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Theory and Methodology

Analysis of upper bounds for the Pallet Loading Problem

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Abstract

This paper is concerned with upper bounds for the well-known Pallet Loading Problem (PLP), which is the problem of packing identical boxes into a rectangular pallet so as to maximize the number of boxes fitted. After giving a comprehensive review of the known upper bounds in the literature, we conduct a detailed analysis to determine which bounds dominate which others. The result is a ranking of the bounds in a partial order. It turns out that two of the bounds dominate all others: a bound due to Nelissen and a bound obtained from the linear programming relaxation of a set packing formulation. Experiments show that the latter is almost always optimal and can be computed quickly.

Keywords: Packing; Pallet loading; Mathematical programming

1. Introduction

The Pallet Loading Problem (PLP) is the problem of packing identical boxes into a rectangular pallet, maximising the number of boxes fitted. Since, in practical applications, boxes have their vertical orientation fixed and are orthogonally placed on the pallet in layers, the problem reduces to that of packing identical rectangles into a large containing rectangle.

The PLP can be defined by a quadruple (L, W, a, b) of positive integers, where L and W denote the pallet length and width, and a and b denote the box length and width, respectively. Without loss of generality we assume that $L \geqslant W \geqslant a \geqslant b$. The objective is to maximize the number of $a \times b$ boxes packed into the $L \times W$ pallet.

The PLP has many practical applications (Carpenter and Dowsland, 1985; Dyckhoff, 1990). It is also interesting from the point of view of computational complexity, because it is a very simple problem which is not known to be polynomially solvable, yet not proven to be NP-hard. Indeed, the decision version of the

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PLP is not known to be in NP at all, because the number of boxes in an optimal packing is $O(LW/ab)$, which is not bounded by a polynomial in the binary encoding length of the instance, which is $O(\log_2 L)$ (see Nelissen (1993), for further discussion on this issue).

An excellent survey of heuristic and exact algorithms for the PLP is given by Nelissen (1993); for two more recent heuristics see also Scheithauer and Terno (1996) and Morabito and Morales (1998). In this paper, however, we are concerned with upper bounds. Good upper bounds are desirable as they can be used both to evaluate the performance of heuristics and to improve the efficiency of exact algorithms. The purpose of the present paper is to review the known upper bounds in the literature and to establish which of these bounds dominate which others.

The structure of the paper is as follows. In Section 2, we review the known upper bounding procedures in the literature. In Section 3 we establish dominance relations between bounds, or, where this is not possible, show that certain bounds are incomparable. It turns out that two of the bounds dominate all others: a bound due to Nelissen (1993) and a bound obtained from the linear programming relaxation of a set packing formulation. In Section 4 we present some computational results to show how the bounds behave in practice. It turns out that the set packing based bound is almost always optimal. Finally, concluding comments are made in Section 5.

2. Known upper bounding procedures

2.1. The area bound

The obvious bound $\lfloor LW/ab \rfloor$, where $\lfloor \cdot \rfloor$ denotes rounding down to the nearest integer, is called the *area* bound. It is part of the folklore.

2.2. The Barnes bound

The upper bound of Barnes (1979) is based on an older result of Barnett and Kynch (1967), which states that the maximum number $n_{a\times 1}$ of $a \times 1$ strips which can be orthogonally packed into an $L \times W$ pallet is equal to:

$$n_{a\times 1} := \begin{cases} (LW - (L \bmod a)(W \bmod a))/a & \text{if } L \bmod a + W \bmod a \leqslant a, \\ (LW - (a - L \bmod a)(a - W \bmod a))/a & \text{otherwise.} \end{cases}$$

(Here, 'mod' is short for 'modulo' – that is, $L \bmod a$ means the integer remainder obtained when L is divided by a, and so on.) Since each box consists of b such strips, the quantity $UB(a) := \lfloor n_{a\times 1}/b \rfloor$ is a valid upper bound for the PLP.

