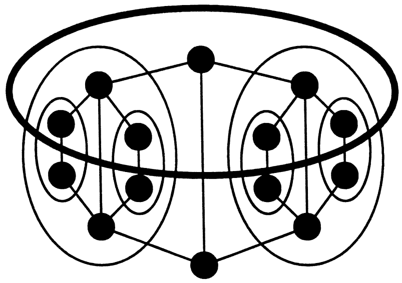

Fig. 14. A binested set.

5. Obtaining new inequalities

A wide variety of new inequalities can be found for the GRP using the results of Sections 2 and 3. Since space is limited, however, only a few examples will be given.

Suppose that a GRP instance is defined on the graph G shown in Fig. 6. The x vector with $x_e = 1$ if $e \notin E_R$, 0 otherwise, satisfies all of the inequalities mentioned in Section 4, yet violates the degree conditions (2). To find an inequality which cuts off this invalid vector, note that the corresponding transformed graph G' (Fig. 7) is identical to the one shown in Fig. 12. This means that the corresponding clique-tree inequality is valid for SGTSP(G', $V_R \cup V^f$). The resulting projected clique-tree inequality, valid for GRP(G, V_R, E_R), is

$$\sum_{e \in E} x_e \geqslant 12.$$

This hitherto unknown inequality cuts off the invalid x vector. It can also be shown to induce a facet of GRP(G, V_R, E_R).

Now suppose that a GRP instance is defined on the graph G shown in Fig. 15 (ignore the stars for the moment). The x vector with $x_e = 1$ if $e \notin E_R$, 0 otherwise, satisfies all of the inequalities mentioned in Section 4, yet violates the degree conditions. To find an inequality which cuts off this invalid vector, note that the corresponding transformed graph G' is similar to the one shown in Fig. 13. This means that the corresponding bipartition inequality is valid for SGTSP(G', $V_R \cup V^f$). The resulting projected integral bipartition inequality, valid for GRP(G, V_R, E_R), is

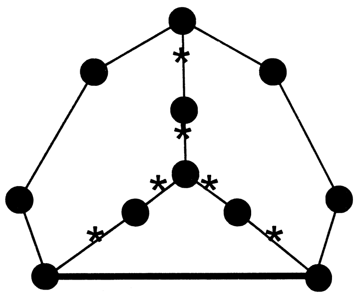

Fig. 15. GRP instance.

$$\sum_{e \in E} \beta_e x_e \geqslant 20,$$

where β_e equals 1 for the 'starred' edges, but 2 for the non-starred edges. Again, this hitherto unknown inequality cuts off the invalid x vector and induces a facet of GRP(G, V_R, E_R).

In a similar way, a (facet-inducing) projected binested inequality cuts off an invalid integral x vector for the GRP instance shown in Fig. 16. It has RHS 15 and is obtained from a binested inequality of the form shown in Fig. 14. It too is facet-inducing.

Finally, we introduce a new inequality which is not of HTC type. Suppose that a GRP instance is defined on the graph G shown in Fig. 17. The corresponding graph G' is as shown in Fig. 18. For the corresponding (S)GTSP instance, the *chain inequality*

$$\sum_{e \in E \cup E^f} x_e \geqslant 12$$

is valid and facet-inducing (see Padberg and Hong, 1980; Jünger et al., 1995, Jünger et al., 1997).

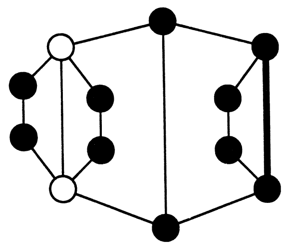

Fig. 16. GRP instance.

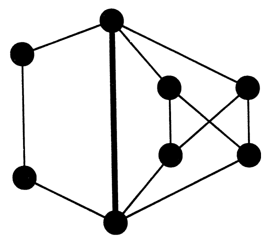

Fig. 17. GRP instance.

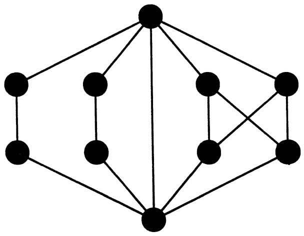

Fig. 18. Chain configuration.

The resulting *projected chain inequality* takes the form

$$\sum_{e \in E} x_e \geqslant 9$$

and can be shown to induce a facet of GRP(G, V_R, E_R).

6. Concluding comments

A strong polyhedral relationship between the GTSP, SGTSP and GRP has been demonstrated. Large new classes of valid inequalities, many of which induce facets, have been given for the SGTSP and GRP. In the case of the GRP, these have been shown to subsume the GTSP-type, R-odd cut and path-bridge inequalities and also to substantially overlap with the class of honeycomb inequalities.

It might be said that the nature of GRP polyhedra is now understood, in the sense that the only area remaining open for further research is to specify conditions under which facets of GTSP polyhedra yield facets of GRP polyhedra. Because of this, it is the author's belief that future research should be directed towards exploiting the known polyhedral results to produce efficient optimisation algorithms for the GRP and its special cases.

In fact, the author has already done something in this direction. In the context of the TSP, Applegate et al. (1995), Fleischer and Tardos (1996) and Caprara et al. (1997) have described separation algorithms for comb inequalities. The author has observed that these results are easily generalised to yield efficient separation algorithms for path-bridge inequalities. Using this result, the author was able to solve all but one of the 26 RPP instances described in Corberán and Sanchis (1994) using only polynomial separation routines, without branching.

Corberán and Sanchis have, independently, produced some efficient heuristic separation routines for honeycomb (and therefore K–C) inequalities (Corberán, 1997) and have solved all 26 RPP instances without branching. The challenge now is to produce an efficient branch-and-cut algorithm for the GRP and its special cases.

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