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On Disjunctive Cuts for Combinatorial Optimization

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Abstract. In the successful *branch-and-cut* approach to combinatorial optimization, linear inequalities are used as cutting planes within a branch-and-bound framework. Although researchers often prefer to use *facet-inducing* inequalities as cutting planes, good computational results have recently been obtained using *disjunctive cuts*, which are not guaranteed to be facet-inducing in general.

A partial explanation for the success of the disjunctive cuts is given in this paper. It is shown that, for six important combinatorial optimization problems (the *clique partitioning*, *max-cut*, *acyclic subdigraph*, *linear ordering*, *asymmetric travelling salesman* and *set covering* problems), certain facet-inducing inequalities can be obtained by simple disjunctive techniques. New polynomial-time *separation algorithms* are obtained for these inequalities as a by-product.

The disjunctive approach is then compared and contrasted with some other 'general-purpose' frameworks for generating cutting planes and some conclusions are made with respect to the potential and limitations of the disjunctive approach.

Keywords: integer programming, cutting planes, max-cut problem, asymmetric travelling salesman problem, set covering problem

1. Introduction

At present, one of the most successful approaches to solving large-scale $\mathcal{NP}$-hard combinatorial optimization problems to optimality is the so-called *branch-and-cut* method (see Padberg and Rinaldi, 1991; Caprara and Fischetti, 1997). In this method, the problem is formulated as an Integer Linear Program (ILP), which is then solved by a combination of cutting planes (valid linear inequalities) and branch-and-bound.

The success of a branch-and-cut algorithm largely depends upon the quality of the cutting planes used. For this and other reasons, a considerable amount of research has been conducted into cutting plane theory. This research can roughly be divided into two strands. First, there is research into 'general-purpose' techniques for showing that inequalities are valid. This has yielded, for example, the *integer rounding* technique (Gomory, 1963; Chvátal, 1973), the *disjunctive* technique (Balas, 1979) and the *lift-and-project* technique (Sherali and Adams, 1990; Lovász and Schrijver, 1991; Balas et al., 1993). Second, there has been the study of ILP formulations of *specific* combinatorial optimization problems, with the goal of finding large classes of inequalities which induce *facets* of the associated integer polyhedra (see, e.g., Nemhauser and Wolsey, 1988 and Sections 2 and 3).

In general, facet-inducing inequalities are more effective as cutting planes than 'general-purpose' inequalities (Padberg and Rinaldi, 1991). However, finding facet-inducing inequalities for a particular problem can be very time-consuming and, even after some have been discovered, one then faces the problem of devising a *separation algorithm* for them, i.e., a procedure for detecting them when they are violated (see Grötschel et al., 1988 and Section 2). Hence, researchers have continued to incorporate 'general-purpose' cuts in branch-and-cut algorithms, sometimes with good results (e.g., Balas et al., 1993, 1996a, 1996b).

Particularly good results have been obtained with the so-called *disjunctive cuts* (see Balas et al., 1993, 1996; Nemhauser and Wolsey, 1988; Balas, 1979). The goal of the present paper is to provide a (partial) explanation for the success of these cutting planes, to explore their potential and also some of their limitations. The main result is that, for many important problems, there are large classes of facet-inducing disjunctive cuts.

The outline of the paper is as follows. In Section 2, we give the necessary notation and definitions. In Section 3, we analyse disjunctive cuts for six important combinatorial optimization problems (the *clique partitioning*, *max-cut*, *acyclic subdigraph*, *linear ordering*, *asymmetric travelling salesman* and *set covering* problems), and show that certain facet-inducing inequalities are in fact rather simple disjunctive cuts. As a by-product we obtain new *polynomial-time* separation algorithms for these inequalities. In Section 4, we compare these disjunctive separation algorithms with some other general techniques for separation proposed recently by Caprara and Fischetti (1996), Müller and Schulz (1996) and Borndörfer and Weismantel (2000). Finally, in Section 5 we conclude with a discussion of the potential and limitations of the disjunctive approach and some directions for further research.

2. Notation and definitions

The majority of combinatorial optimization problems, even some non-linear ones, can be formulated as *Integer Linear Programs* (ILPs)—see Nemhauser and Wolsey (1988). For the purposes of this paper, an ILP is a problem of the form $\min\{c^T x : Ax \leq b, x \in Z^n\}$, where $x \in Z^n$ is a vector of integer decision variables, $c \in \mathbb{R}^n$ is a vector of objective function coefficients, A is an $m \times n$ integer matrix and b is a right hand side vector. We assume that, if a variable x_i must be non-negative, then the inequality $-x_i \leq 0$ is either included explicitly in the system $Ax \leq b$ or implied by it. We also assume that n is bounded by a polynomial in the size of the problem data, although we allow m to be exponentially large (see Subsections 3.3 and 3.5).

Associated with a given ILP are the two polyhedra:

- $P := \{x \in \mathbb{R}^n : Ax \leq b\}$ (the feasible region of the linear programming relaxation), and
- $P_I := \text{conv}\{x \in Z^n : Ax \leq b\} = \text{conv}\{x \in P \cap Z^n\}$ (the convex hull of feasible integer solutions).

We have $P_I \subseteq P$ and we assume in this paper that containment is strict.

A *cutting plane* is a linear inequality which is *valid* for P_I (i.e., satisfied by all $x \in P_I$), but violated by some $x \in P \setminus P_I$. A linear inequality which induces a face of P_I of maximal dimension is said to be *facet-inducing*. In general, it is $\mathcal{NP}$-hard to determine whether an inequality is facet-inducing, although, for some problems, it may be possible to characterize some facets.

Disjunctive cutting planes were introduced by Balas (1979). Suppose we know that any feasible integer solution x, in addition to belonging to P, must also satisfy *at least one* of q systems of inequalities $C^1x \leq d^1$, $C^2x \leq d^2, \ldots, C^qx \leq d^q$. That is, each integer solution must satisfy the *disjunction*

$$\bigvee_{j=1}^{q}(Ax \leq b, C^jx \leq d^j), \tag{1}$$

where '$\vee$' denotes logical 'or'. Then, if we define for $j = 1, \ldots, q$ the polyhedron $P^j := \{x \in I\!R^n : Ax \leq b, C^jx \leq d^j\}$, we have that any inequality which is valid for $P^1, \ldots, P^q$ is also valid for P_I. Such an inequality is called a *disjunctive cut*.