Analogously, the maximum number $n_{b\times 1}$ of $b \times 1$ strips which can be orthogonally packed into an $L \times W$ pallet is equal to:

$$n_{b\times 1} := \begin{cases} (LW - (L \bmod b)(W \bmod b))/b & \text{if } L \bmod b + W \bmod b \leqslant b, \\ (LW - (b - L \bmod b)(b - W \bmod b))/b & \text{otherwise,} \end{cases}$$

and therefore the quantity $UB(b) := \lfloor n_{b\times 1}/a \rfloor$ is also a valid upper bound for the PLP. The minimum of $UB(a)$ and $UB(b)$ yields the Barnes bound.

Note that the area and Barnes bounds can be computed in constant time.

2.3. Bounds from equivalent instances

The area and Barnes bounds can be improved using the notions of *feasible partitions* and *equivalent instances* (Bischoff and Dowsland, 1982). A feasible partition of L is an ordered pair (n, m) of non-negative integers such that $na + mb \leqslant L$; feasible partitions of W are defined similarly. Two instances of the PLP are *equivalent* if the sets of feasible partitions for L and W are the same for both problems. Equivalent instances have the same optimal solutions.

Even though two PLP instances may be equivalent, they may have different area bounds, or different Barnes bounds, or both. For example, the area and Barnes bounds for the instance $(9, 9, 5, 5)$ are both 3, but for the equivalent instance $(1, 1, 1, 1)$ they are both 1. This phenomenon is used by Dowsland (1985b) to improve the area and Barnes bounds as follows. Given an instance (L, W, a, b), define:

$$L' = \max\{na + mb \mid (n, m) \text{ is a feasible partition of } L\},$$
$$W' = \max\{na + mb \mid (n, m) \text{ is a feasible partition of } W\}.$$

Then (L, W, a, b) is equivalent to (L', W', a, b), but $L' \leqslant L, W' \leqslant W$. This reduction is known as the *perfect partition reduction*. For the example given above, $(9, 9, 5, 5)$, the perfect partition reduction yields $(5, 5, 5, 5)$; the effect is that the area and Barnes bounds drop from 3 to 1, which is optimal.

Although not noted by Dowsland (1985b), perfect partition reduction can be performed in a time which is polynomial in the encoding length of the PLP instance, because knapsack problems with only two variables can be solved in polynomial time (see Hirschberg and Wong, 1976).

A related, but in one sense superior, procedure is given by Dowsland (1984). Given a PLP instance, this procedure finds an instance in the same equivalence class with *minimum* area bound (see also Nelissen, 1993). This is more powerful than the perfect partition reduction. For example, consider the instance $(200, 200, 20, 19)$, for which the area bound is 105. No perfect partition reduction is possible for this instance, because $L' = L$ and $W' = W$. Yet, the equivalent instance $(10, 10, 1, 1)$ has an area bound of 100, which is optimal.

Unfortunately, the procedure of Dowsland (1984) is only *pseudo*-polynomial (that is, polynomial in L but not in $\log_2 L$). Moreover, the reduction can yield an equivalent instance for which W is non-integer, or even *irrational*. This can lead to difficulties if one wishes to use the reduced instance for further computations and, in practice, some kind of rational (diophantine) approximation must be made. (We will mention this issue again in Section 4.)

A third transformation is based on the idea of viewing the PLP as a problem of finding a stable set of maximum cardinality in a certain graph (Dowsland, 1987). This graph is formally defined in Section 2.5 below, so we do not give details here. Basically, the transformation is a pseudo-polynomial procedure which, given a PLP instance (L, W, a, b), finds an equivalent instance which minimizes the number of vertices in the associated graph. This often leads to a significant improvement in the Barnes bound, as well as a reduction in the size of L, W, a and b.

2.4. Bounds from relaxed instances

In fact, it is not necessary to transform an instance into an *equivalent* instance to obtain a valid bound. An instance which is a *relaxation* of the original can also be used. A PLP instance I_1 is a relaxation of an instance I_2 if the set of feasible partitions for I_2 is a *subset* of the set of feasible partitions of I_1. Oddly, a bound for a relaxed instance may be stronger than the equivalent bound for the original instance. For example, the Barnes bound for the instance $(19, 19, 7, 2)$ is 25, but for the relaxed instance $(10, 10, 4, 1)$ it is

24. Dowsland (1985a) and Exeler (1991) give pseudo-polynomial procedures for exploiting this idea to produce good bounds, but we do not go into details here for the sake of brevity.

2.5. The packing bound

The next upper bound is based on the following formulation of the PLP as a zero–one Integer Linear Program (0-1 ILP), which has appeared several times in the literature (Beasley, 1985; Dowsland, 1985a; Hadjiconstantinou and Christofides, 1995; Morabito and Morales, 1998):

$$\text{Maximize} \quad \sum_{p=1}^{L-a+1} \sum_{q=1}^{W-b+1} X_{1pq} + \sum_{p=1}^{L-b+1} \sum_{q=1}^{W-a+1} X_{2pq} \tag{1}$$

subject to:

$$\sum_{p=\max\{1,r-a+1\}}^{\min\{r,L-a+1\}} \sum_{q=\max\{1,s-b+1\}}^{\min\{s,W-b+1\}} X_{1pq} + \sum_{p=\max\{1,r-b+1\}}^{\min\{r,L-b+1\}} \sum_{q=\max\{1,s-a+1\}}^{\min\{s,W-a+1\}} X_{2pq} \leqslant 1 \quad (r = 1,\ldots,L;\ s = 1,\ldots,W), \tag{2}$$

$$X_{1pq} \in \{0,1\} \quad (1 \leqslant p \leqslant L-a+1;\ 1 \leqslant q \leqslant W-b+1), \tag{3}$$

$$X_{2pq} \in \{0,1\} \quad (1 \leqslant p \leqslant L-b+1;\ 1 \leqslant q \leqslant W-a+1). \tag{4}$$

To understand this formulation, it is helpful to regard the pallet as made of LW small 'squares'. Each square is identified by two co-ordinates. The co-ordinates of the squares at the bottom-left, bottom-right, top-left and top-right corners of the pallet are $(1,1)$, $(L,1)$, $(1,W)$ and (L,W), respectively. The variable X_{1pq} takes the value 1 if and only if a box is placed 'horizontally' (with its edge of length b parallel to the pallet edge of length L), in such a way that its bottom-left corner covers the square (p,q). Similarly, the variable X_{2pq} takes the value 1 if and only if a box is placed 'vertically' (with its edge of length b parallel to the pallet edge of length W), so that its bottom-left corner covers the square (p,q). Then the constraints (2) ensure that no two boxes overlap. This is because each individual constraint of the form (2) ensures that a particular 'square' is covered by at most one box.

We can associate a graph with the formulation (1)–(4), by defining a vertex for each variable and an edge between two vertices if and only if the two variables appear together in a constraint of the form (2). Then, the PLP is a maximum cardinality stable set problem, as mentioned in Section 2.3, and each individual constraint of the form (2) corresponds to a clique in the graph.

If we replace the integrality conditions (3) and (4) with the following weaker non-negativity conditions:

$$X_{1pq} \geqslant 0 \quad (1 \leqslant p \leqslant L-a+1;\ 1 \leqslant q \leqslant W-b+1), \tag{3'}$$
$$X_{2pq} \geqslant 0 \quad (1 \leqslant p \leqslant L-b+1;\ 1 \leqslant q \leqslant W-a+1), \tag{4'}$$

we obtain the LP relaxation of the integer program. Obviously, rounding down the optimal objective value of the LP relaxation gives a valid upper bound for the PLP. We will call it the *packing bound*, because the formulation is of standard set packing type. It can be shown, using the ellipsoid method for linear programming, that the packing bound can be computed in pseudo-polynomial time.

The LP (1), (2), (3′), and (4′) can be rather large in practice and this has led some authors to compute only an approximation to the packing bound, using Lagrangian relaxation and subgradient optimization (Beasley, 1985; Dowsland, 1985a; Hadjiconstantinou and Christofides, 1995). However, the LP can be reduced dramatically in size, without losing any optimal solution, by eliminating variables using the concept of *raster points* (see Scheithauer and Terno, 1996; Morabito and Morales, 1998). Alternatively, one of the transformations mentioned in Section 2.3 could be applied to reduce L and W.