In theory, *any* valid inequality for an ILP is a disjunctive cut, provided that we allow q to be arbitrarily large (Balas, 1979). However, we will only label an inequality disjunctive if it can be derived from a disjunction in which q is 'reasonably small', i.e., polynomial in n. When this condition is satisfied, there is a short proof that the inequality is valid: use linear programming duality to show that the inequality is valid for each of the P^j.

Balas (1979) considered three main kinds of disjunction:

- Disjunctions of the form $(x_i \leq 0) \vee (-x_i \leq -1)$, where x_i is a 0-1 variable,
- Disjunctions of the form $\bigvee_{i \in S}(x_i = 1)$, valid whenever the ILP contains a 'multiple-choice' constraint of the form $\sum_{i \in S} x_i = 1$,
- Disjunctions of the form $(\alpha x \leq \beta - 1) \vee (\alpha x \geq \beta)$, where $\alpha \in Z^n$ and $\beta \in Z$, valid for any ILP.

In this paper we will call cutting planes derived from these three kinds of disjunction *simple* cuts, *multiple choice* cuts and *split* cuts, respectively. Split cuts (named by Cook et al., 1990) are of particular interest as they generalize not only the simple cuts, but also many other well-known cutting planes—see Nemhauser and Wolsey (1988, 1990) and Section 4.

Next we consider the question of *separation* (Grötschel et al., 1988). For a given class of linear inequalities $\mathcal{F}$ and a given point $x^* \in I\!R^n$, the *separation problem* is to find an inequality in $\mathcal{F}$ which is violated by x^*, or to prove that none exist.

In the case of 0-1 ILPs in which m is polynomial in the problem size, simple cuts can be separated in polynomial time by the so-called *lift-and-project* technique, in which the original linear program is lifted into a higher-dimensional space using auxiliary variables. See, for example, Balas (1998), Balas et al. (1993), Lovász and Schrijver (1991) or Sherali and Lee (1990).

In fact, it was pointed out to the author by Alberto Caprara that the following more general proposition is a consequence of results in Balas (1979, 1998):

Proposition 1. *Let a polyhedron $P := \{x \in I\!R^n : Ax \leq b\}$ and a fixed disjunction of the form* (1) *be given. If the number of rows in the system $Ax \leq b$ or any of the systems $C^jx \leq d^j$ is exponential in the number of variables n, then assume that the systems are represented* *implicitly via separation oracles. Then provided that P is bounded (i.e., a polytope), the separation of disjunctive cuts can be carried out in polynomial time.*

Proof: Separation of disjunctive cuts means testing, for a given $x^* \in P$, if x^* lies in the convex hull of $P^1 \cup \cdots \cup P^q$. This is so only if x^* can be expressed as a convex combination of vectors lying within the P^j. Following Balas (1998), this is equivalent to finding a solution to the linear system

$$\begin{aligned}
&y^1 + \cdots + y^q = x^*, \\
&z^1 + \cdots + z^q = 1, \\
&Ay^j \le bz^j, \quad (j = 1, \ldots, q), \\
&C^j y^j \le d^j z^j, \quad (j = 1, \ldots, q), \\
&y^j \in \mathbb{R}^n \quad (j = 1, \ldots, q), \\
&z^j \in \mathbb{R}_+ \quad (j = 1, \ldots, q).
\end{aligned}$$

By the polynomial equivalence of separation and optimization (Grötschel et al., 1988), this system can be solved efficiently if one can test efficiently, for a given vector $(y^j)^*$ and scalar $(z^j)^*$, if $A(y^j)^* > b(z^j)^*$ or $C^j(y^j)^* > b^j(z^j)^*$. If $(z^j)^* = 0$, then because P is bounded we have $(y^j)^* = 0$ and we are done. Otherwise, if $(z^j)^* > 0$, we can test whether $A\frac{(y^j)^*}{(z^j)^*} > b$ or $C^j\frac{(y^j)^*}{(z^j)^*} > d^j$ using the separation oracles. If a solution to the system exists, no disjunctive cut is violated; otherwise a violated cut can be constructed via the well-known Farkas' Lemma (see, e.g., Nemhauser and Wolsey, 1988). □

This immediately implies that *multiple-choice cuts* can be separated efficiently, provided that P is bounded and that there are only a polynomial number of multiple-choice constraints (which is true for any ILP of interest). It does not however imply that *split cut* separation is polynomial. Indeed, the complexity of split cut separation is open and it may well be $\mathcal{NP}$-hard.

However, there is a fast and effective procedure which, given a simple cut, produces a (not necessarily simple) split cut which is at least as strong and may be considerably stronger (see Balas et al., 1993, 1996; Balas and Jeroslow, 1980). In this way one can obtain a reasonable *heuristic* for split cut separation.

Surprisingly, there are very few papers which analyse disjunctive cuts for ILPs associated with particular combinatorial optimization problems. Those which do exist are mainly concerned with set packing and set partitioning or related problems (e.g., Lovász and Schrijver, 1991; Balas et al., Pataki, 1994; Sherali and Lee, 1996).

In the next section we perform such an analysis for six other important problems.

3. Analysis of particular problems

3.1. The clique partitioning problem

Given a complete undirected graph $G = (V, E)$, define for any $S \subseteq V$ the edge set $E(S) := \{\{i, j\} \in E : i, j \in S\}$. Note that S together with $E(S)$ forms a clique in G. Now suppose that weights w_{ij} are given for each $\{i, j\} \in E$. The *Clique Partitioning Problem* (CPP) is that of partitioning V into sets $S_1, \ldots, S_q$ such that the sum of the weights of the edges in $E(S_1) \cup \cdots \cup E(S_q)$ is maximized.

The CPP can be formulated as an ILP in the following way. Define a binary variable x_{ij} for each edge $\{i, j\} \in E$, taking the value 1 if $\{i, j\} \in E(S_k)$ for some k, 0 otherwise. (In this subsection and the next, we assume that x_{ij} and x_{ji} refer to the same variable.) Then maximize $\sum_{\{i,j\} \in E} w_{ij} x_{ij}$ subject to:

$$x_{ij} + x_{ik} - x_{jk} \leq 1 \quad (\forall i, j, k \in V : |\{i, j, k\}| = 3) \tag{2}$$

$$x_{ij} \in \{0, 1\} \quad (\forall \{i, j\} \in E) \tag{3}$$

We define $P := \{x \in \mathbb{R}_+^{|E|} : \text{(2) holds}\}$. There is no need to include explicit upper bounds $x_{ij} \leq 1$ in the definition of P, since these are easily shown to be implied by the inequalities (2).

The associated integer polytope, $P_I := conv\{x \in P \cap Z_+^{|E|}\}$, is called the *clique partitioning polytope*. Among the inequalities known to induce facets of P_I, apart from the inequalities defining P themselves, are the *2-chorded odd cycle* inequalities (Grötschel and Wakabayashi, 1990) and the *odd wheel* inequalities (Chopra and Rao, 1993).

Let $p \geq 2$ be an integer and let $v_1, \ldots, v_{2p+1}$ be distinct vertices in G. For ease of notation, we also identify $v_{2p+2} := v_1$ and $v_{2p+3} := v_2$. The *2-chorded odd cycle* inequality associated with $v_1, \ldots, v_{2p+1}$ takes the form

$$\sum_{i=1}^{2p+1} x_{v_i, v_{i+1}} - \sum_{i=1}^{2p+1} x_{v_i, v_{i+2}} \leq p. \tag{4}$$

Now let $p \geq 1$ be an integer and let $v_1, \ldots, v_{2p+1}$ and u be distinct vertices in G. Again, for ease of notation, we identify $v_{2p+2} := v_1$. The *odd wheel* inequality associated with $v_1, \ldots, v_{2p+1}$ and u takes the form

$$\sum_{i=1}^{2p+1} x_{u, v_i} - \sum_{i=1}^{2p+1} x_{v_i, v_{i+1}} \leq p. \tag{5}$$

Theorem 1. *2-chorded odd cycle inequalities are simple cuts.*

Proof: We use the disjunction $(x_{v_1, v_2} \leq 0) \vee (x_{v_1, v_2} \geq 1)$.
First term of disjunction: we obtain (4) by summing together the inequality $x_{v_1, v_2} \leq 0$, the inequalities $x_{v_i, v_{i+1}} + x_{v_{i+1}, v_{i+2}} - x_{v_i, v_{i+2}} \leq 1$ for $i = 2, 4, \ldots, 2p$, and the non-negativity inequalities $-x_{v_i, v_{i+2}} \leq 0$ for $i = 1, 3, \ldots, 2p+1$.
Second term of disjunction: we obtain (4) by summing together the inequality $-x_{v_1, v_2} \leq -1$, the inequalities $x_{v_i, v_{i+1}} + x_{v_{i+1}, v_{i+2}} - x_{v_i, v_{i+2}} \leq 1$ for $i = 1, 3, \ldots, 2p+1$, and the non-negativity inequalities $-x_{v_i, v_{i+2}} \leq 0$ for $i = 2, 4, \ldots 2p$. □

Theorem 2. *Odd wheel inequalities are simple cuts.*

Proof: We use the disjunction $(x_{v_1,u} \leq 0) \vee (x_{v_1,u} \geq 1)$.
First term of disjunction: we obtain (5) by summing together the inequality $x_{v_1,u} \leq 0$, the inequalities $x_{v_i,u} + x_{v_{i+1},u} - x_{v_i,v_{i+1}} \leq 1$ for $i = 2, 4, \dots, 2p$, and the non-negativity inequalities $-x_{v_i,v_{i+1}} \leq 0$ for $i = 1, 3, \dots, 2p+1$.
Second term of disjunction: we obtain (5) by summing together the inequality $-x_{v_1,u} \leq -1$, the inequalities $x_{v_i,u} + x_{v_{i+1},u} - x_{v_i,v_{i+1}} \leq 1$ for $i = 1, 3, \dots, 2p+1$, and the non-negativity inequalities $-x_{v_i,v_{i+1}} \leq 0$ for $i = 2, 4, \dots, 2p$. □

Thus, we obtain the following corollary:

Corollary 1. *There is a polynomial-time separation algorithm for a class of inequalities which includes all* 2*-chorded odd cycle and all odd wheel inequalities.*

The same result has been obtained by other authors using completely different techniques, see Caprara and Fischetti (1996), Borndörfer and Weismantel (2000) and Müller and Schulz (1996). Polynomial separation of the *exact* class of odd wheel inequalities was obtained by Deza et al. (1992).

3.2. The max-cut problem

Given a complete undirected graph $G = (V, E)$, define for any $S \subseteq V$ the edge set $\delta(S) := \{\{i, j\} \in E : |\{i, j\} \cap S| = 1\}$. Note that the edges in $\delta(S)$ form a cut in G. Now suppose that (not necessarily non-negative) weights w_{ij} are given for each $\{i, j\} \in E$. The *Max-Cut Problem* (MCP) is that of finding a vertex set $S \subseteq V$ such that the sum of the weights of the edges in $\delta(S)$ is maximized.

The MCP can be formulated as an ILP in the following way. Define a binary variable x_{ij} for each edge $\{i, j\} \in E$, taking the value 1 if $\{i, j\} \in \delta(S)$ 0 otherwise. Then maximize $\sum_{\{i,j\} \in E} w_{ij} x_{ij}$ subject to:

$$x_{ij} + x_{ik} + x_{jk} \leq 2 \quad (\forall \{i, j\} \in E, k \in V \setminus \{i, j\}) \tag{6}$$

$$x_{ij} - x_{ik} - x_{jk} \leq 0 \quad (\forall \{i, j\} \in E, k \in V \setminus \{i, j\}) \tag{7}$$

$$x_{ij} \in \{0, 1\} \quad (\forall \{i, j\} \in E) \tag{8}$$

We define $P := \{x \in \mathbb{R}^{|E|} : (6), (7) \text{ hold}\}$. Again, there is no need to include explicit bounds $0 \leq x_{ij} \leq 1$ in the definition of P, since these are easily shown to be implied by the inequalities (6), (7).

The associated integer polytope, $P_I := conv\{x \in P \cap Z^{|E|}\}$, is called the *cut polytope*. Among the inequalities known to induce facets of P_I, apart from the inequalities defining P themselves, are the *odd bicycle wheel* inequalities (Barahona and Mahjoub, 1986) and the *circulant* inequalities (Poljak and Turzik, 1992).