2.6. Other LP-based bounds

Another upper bound for the PLP based on linear programming is due to Isermann (1987). Let $F(L,a,b)$ denote the set of all feasible partitions of L and $F(W,a,b)$ denote the same for W. Define a variable x_{ij} for each $(i,j) \in F(L,a,b)$ and a variable y_{fg} for each $(f,g) \in F(W,a,b)$. Then the solution to the following LP (rounded down if possible) gives a valid upper bound to the PLP:

$$\text{Maximize} \quad \sum_{(i,j)\in F(L,a,b)} (i/b)x_{ij} + \sum_{(f,g)\in F(W,a,b)} (f/b)y_{fg} \tag{5}$$

subject to:

$$\sum_{(i,j)\in F(L,a,b)} x_{ij} \leqslant W, \tag{6}$$

$$\sum_{(f,g)\in F(W,a,b)} y_{ij} \leqslant L, \tag{7}$$

$$\sum_{(i,j)\in F(L,a,b)} aix_{ij} - \sum_{(f,g)\in F(W,a,b)} bgy_{fg} = 0, \tag{8}$$

$$\sum_{(i,j)\in F(L,a,b)} bjx_{ij} - \sum_{(f,g)\in F(W,a,b)} afy_{fg} = 0, \tag{9}$$

$$x_{ij} \geqslant 0 \quad (i,j) \in F(L,a,b), \tag{10}$$

$$y_{ij} \geqslant 0 \quad (f,g) \in F(W,a,b). \tag{11}$$

Again, one could adjust this LP by replacing (L,W,a,b) with some equivalent or relaxed quadruple and still obtain a valid upper bound. In particular, Isermann recommends replacing L and W in (6) and (7) with their perfect partition equivalents L' and W' (Section 2.3).

Naujoks (1989) shows that a stronger bound can sometimes be obtained by adding an additional constraint and optimising each term in the objective function (5) separately. Nelissen (1993) obtains a still stronger bound by adding additional constraints and computing tight lower and upper bounds on the value of each variable. All of these bounds can be obtained in pseudo-polynomial time. However, we do not go into more detail here for the sake of brevity.

3. Dominance relations

In this section, we give a number of theorems which establish dominance relations between the area, Barnes, Isermann and packing bounds. Before doing this, however, we define two PLP instances which will turn out to be particularly useful.

Instance 1 $(6, 6, 4, 1)$**.** The area and Isermann bounds are both 9, whereas the Barnes and packing bounds are both 8. The optimum is 8.

Instance 2 $(9, 8, 5, 2)$**.** The area and Barnes bounds are both 7, whereas the Isermann and packing bounds are both 6. The optimum is 6.

Note that neither of these instances is affected by perfect partition reduction.

Theorem 1. *The Barnes bound dominates the area bound, and this dominance can be strict, even when L and W have been reduced to their perfect partition equivalents.*

Proof. The quantity $n_{a\times 1}$ defined in Section 2.2 is clearly less than or equal to LW/a. Therefore, $UB(a) = \lfloor n_{a\times 1}/b \rfloor$ is less than or equal to the area bound. Exactly the same argument applies to $n_{b\times 1}$ and $UB(b)$. Since the Barnes bound is the minimum of $UB(a)$ and $UB(b)$, the Barnes bound dominates the area bound. Strict dominance is shown by Instance 1. □

Theorem 2. *The packing bound dominates the Barnes bound, and this dominance can be strict, even when L and W have been reduced to their perfect partition equivalents.*