Let $p \geq 1$ be an integer and let $v_1, \dots, v_{2p+1}$, s and t be distinct vertices in G. As in the previous subsection, we also identify $v_{2p+2} := v_1$. The *odd bicycle wheel* inequality associated with $v_1, \ldots, v_{2p+1}$, s and t takes the form

$$\sum_{i=1}^{2p+1} x_{v_i,v_{i+1}} + \sum_{i=1}^{2p+1} x_{s,v_i} + \sum_{i=1}^{2p+1} x_{t,v_i} + x_{st} \leq 4p + 2. \tag{9}$$

Now let $p \geq 2$ be an even integer and let $v_1, \ldots, v_{2p+1}$ be distinct vertices in G. Again, we identify $v_{2p+2} := v_1$ and $v_{2p+3} := v_2$. The $(2p+1, 2)-$*circulant* inequality associated with $v_1, \ldots, v_{2p+1}$ takes the form

$$\sum_{i=1}^{2p+1} x_{v_i,v_{i+1}} + \sum_{i=1}^{2p+1} x_{v_i,v_{i+2}} \leq 3p. \tag{10}$$

Theorem 3. *Odd bicycle wheel inequalities are simple cuts.*

Proof: We use the disjunction $(x_{v_1,s} \leq 0) \vee (x_{v_1,s} \geq 1)$.
First term of disjunction: we obtain (9) by summing together twice the inequality $x_{v_1,s} \leq 0$, the inequalities $x_{s,v_i} + x_{s,v_{i+1}} + x_{v_i,v_{i+1}} \leq 2$ for $i = 2, 4, \ldots, 2p$, the inequalities $x_{t,v_i} + x_{t,v_{i+1}} + x_{v_i,v_{i+1}} \leq 2$ for $i = 1, 3, \ldots, 2p+1$, and the inequality $x_{st} - x_{v_1,s} - x_{v_1,t} \leq 0$.
Second term of disjunction: we obtain (9) by summing together twice the inequality $-x_{v_1,s} \leq -1$, the inequalities $x_{s,v_i} + x_{s,v_{i+1}} + x_{v_i,v_{i+1}} \leq 2$ for $i = 1, 3, \ldots, 2p+1$, the inequalities $x_{t,v_i} + x_{t,v_{i+1}} + x_{v_i,v_{i+1}} \leq 2$ for $i = 2, 4, \ldots, 2p$ and the inequality $x_{st} + x_{v_1,s} + x_{v_1,t} \leq 2$. □

Theorem 4. $(2p+1, 2)$-*circulant inequalities are simple cuts.*

Proof: We use the disjunction $(x_{v_1,v_2} \leq 0) \vee (x_{v_1,v_2} \geq 1)$. The rest of the proof is similar to the previous proofs. □

Thus, we obtain the following corollary:

Corollary 2. *There is a polynomial-time separation algorithm for a class of inequalities which includes all odd bicycle wheel inequalities and all* $(2p+1, 2)$-*circulant inequalities.*

Polynomial separation of odd bicycle wheel inequalities was shown by Gerards (1985). However, before the writing of the present paper, no way of separating $(2p+1, 2)$-circulant inequalities was known. Indeed, the separation problem for the *exact* class of $(2p+1,$ $2)$-circulant inequalities is known to be $\mathcal{NP}$-hard (Poljak and Turzik, 1992).

We would also like to mention that much larger classes of facet-inducing inequalities can be derived from the odd bicycle wheel and $(2p+1, 2)$-circulant inequalities by the so-called *switching* operation, which involves changing the sign of the coefficients for the variables in any cut and changing the right hand side accordingly—see for example Barahona and Mahjoub (1986). It is not hard to prove that the resulting inequalities are still simple cuts and therefore the separation result holds for them also.

3.3. The acyclic subdigraph problem

When dealing with digraphs (directed graphs), we let (i, j) denote the arc from vertex i to vertex j.

A (simple) *dipath* in a digraph is a set of arcs which can be represented as $\{(v_1, v_2), (v_2, v_3) \dots (v_{q-1}, v_q)\}$, where $v_1, \dots, v_q$ are distinct vertices. A (simple) *dicycle* in a digraph is a set of arcs which can be represented as $\{(v_1, v_2), (v_2, v_3), \dots, (v_{q-1}, v_q), (v_q, v_1)\}$, where $v_1, \dots, v_q$ are distinct vertices. A digraph containing no dicycles is said to be *acyclic*. Given a *complete* digraph $G = (V, A)$, along with (not necessarily non-negative) weights w_{ij} for each $(i, j) \in A$, the *Acyclic Subdigraph Problem* (ASP) is that of finding an acyclic subdigraph of maximum weight.

The most natural ILP formulation of the ASP is the following (Grötschel et al., 1985a). Define a binary variable x_{ij} for each arc $(i, j) \in A$, taking the value 1 if (i, j) is in the subdigraph, 0 otherwise. Then maximize $\sum_{(i,j)\in A} w_{ij}x_{ij}$ subject to:

$$x(C) \leq |C| - 1 \quad (\forall \textit{dicycles } C) \tag{11}$$

$$x_{ij} \in \{0, 1\} \quad (\forall (i, j) \in A). \tag{12}$$

(We are using the standard shorthand in constraints (11), in that $x(C)$ denotes $\sum_{(i,j)\in C} x_{ij}$.)

We can now define $P := \{x \in \mathbb{R}_+^{|A|} : \text{(11) holds}\}$. Again, there is no need to include explicit bounds $x_{ij} \leq 1$ in the definition of P, since these are implied by the dicycle inequalities (11) when $|C| = 2$. Moreover, note that, although there are an exponential number of inequalities (11), the associated separation problem can be solved in polynomial time by shortest path techniques (Grötschel et al., 1985a).

The associated integer polytope, $P_I := conv\{x \in P \cap Z^{|A|}\}$, is called the *acyclic subdigraph polytope*. Among the inequalities known to induce facets of P_I, apart from those inducing facets of P, are the following *Möbius ladder* inequalities (Grötschel et al., 1985a).

Let $k \geq 1$ be an integer and let $C_0, \dots, C_{2k}$ each be dicycles such that, for $j = 0, \dots, 2k$, C_j and C_{j+1} have a dipath D_j in common (indices are interpreted modulo k), but that there are no other arcs in common between the C_j. Then the associated Möbius ladder inequality takes the form:

$$x(C_0 \cup \dots \cup C_{2k}) \leq \sum_{j=0}^{2k} |C_j \backslash D_j| - k - 1. \tag{13}$$

Theorem 5. *Any Möbius ladder inequality of the form* (13) *can be derived as a split cut from the disjunction* $(x(D_0) \leq |D_0| - 1) \vee (x(D_0) \geq |D_0|)$.

Proof: *First term of disjunction*: we obtain (13) by summing $x(D_0) \leq |D_0| - 1$ together with the inequalities $x(C_k) \leq |C_k| - 1$ for $1, 3, \dots, 2k - 1$ and the upper bounds $x_{ij} \leq 1$ for $(i, j) \in (C_0 \cup C_2 \cup \dots \cup C_{2k}) \backslash (D_0 \cup D_1 \cup \dots \cup D_{2k})$.
Second term of disjunction: we obtain (13) by summing $-x(D_0) \leq -|D_0|$ together with the inequalities $x(C_k) \leq |C_k| - 1$ for $0, 2, \dots, 2k$ and the upper bounds $x_{ij} \leq 1$ for $(i, j) \in (C_1 \cup C_3 \cup \dots \cup C_{2k-1}) \backslash (D_0 \cup D_1 \cup \dots \cup D_{2k})$. □

This does not immediately lead to a polynomial-time separation algorithm, because there are an exponential number of dipaths on which one might perform a disjunction. However, such an algorithm does indeed exist, as expressed in the following theorem.

Theorem 6. *There is a polynomial-time separation algorithm for a class of inequalities which includes all Möbius ladder inequalities.*

Sketch of Proof: The proof of Theorem 5 can be used to show that the Möbius ladder inequality (13) is valid even when dicycles have arcs in common other than the D_j. So it suffices to separate these more general inequalities. Let x^* be the point to be separated. Using a shortest path algorithm find, for every pair of vertices $(i, j) \in A$, a dipath D^*_{ij} going from i to j such that $|D^*_{ij}| - x^*(D^*_{ij})$ is as small as possible. Now suppose that a Möbius ladder inequality of the form (13) is violated by x^* and that the dipath D_0 goes from vertex i to vertex j. Then we can obtain a (generalized) Möbius ladder inequality which is violated by at least as much if we replace D_0 by D^*_{ij} and adjust C_0 and C_{2k} accordingly. Therefore, it suffices to consider only $|A|$ disjunctions, each of the form $(x(D^*_{ij}) \leq |D^*_{ij}| - 1) \vee (x(D^*_{ij}) \geq |D^*_{ij}|)$ for some $(i, j) \in A$. The $|A|$ associated separation problems can be solved in polynomial time by Proposition 1. □

The same result has been obtained by Borndörfer and Weismantel (1997a) and Müller and Schulz (1996) using different techniques. Caprara and Fischetti (1996) give a different separation algorithm for the special case where the $|C_j|$ are bounded by a constant.

3.4. *The linear ordering problem*

Closely related to the ASP is the so-called *Linear Ordering Problem* (LOP). The LOP is identical to the ASP, but with the added restriction that the subdigraph must be a *tournament*, i.e., it must include, for all vertex pairs $\{i, j\} \subseteq V$, exactly one of the arcs $(i, j), (j, i)$.

The obvious way of modelling the LOP is to add the following constraints to (11)–(12):

$$x_{ij} + x_{ji} = 1. \quad (\forall \{i, j\} \subset V) \tag{14}$$

We can then define $P := \{x \in \mathbb{R}^{|A|}_+ : (11), (14) \text{ hold}\}$. However, as shown by Grötschel et al. (1985b), most of the inequalities (11) are then redundant: they only induce facets of P when $|D| = 3$. Therefore, if we replace the constraint set (11) with

$$x_{ij} + x_{jk} + x_{ki} \leq 2 \quad (\forall (i, j) \in A, k \in V \setminus \{i, j\}), \tag{15}$$

we have that $P = \{x \in \mathbb{R}^{|A|}_+ : (14), (15) \text{ hold}\}$. Thus P has only a polynomial number of facets.

The associated integer polytope, $P_I := conv\{x \in P \cap Z^{|A|}\}$, is called the *linear ordering polytope*. By definition, all of the Möbius ladder inequalities presented in the previous subsection are valid for the linear ordering polytope. However, it is shown in Grötschel et al. (1985b) that most of these are redundant: the only ones which induce facets have $|C_j| \in \{3, 4\}$ and $|D_j| = 1$ for $j = 0, \ldots, 2k$. This immediately gives the following results:

Theorem 7. *Any non-redundant Möbius ladder inequality for the LOP is a simple cut.*

Proof: Immediate from Theorem 5 and the fact that, in the case of the LOP, $|D_0| = 1$ in non-redundant Möbius ladder inequalities. □

Corollary 3. *There is a polynomial-time separation algorithm for a class of inequalities which includes all Möbius ladder inequalities for the LOP.*

Again, the same result has been obtained by Müller and Schulz (1996), Caprara and Fischetti (1996) and Borndörfer and Weismantel (2000). using different techniques.

3.5. The asymmetric travelling salesman problem

Given a complete digraph $G = (V, A)$ with (not necessarily non-negative) weights w_{ij} for each $(i, j) \in A$, the well-known *Asymmetric Travelling Salesman Problem* (ATSP) is that of finding a minimum weight hamiltonian circuit (tour) in G.

The standard ILP formulation of this problem (see, e.g., Grötschel and Padberg, 1985) is to define a binary variable x_{ij} for each arc $(i, j) \in A$, taking the value 1 if (i, j) is in the tour, 0 otherwise. Then, define for any $S \subset V$ the arc set $A(S) := \{(i, j) \in A : i, j \in S\}$ and maximize $\sum_{(i,j) \in A} w_{ij} x_{ij}$ subject to:

$$\sum_{j \neq i} x_{ij} = 1 \qquad (\forall i \in V) \tag{16}$$

$$\sum_{j \neq i} x_{ji} = 1 \qquad (\forall i \in V) \tag{17}$$

$$x(A(S)) \leq |S| - 1 \qquad (\forall S \subset V : 2 \leq |S| \leq V - 2) \tag{18}$$

$$x_{ij} \in \{0, 1\} \qquad (\forall (i, j) \in A). \tag{19}$$

Constraints (16) and (17) are called *out-degree* and *in-degree* equations, repectively. Constraints (18) are the well-known *subtour elimination inequalities*. We define $P := \{x \in \mathbb{R}_+^{|A|} : (16)\text{–}(18) \text{ hold}\}$. Although there are an exponential number of subtour elimination inequalities, the separation problem for P can still be solved in polynomial time by maximum flow techiques (see, e.g., Padberg and Grötschel, 1985).