Proof. By elementary LP duality theory, to show dominance it suffices to give a feasible solution to the dual of the LP relaxation with an objective value which, when rounded down, equals the Barnes bound. The dual is

$$\text{Minimize} \quad \sum_{r=1}^{L} \sum_{s=1}^{W} \mu_{rs} \tag{12}$$

subject to:

$$\sum_{r=p}^{p+a-1} \sum_{s=q}^{q+b-1} \mu_{rs} \geqslant 1 \quad (p = 1, \ldots, L - a + 1;\ q = 1, \ldots, W - b + 1), \tag{13}$$

$$\sum_{r=p}^{p+b-1} \sum_{s=q}^{q+a-1} \mu_{rs} \geqslant 1 \quad (p = 1, \ldots, L - b + 1;\ q = 1, \ldots, W - a + 1). \tag{14}$$

$$\mu_{rs} \geqslant 0 \quad (r = 1, \ldots, L;\ s = 1, \ldots, W). \tag{15}$$

There is one dual variable for each 'square' in the pallet (Section 2.5). It is not hard to show that a feasible solution to the dual is obtained by setting $\mu_{pq} = b^{-1}$ if $p + q$ is a multiple of a, $\mu_{pq} = 0$ otherwise. It is also not hard to show (see Barnett and Kynch, 1967) that the objective function in this case is

$$\begin{array}{ll} (LW-(L \bmod a)(W \bmod a))/ab & \text{if } L \bmod a + W \bmod a \leqslant a, \\ (LW-(a-L \bmod a)(a-W \bmod a))/ab & \text{otherwise.} \end{array}$$

Rounding this value down gives the upper bound $UB(a)$ mentioned in Section 2.2.

Similarly, setting $\mu_{pq} = a^{-1}$ if $p+q$ is a multiple of b, 0 otherwise give a feasible dual solution whose objective value, rounded down, is the upper bound $UB(b)$ mentioned in Section 2.2. Thus, the packing bound dominates the Barnes bound. Strict dominance is shown by Instance 2. □

Theorem 3. *The Isermann bound dominates the area bound, and this dominance can be strict, even when L and W have been reduced to their perfect partition equivalents.*

Proof. If (i,j) is a feasible partition of L, then, by definition, $ia+jb \leqslant L$. Equivalently, $j \leqslant (L-ia)/b$. Thus, from Eq. (9) we obtain

$$\sum_{(i,j)\in F(L,a,b)} (L-ia)x_{ij} - \sum_{(f,g)\in F(W,a,b)} afy_{fg} \geqslant 0. \tag{16}$$

Rearranging (16) yields

$$\sum_{(i,j)\in F(L,a,b)} (i/b)x_{ij} + \sum_{(f,g)\in F(W,a,b)} (f/b)y_{fg} \leqslant (L/ab) \sum_{(i,j)\in F(L,a,b)} x_{ij}. \tag{17}$$

Using (6) to weaken the right-hand side of (17) we obtain

$$\sum_{(i,j)\in F(L,a,b)} (i/b)x_{ij} + \sum_{(f,g)\in F(W,a,b)} (f/b)y_{fg} \leqslant LW/ab. \tag{18}$$

Since the left-hand side of (18) is identical to the objective function (5), we have proved dominance. Strict dominance is shown by Instance 2. □

We note in passing that a different proof of Theorem 2 can be obtained by using the fact that all feasible partitions (f,g) of W satisfy $fa+gb \leqslant W$.

Theorem 4. *The Barnes and Isermann bounds are incomparable (neither dominates the other), even when L and W have been reduced to their perfect partition equivalents.*

Proof. For Instance 1, the Barnes bound is smaller than the Isermann bound. For Instance 2, however, the Isermann bound is smaller than the Barnes bound. □

Next we will show that the packing bound dominates the Isermann bound. The proof of this is much more involved. We begin by defining, for any feasible solution X^* to the packing LP and any $r = 1, \ldots, L$, the following quantities:

$$\lambda_{1r} = \sum_{p=\max\{1,r-a+1\}}^{\min\{r,L-a+1\}} \sum_{q=1}^{W-b+1} X^*_{1pq}, \quad \lambda_{2r} = \sum_{p=\max\{1,r-b+1\}}^{\min\{r,L-b+1\}} \sum_{q=1}^{W-a+1} X^*_{2pq}.$$

Note that, if X^* was integral, then λ_{1r} would equal the number of boxes packed horizontally which overlap the rth vertical 'strip' in the pallet and λ_{2r} would equal the same for boxes packed vertically.