The associated integer polytope, $P_I := conv\{x \in P \cap Z^{|A|}\}$, is called the *asymmetric travelling salesman polytope*. Among the inequalities known to induce facets of P_I, apart from those inducing facets of P, are the D_k^+ and D_k^- inequalities (see Grötschel and Padberg, 1985; Fischetti, 1991).

Let $k \geq 3$ be an integer and let $v_1, \ldots, v_k$ be distinct vertices in G. The D_k^+ inequality associated with $v_1, \ldots, v_k$ is:

$$\sum_{i=1}^{k-1} x_{v_i,v_{i+1}} + x_{v_k,v_1} + 2\sum_{i=3}^{k} x_{v_1,v_i} + \sum_{i=4}^{k}\sum_{j=3}^{i-1} x_{v_i,v_j} \leq k-1 \tag{20}$$

and the D_k^- inequality is:

$$\sum_{i=1}^{k-1} x_{v_i,v_{i+1}} + x_{v_k,v_1} + 2\sum_{i=2}^{k-1} x_{v_i,v_1} + \sum_{i=3}^{k-1}\sum_{j=2}^{i-1} x_{v_i,v_j} \leq k-1. \tag{21}$$

Theorem 8. *D_k^+ and D_k^- inequalities are multiple-choice cuts.*

Sketch of Proof: By symmetry, it suffices to show the result for the D_k^+ inequalities. We use the disjunction $\bigvee_{s \in V\setminus\{v_1\}}(x_{v_1,k} = 1)$.

Case One: $s \in V \setminus \{v_3, \ldots, v_k\}$. We obtain (20) by summing together the inequality $-x_{v_1,s} \leq -1$, the out-degree equation for vertex v_1 (in less-than-or-equal-to form), the subtour elimination inequality for $\{v_1, \ldots, v_k\}$ and an appropriate set of non-negativity inequalities.

Case Two: $s = v_t$ for some $t \in \{v_3, \ldots, v_k\}$. We obtain (20) by summing together the inequality $-x_{v_1,v_t} \leq -1$ (twice), the out-degree equation for vertex v_1 (twice, in less-than-or-equal-to form), the in-degree equation for vertices $v_3, \ldots, v_t$ (also in less-than-or-equal-to form), the subtour elimination inequality for $\{v_1, v_t, v_{t+1}, \ldots, v_k\}$ and an appropriate set of non-negativity inequalities. □

Corollary 4. *There is a polynomial-time separation algorithm for a class of inequalities which includes all D_k^+ and D_k^- inequalities.*

Proof: There are only $2|V|$ multiple-choice disjunctions to consider, one for each in- and out-degree equation. The result then follows from Proposition 1. □

This separation result is new. (A fast and effective algorithm for D_k^+ and D_k^- separation was given in Fischetti and Toth (1997), but it was not shown to run in polynomial time.)

Polynomial-time disjunctive separation algorithms can also be obtained for various more complex inequalities obtained from D_k^+ and D_k^- inequalities by the so-called *clique-lifting* operation (see Fischetti and Toth, 1997). Details are omitted for the sake of brevity.

3.6. The set covering problem

The last problem we consider is the well-known *Set Covering Problem* (SCP). (See Ceria et al., 1997 for a survey of the literature on this problem.) An SCP instance with m constraints and n variables takes the form

$$\min\{cx : Ax \geq \mathbf{1},\ x \in \{0,1\}^n\}, \tag{22}$$

where $A \in \{0,1\}^{m\times n}$ and $\mathbf{1}$ is a (column) m-vector with all components equal to 1.

We define $P := \{x \in [0,1]^n : Ax \geq \mathbf{1}\}$. The associated P_I is called the *set covering polytope*.

The next theorem shows the validity of a large class of disjunctive cuts for the SCP. To present it, we need some further notation. Let $N := \{1, \ldots, n\}$. For $i = 1, \ldots, m$, we define the set $S(i) := \{j \in N : A_{ij} = 1\}$, the so-called *support* of row i. Then, the ith constraint in the SCP instance is of the form $x(S(i)) \geq 1$.

Theorem 9. *Let p and q be integers with $p > q \geq 2$, such that p is not a multiple of q. Let $N_0, N_1, \ldots, N_{p-1}$ be non-empty subsets of N (not necessarily disjoint). Suppose that there are row indices r_k for $k = 0, \ldots, p-1$ such that $S(r_k) \subseteq N_k \cup N_{k+1} \cup \cdots \cup N_{k+q-1}$ (indices taken modulo p). Then the inequality*

$$\sum_{k=0}^{p-1} x(N_k) \geq \lceil p/q \rceil, \tag{23}$$

where $\lceil \cdot \rceil$ denotes integer rounding upward, is valid for P_I and implied by the disjunction $\bigvee_{j \in S(r_0)} (x_j \geq 1)$.

Proof: If $j \in S(r_0)$, then $j \in N_t$ for some $0 \leq t \leq q-1$. We obtain the inequality (23) by summing together the inequality $x_j \geq 1$, the inequalities $x(S(r_k)) \geq 1$ for $k = t+1, t+q+1, \ldots, t+(\lfloor p/q \rfloor - 1)q$, and an appropriate set of non-negativity inequalities. □

We will call inequalities of the form (23) *generalized cycle* inequalities, because they are analogous to the generalized cycle inequalities for the independence system polytope (see Euler et al., 1987; Müller and Schulz, 1996).

When the N_k are all pairwise disjoint and all of cardinality 1, generalized cycle inequalities reduce to *q-rose inequalities of order p*, as defined by Sassano (1989). When $q = 2$, they reduce to what Borndörfer and Weismantel (1997) call *aggregated odd cycle* inequalities. If both of these conditions hold, we obtain what Hoffman and Padberg (1993) call *odd hole* inequalities. Finally, generalized cycle inequalities with $q < p < 2q$ (i.e., with right hand side 2) are a special case of the inequalities studied in Balas and Ng (1989).

It is difficult to specify conditions under which a generalized cycle inequality induces a facet of P_I, as it depends upon the other rows present in A. However, it is easy to produce facet-inducing examples. Moreover, we have:

Corollary 5. *There is a polynomial-time separation algorithm for a class of inequalities which includes all generalized cycle inequalities.*

Proof: Only m disjunctions need to be used, one for each row of A. The result then follows from Proposition 1. □

This separation result is new, although a separation *heuristic* for the case $q = 2$ appears in Borndörfer and Weismantel (1997).

It is also possible to show that certain more complex inequalities, obtained from the generalized cycle inequalities by *lifting* (see Nemhauser and Wolsey, 1988; Sassano, 1989), are disjunctive cuts. Details are omitted for the sake of brevity.

4. Other separation techniques

Some of the classes of facet-inducing inequalities proven to be polynomial-time separable in the previous section were also proven to be polynomial-time separable in rather different ways by Caprara and Fischetti (1996), Müller and Schulz (1996) and Borndörfer and Weismantel (1997, 2000).

Caprara and Fischetti proposed a separation scheme based on *Chvátal-Gomory* (CG) cuts. CG cuts are (see Chvátal, 1973; Gomory, 1963; Nemhauser and Wolsey, 1988) valid inequalities of the form $(\lambda^T A)x \leq \lfloor \lambda^T b \rfloor$, where $\lambda \in \mathbb{R}^m_+$ is a vector such that $\lambda^T A$ is integral and $\lfloor \cdot \rfloor$ represents integer rounding downward. It can be shown that *all* of the cuts mentioned in Section 3 are CG cuts.

CG cuts are split cuts. To see this, use the disjunction $((\lambda^T A)x \leq \lfloor \lambda^T b \rfloor) \vee ((\lambda^T A)x \geq \lfloor \lambda^T b \rfloor + 1)$ and note that in this case P^2 is empty.

Caprara and Fischetti call a CG cut with $\lambda \in \{0, \frac{1}{2}\}^m$ a $\{0, \frac{1}{2}\}$-*cut*. They showed that, despite the fact that $\{0, \frac{1}{2}\}$-cuts are a very special kind of cutting plane, many facet-inducing inequalities for certain combinatorial optimization problems are in fact $\{0, \frac{1}{2}\}$-cuts. They also showed that the separation problem for $\{0, \frac{1}{2}\}$-cuts is $\mathcal{NP}$-hard in general, but polynomially solvable in certain special cases, e.g., when A has at most two odd coefficients per row or per column.

Caprara and Fischetti then suggest the following approach to separating $\{0, \frac{1}{2}\}$-cuts. First, *weaken* the system $Ax \leq b$ in such a way that there are at most two odd coefficients per row in the resulting system. Then, separate $\{0, \frac{1}{2}\}$-cuts for this weaker system. Finally, strengthen any violated cuts found by replacing the weakened constraints in the CG-derivation with the original, 'strong' counterparts.

It turns out that the 2-chorded odd cycle inequalities and odd wheel inequalities for the CPP, and the Möbius ladder inequalities for the ASP and LOP, are $\{0, \frac{1}{2}\}$-cuts. Using this fact, Caprara and Fischetti were able to derive polynomial time separation algorithms for these inequalities. Independently, and at about the same time, Müller and Schulz (1996) found polynomial time separation algorithms for essentially the same problems and the same inequalities.

As seen in this paper, we were able to duplicate these results using the disjunctive approach. However, it can be shown that none of the other inequalities mentioned in Section 3 are $\{0, \frac{1}{2}\}$-cuts (apart from the $(p, 2)$-cycle inequalities for the SCP and the D_3^+ and D_3^- inequalities for the ATSP.) Thus, the disjunctive approach appears to be superior in some cases.

On the other hand, there are simple ILP instances where the Caprara–Fischetti approach is superior to the disjunctive approach. Consider the polyhedron $P = \{x \in \mathbb{R}^6 : x_1 + x_2 + 2x_4 \leq 3, x_1 + x_3 + 2x_5 \leq 3, x_2 + x_3 + 2x_6 \leq 3\}$. Note that the inequalities defining P have two odd left hand side entries each. The $\{0, \frac{1}{2}\}$-cut $x_1 + \cdots + x_6 \leq 4$ is facet-inducing for the associated P_I and is easily separated by the Caprara–Fischetti approach. Yet, it is hard to separate by the disjunctive approach (it is not a simple or multiple-choice cut).

Thus neither of the two approaches dominates the other.

Next, we consider the separation results of Borndörfer and Weismantel (2000). Their approach is based on a so-called *conflict graph*.

Given an arbitrary ILP instance, suppose that for any $\alpha \in Z^n$ we can quickly compute a quantity $r(\alpha) \in Z$ such that $\max\{\alpha x : x \in P_I\} \leq r(\alpha)$. (This can be done, e.g., by using LP relaxation and rounding.) We then say that two integer vectors $\alpha^1, \alpha^2 \in Z^n$ are *in conflict* if $r(\alpha^1 + \alpha^2) \leq r(\alpha^1) + r(\alpha^2) - 1$.

Now suppose that we have several vectors $\alpha^1, \ldots, \alpha^q$. (The question of how to choose these is deferred until later.) The *conflict graph* $\bar{G} = (\bar{V}, \bar{E})$ associated with these vectors is the graph with vertex set $\bar{V} = \{1, \ldots, q\}$ and an edge between two vertices if and only if the corresponding vectors are in conflict. That is, the edge $\{i, j\}$ represents the fact that the inequality $(\alpha^i + \alpha^j)x \leq r(\alpha^i) + r(\alpha^j) - 1$ is valid for P_I.