Lemma 1. *If X^* is a feasible solution to the packing LP, then for any $1 \leqslant r \leqslant L$, the pair $(\lambda_{2r}, \lambda_{1r})$ is a convex combination of feasible partitions of W. That is, one can define a weight π_{fgr} for each feasible partition $(f,g) \in F(W,a,b)$ such that*

$$\lambda_{1r} = \sum_{(f,g)\in F(W,a,b)} g\pi_{fgr}, \quad \lambda_{2r} = \sum_{(f,g)\in F(W,a,b)} f\pi_{fgr}, \quad \text{and} \quad \sum_{(f,g)\in F(W,a,b)} \pi_{fgr} = 1.$$

Proof. Consider the set of constraints (2) for fixed r. The matrix of left-hand side coefficients for these constraints has the well-known 'consecutive 1's property', which means that in each column, the 1's appear consecutively. This implies (Fulkerson and Gross, 1965) that, if only those constraints were present (along with the non-negativity inequalities), then the feasible region would be an integral polyhedron. From the definition of λ_{1r} and λ_{2r}, it follows that, even if *only those* constraints were present, then for any extreme point X^* of the feasible region, $(\lambda_{2r}, \lambda_{1r})$ would be a feasible partition of W. Thus, if X^* satisfies *all* of the constraints in the packing LP, the pair $(\lambda_{2r}, \lambda_{1r})$ must be a convex combination of feasible partitions of W. □

Now we define, for any X^* and any $s = 1, \dots, W$, the quantities

$$\mu_{1s} = \sum_{p=1}^{L-a+1} \sum_{q=\max\{1,s-b+1\}}^{\min\{s,W-b+1\}} X^*_{1pq}, \quad \mu_{2s} = \sum_{p=1}^{L-b+1} \sum_{q=\max\{1,s-a+1\}}^{\min\{s,W-a+1\}} X^*_{2pq}.$$

These are analogous to λ_{1r} and λ_{2r}. If X^* was integral, then μ_{1r} would equal the number of boxes packed horizontally which overlap the sth *horizontal* 'strip' in the pallet and μ_{2r} would equal the same for boxes packed vertically.

Lemma 2. *If X^* is a feasible solution to the packing LP, then for any $1 \leqslant s \leqslant W$, the pair (μ_{1s}, μ_{2s}) is a convex combination of feasible partitions of L. That is, one can define a weight σ_{ijs} for each feasible partition $(i,j) \in F(L,a,b)$ such that*

$$\mu_{1s} = \sum_{(i,j)\in F(L,a,b)} i\sigma_{ijs}, \quad \mu_{2s} = \sum_{(i,j)\in F(L,a,b)} j\sigma_{ijs}, \quad \text{and} \quad \sum_{(i,j)\in F(L,a,b)} \sigma_{ijs} = 1.$$

Proof. Similar to the proof of Lemma 1. □

We are now ready to prove another theorem.

Theorem 5. *The packing bound dominates the Isermann bound, and this dominance can be strict, even when L and W have been reduced to their perfect partition equivalents.*

Proof. Instance 1 shows that the packing bound can be better than the Isermann bound. It remains only to show that, given any solution X^* to the packing LP, there is a solution x to the Isermann LP with same objective value. This is done using the quantities π_{fgr} and σ_{ijs} defined in Lemmas 1 and 2, by setting

$$x_{ij} = \sum_{s=1}^{W} \sigma_{ijs}, \quad y_{fg} = \sum_{r=1}^{L} \pi_{fgr}.$$

This solution satisfies (6), because

$$\sum_{(i,j)\in F(L,a,b)} x_{ij} = \sum_{s=1}^{W} \sum_{(i,j)\in F(L,a,b)} \sigma_{ijs} = \sum_{s=1}^{W} 1 = W.$$

By an analogous argument (7) is also satisfied. Moreover, (8) is satisfied because

$$\begin{aligned}\sum_{(i,j)\in F(L,a,b)} aix_{ij} &= a\sum_{s=1}^{W} \sum_{(i,j)\in F(L,a,b)} i\sigma_{ijs} = a\sum_{s=1}^{W}\mu_{1s}\\ &= b\sum_{r=1}^{L}\lambda_{1r} \text{ (by simple counting arguments)}\\ &= b\sum_{r=1}^{L}\sum_{(f,g)\in F(W,a,b)} g\pi_{fgr} = \sum_{(f,g)\in F(W,a,b)} bgy_{fg}.\end{aligned}$$

By an analogous argument (9) is also satisfied.

This solution to the Isermann LP has the same objective function value as X^* because

$$\begin{aligned}\sum_{(i,j)\in F(L,a,b)} (i/b)x_{ij} + \sum_{(f,g)\in F(W,a,b)} (f/b)y_{fg} &= \sum_{s=1}^{W}\sum_{(i,j)\in F(L,a,b)} (i/b)\sigma_{ijs} + \sum_{r=1}^{L}\sum_{(f,g)\in F(W,a,b)} (f/b)\pi_{fgr}\\ &= \sum_{s=1}^{W}\mu_{1s}/b + \sum_{r=1}^{L}\lambda_{2r}/b\\ &= \sum_{p=1}^{L-a+1}\sum_{q=1}^{W-b+1} X^*_{1pq} + \sum_{p=1}^{L-b+1}\sum_{q=1}^{W-a+1} X^*_{2pq}\\ &\quad \text{(again, by simple counting arguments).} \qquad \square\end{aligned}$$

To end this section, we give one final result.

Theorem 6. *The packing bound dominates the minimum of the Barnes and Isermann bounds, and this dominance can be strict, even when L and W have been reduced to their perfect partition equivalents.*

Proof. Theorems 2 and 5 show that the packing bound dominates both of the other bounds. Strict dominance is shown by the PLP instance (14, 13, 4, 3). For this instance, which cannot be reduced using perfect partitions, the area, Barnes and Isermann bounds are all 15, whereas the packing bound is 14. The optimum is 14. $\square$

4. Computational experiments

In this section we give some computational results to give a feel for the behaviour of the bounds in practice. We use three sets of test instances. The first set contains 1000 random instances with $L = 100$ generated according to the scheme in Dowsland (1985b) to resemble instances occurring in practical applications. These have optimal values ranging between 3 and 52. The second set contains 100 random instances with $L = 200$ obtained by using the same scheme, but doubling L and W. These have optimal values between 51 and 58. (We also created some larger instances, but these were discarded due to huge running times in our experiments.)

The instances in our third set were donated to us by Josef Nelissen. This test set, called 'Cover I', contains one representative from each of the 8274 distinct equivalence classes for which the optimum lies between 1 and 50, each of which has already been reduced with the procedure of Dowsland (1984) mentioned in Section 2.3. However, we only used 3176 of these 8274 instances, namely those with $L \leqslant 99$. The remaining 5098 instances had $933 \leqslant L \leqslant 32673$, probably due to approximating irrational ratios as mentioned in Section 2.3.

Algorithms were coded using Microsoft Visual C and run on a 350 MHz Pentium II PC with 128 MB of RAM under Windows NT. For computing the LP-based bounds, we used the dual simplex routine of the ILOG CPLEX 6.0 callable library. We also used CPLEX to obtain the optimal solution values for the instances in the first two data sets. (This took about a day for each of the two sets. A specialized PLP algorithm would probably have been quicker for this purpose, but none was available.)

Results for the first test set are displayed in Table 1. We give the total computing time in seconds (over all 1000 instances), the number of instances on which the bound is optimal, the average error and the maximum error for *seven* bounding procedures. 'PPR' stands for *perfect partition reduction* and 'Hybrid' gives the results for the minimum of the 'PPR + Barnes' and Isermann bounds.