Proposition (Borndörfer and Weismantel, 2000)**.** *Let P, $\alpha^1, \ldots, \alpha^q$ and $\bar{G}$ be given. Then:*

- *if $C \subseteq \bar{V}$ induces a clique in $\bar{G}$ then the aggregated clique inequality $(\sum_{j \in C} \alpha^j)x \leq \sum_{j \in C} r(\alpha^j) - |C| + 1$ is valid for P_I.*
- *If $v_0, \ldots, v_{2c}$ induces a cycle in $\bar{G}$, then the aggregated odd cycle inequality*

$$\left(\sum_{j=0}^{2c} \alpha^{v_j}\right) x \leq \sum_{j=0}^{2c} r(\alpha^{v_j}) - c - 1 \tag{24}$$

  is valid for P_I.

Moreover, if q is polynomial in the original problem size, then:

- *A generalization of aggregated clique inequalities can be separated in polynomial time.*
- *Aggregated odd cycle inequalities can be separated in polynomial time.*

These results are based on some analogous results for the *Stable Set Problem*, see for example Grötschel et al. (1988) and Lovász and Schrijver (1991).

There remains the question of how to choose the α^j. Borndörfer and Weismantel do not give any a priori procedure for doing this in general. Instead, they consider ILP formulations of some specific combinatorial optimization problems and show that, for these problems, certain conflict graphs can be constructed in polynomial time in such a way that an exponentially large number of inequalities inducing facets of P_I are aggregated clique or odd cycle inequalities. In this way they also obtained polynomial-time separation algorithms for the 2-chorded odd cycle, odd wheel and Möbius ladder inequalities. This is not a coincidence, as we now show.

Recall that an edge $\{i, j\}$ of the conflict graph represents the inequality $(\alpha^i + \alpha^j)x \leq r(\alpha^i) + r(\alpha^j) - 1$. We call such an inequality a *primitive* inequality. Now suppose that $v_0, \ldots, v_{2c}$ is an odd cycle in $\bar{G}$. If we sum together the $2c + 1$ corresponding primitive inequalities, we obtain $2(\sum_{j=0}^{j=2c} \alpha^{v_j})x \leq 2\sum_{j=0}^{j=2c} r(\alpha^{v_j}) - 2c - 1$. Dividing this by two and rounding down the right hand side, we obtain the aggregated odd cycle inequality (24). This implies that aggregated odd cycle inequalites are 'merely' $\{0, \frac{1}{2}\}$-cuts (with respect to the primitive inequalities). In this sense, the separation algorithm for aggregated odd cycle inequalities can be regarded as another scheme for separating $\{0, \frac{1}{2}\}$-cuts.

Next we show that the aggregated odd cycle inequalities are also disjunctive cuts (in fact split cuts) with respect to the primitive inequalities:

Theorem 10. *The aggregated odd cycle inequality* (24) *is implied by the primitive inequalities and by the disjunction* $(\alpha^{v_0}x \le r(\alpha^{v_0}) - 1) \vee (\alpha^{v_0}x \ge r(\alpha^{v_0}))$.

Proof: In this proof indices are taken modulo $2c$.
First term of disjunction: we obtain (24) by summing together the inequality $\alpha^{v_0}x \le r(\alpha^{v_0}) - 1$ and the primitive inequalities $(\alpha^{v_i} + \alpha^{v_{i+1}})x \le r(\alpha^{v_i}) + r(\alpha^{v_{i+1}}) - 1$ for $i = 1, 3, \ldots, 2c - 1$.
Second term of disjunction: we obtain (24) by summing together the inequality $-\alpha^{v_0}x \le -r(\alpha^{v_0})$ and the primitive inequalities $(\alpha^{v_i} + \alpha^{v_{i+1}})x \le r(\alpha^{v_i}) + r(\alpha^{v_{i+1}}) - 1$ for $i = 0, 2, \ldots, 2c$. □

This implies the following corollary:

Corollary 6. *If, for a given class of ILPs, there is a class $\mathcal{F}$ of inequalities which can be separated in polynomial time because they are aggregated odd cycle inequalities in an appropriate conflict graph, then there is a polynomial-time disjunctive separation algorithm for a class $\mathcal{F}'$ of inequalities with $\mathcal{F} \subseteq \mathcal{F}'$.*

Corollary 6 explains why we have been able to duplicate most of the separation results of Borndörfer and Weismantel (2000), though by somewhat different methods.

It would be interesting to perform an analytical comparison between the Caprara–Fischetti, disjunctive and conflict graph approaches, with a view to understanding their comparative strengths and weaknesses, or even (somehow) integrating them within a more general framework. An interesting feature which they all have in common is that they may yield violated inequalities which are not facet-inducing. In one sense this is theoretically 'inelegant'. Yet, given the $\mathcal{NP}$-hardness of $(2p + 1, 2)$-circulant separation mentioned in Subsection 3.2, it appears that it is not always a good idea to restrict attention to facet-inducing inequalities.

5. Discussion and conclusions

Although disjunctive cutting planes were introduced over two decades ago, their properties are still not well understood. We have shown that many facet-inducing inequalities, for some important combinatorial optimization problems, can be obtained from rather simple disjunctions and therefore separated in polynomial time. This includes, in principle, any inequalities which can be derived from odd cycles in a conflict graph (Borndörfer and Weismantel, 1997), together with many (but not all) $\{0, \frac{1}{2}\}$-cuts (Caprara and Fischetti, 1996).

There are however limitations to the disjunctive approach. For example, consider the well-known *Matching Problem* (Edmonds, 1965). This problem is solvable in polynomial time and a complete description of the polyhedron is obtained using the *blossom* inequalities (see Edmonds, 1965). The blossom inequalities are 'mere' $\{0, \frac{1}{2}\}$-cuts (Caprara and Fischetti, 1996) and their associated separation problem can be solved in polynomial time by maximum flow techniques (Padberg and Rao, 1982). Yet, despite all this, it does not appear to be possible to derive (or separate) blossom inequalities using only a polynomial number of disjunctions. (Indeed, if such a derivation existed, it would yield a compact representation of the matching polytope, the existence of which is a long-standing open problem—see Yannakakis, 1991.)

Moreover, from a practical point of view, disjunctive separation algorithms may not always be wholly desirable. Although they are theoretically polynomial, they rely on the solution of (sometimes large) linear programs. Producing effective practical implementations of the separation algorithms proposed in this paper is a challenging problem in itself.

Nevertheless, the disjunctive approach is certainly worthy of more attention. The author is currently studying disjunctive cuts for the *Symmetric Travelling Salesman* and *Vehicle Routing* problems.

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