From Table 1 a number of conclusions can be drawn. First, it is clear that PPR leads to a major improvement in the area and Barnes bounds. Second, the Isermann bound is usually, but not always, better than the 'PPR + Barnes' bound. (It was better on 91 instances and worse on 18.) Third, the packing bound is clearly superior to all other bounds; it was optimal in 997 cases out of 1000 and for the remaining three instances the gap was only one. Fourth, the extra accuracy of the Isermann and packing bounds comes at the cost of increased running time.

Table 2 gives analogous statistics for the second data set. Similar conclusions apply here.

For the third data set, the reduction of Dowsland (1984) has already been applied and therefore PPR achieves nothing. So we did not compute PPR versions of the area and Barnes bounds. Thus, Table 3 shows the results for five bounds only.

The results in Table 3 are also informative. For these *reduced* instances the Isermann bound was always equal to the area bound. This in turn meant that the Barnes bound was better than both and that the Hybrid bound was always equal to the Barnes bound. However, once again the packing bound is clearly superior to all other bounds (at the cost of increased running time). It was optimal in 3169 cases out of 3176 and for the remaining seven instances the gap was only one.

Table 1
Results for 1000 small/medium random PLP instances

	Area	Barnes	PPR + Area	PPR + Barnes	Isermann	Hybrid	Packing
Time (seconds)	<1	<1	<1	<1	201	201	1551
Successes	264	344	717	811	884	902	997
Mean error	1.250	1.079	0.320	0.213	0.116	0.098	0.003
Max error	7	7	3	3	1	1	1

Table 2
Results for 100 large random PLP instances

	Area	Barnes	PPR + Area	PPR + Barnes	Isermann	Hybrid	Packing
Time (seconds)	<1	<1	<1	<1	112	112	1058
Successes	18	25	60	76	84	89	100
Mean error	1.48	1.26	0.48	0.31	0.16	0.11	0
Max error	6	5	3	3	1	1	0

Table 3
Performance of upper bounds on 3176 transformed PLP instances

	Area	Barnes	Isermann	Hybrid	Packing
Time (seconds)	<1	<1	647	647	2632
Successes	3010	3145	3010	3145	3169
Mean error	0.052	0.010	0.052	0.010	0.002
Max error	1	1	1	1	1

For the sake of interest, and to aid other researchers, we give here the instances on which the packing bound failed to be optimal. The three instances in the first data set were $(100, 64, 17, 10)$, $(100, 82, 22, 8)$ and $(100, 83, 22, 8)$; the seven instances in the third data set were $(32, 22, 5, 4)$, $(32, 27, 5, 4)$, $(40, 26, 7, 4)$, $(40, 33, 7, 4)$, $(53, 26, 7, 4)$, $(37, 30, 8, 3)$ and $(81, 39, 9, 7)$. The optimal solutions for these 10 problems are 36, 45, 45, 34, 42, 36, 46, 48, 45 and 49, respectively. Note that the instances $(100, 82, 22, 8)$ and $(100, 83, 22, 8)$ are equivalent.

5. Conclusions

We have compared four bounds for the PLP and given dominance relationships between them. One of them, the packing bound, has been shown to dominate all the others. An empirical study showed that this bound was optimal in almost every case and that even when it was not, it only exceeded the optimum by one.

In fact, closer examination revealed that the cost of the solution to the packing LP (*before* rounding down) never exceeded the optimum by more than one. It is thus tempting to conjecture that the gap between the packing bound and the optimum *never* exceeds one. However, a similar conjecture about a bound for the *cutting stock problem* (the column generation bound due to Gilmore and Gomory, 1961, 1963) stood for two decades until it was disproved by Marcotte (1986). Thus, we leave an examination of this question for future research.

One final point. Although we did not prove this formally, it can be shown that the packing bound and the upper bound of Nelissen (1993) are incomparable (neither dominates the other). For example, Nelissen's bound is optimal for the instance $(32, 22, 5, 4)$, whereas the packing bound is not; in contrast, the packing bound is optimal for the instance $(24, 22, 5, 3)$, whereas Nelissen's is not (Nelissen, private communication). Perhaps a still stronger bound could be obtained by combining the ideas behind the Nelissen and packing bounds in some way.

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