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Totally tight Chvátal–Gomory cuts ☆

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Abstract

Let $P := \{x \in \mathbb{R}^n\colon Ax \leqslant b\}$ be a polyhedron and P_I its integral hull. A *Chvátal–Gomory* (CG) cut is a valid inequality for P_I of the form $(\lambda^T A)x \leqslant \lfloor \lambda^T b \rfloor$, with $\lambda \in \mathbb{R}^m_+$, $\lambda^T A \in Z^n$ and $\lambda^T b \notin Z$. We give a polynomial-time algorithm which, given some $x^* \in P$, detects whether a *totally tight* CG cut exists, i.e., whether there is a CG cut such that $(\lambda^T A)x^* = \lambda^T b$. Such a CG cut is violated by as much as possible under the assumption that $x^* \in P$.

Keywords: Integer programming; Cutting planes; Separation; Hermite normal form; Chvátal–Gomory cuts

1. Introduction

This paper is concerned with the *separation problem for Chvátal–Gomory cuts*. Given a polyhedron $P := \{x \in \mathbb{R}^n \,:\, Ax \leqslant b\}$, where $A \in Z^{m\times n}$ and $b \in Z^m$, a *Chvátal–Gomory cut* is an inequality of the form

$$(\lambda^T A)x \leqslant \lfloor \lambda^T b \rfloor, \tag{1}$$

where $\lambda \in \mathbb{R}^m_+$ is such that $\lambda^T A \in Z^n$ and $\lambda^T b \notin Z$, and $\lfloor\cdot\rfloor$ denotes lower integer part (see [5,9,13,14]).

Chvátal–Gomory cuts are valid for the integral polyhedron $P_I := \text{conv}\{x \in P \cap Z^n\}$ and, indeed, many important facet-inducing inequalities, for polyhedra associated with many important combinatorial optimization problems, are Chvátal–Gomory cuts, see Caprara et al. [2,3]. For this reason (see, e.g. [10]), we might want to find an efficient algorithm to solve the following problem:

The Chvátal–Gomory separation problem (CG-SEP): Given some $x^* \in P := \{x \in \mathbb{R}^n\colon Ax \leqslant b\}$, find a Chvátal–Gomory cut which is violated by x^*, or prove that none exists.

The complexity of CG-SEP was posed as an open problem in [14]. Unfortunately, CG-SEP was recently shown to be strongly $\mathcal{NP}$-hard by Eisenbrand [8].

A problem related to CG-SEP was recently studied in [3]. Call a Chvátal–Gomory cut a mod-*k cut* if $\lambda \in \{0, 1/k, \ldots, (k-1)/k\}^m$ for some integer $k \geqslant 2$. It is easy to show that every (nondominated) Chvátal–Gomory cut is a mod-k cut for some k. Then, CG-SEP can be decomposed into an infinite number of subproblems: for each k, solve

The mod-k separation problem (mod-k-SEP): Given some $x^* \in P = \{x \in \mathbb{R}^n\colon Ax \leqslant b\}$ and some integer $k \geqslant 2$, find a mod-k cut which is violated by x^*, or prove that none exists.

Unfortunately, this does not seem to help much. In [2] it is shown that mod-k-SEP is strongly $\mathcal{NP}$-hard even for $k = 2$.

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Some more useful concepts can be found in [3]. Given some $x^* \in P$, the vector $s^* := b - Ax^*$ is called the *slack vector*. (Note that the components of s^* are non-negative.) It is not difficult to show that the slack of a Chvátal–Gomory cut (1), computed with respect to x^*, equals $\lfloor \lambda^T b \rfloor - \lambda^T b + \lambda^T s^*$. Therefore, for any given $x^* \in P$, an upper bound on the violation of a Chvátal–Gomory cut is $\lambda^T b - \lfloor \lambda^T b \rfloor$ ($\leqslant 1$). For this upper bound to be achieved, it must be possible for $s_j^* \lambda_j = 0$ to hold for $j = 1, \dots, m$, that is, for all inequalities used in the derivation of the cut to be tight at x^*. When this occurs, the Chvátal–Gomory cut is said to be *totally tight* (TT) at x^* [3].

In [3], it is shown how to identify TT mod-k cuts for a fixed k, or prove that none exist, in polynomial time (provided that the prime factorization of k is known). If a TT mod-k cut is found, this provides a positive solution to mod-k-SEP and therefore also for CG-SEP.

In Section 2 of this paper we present a related positive result. We show that it is possible to identify a TT Chvátal–Gomory cut, or prove that none exists, in polynomial time. That is, it is not necessary to specify k when looking for a TT cut. Clearly, if a TT cut is found, this provides a positive solution to CG-SEP, and therefore to mod-k-SEP for some integer k.

2. Finding TT cuts

In this section, we give a polynomial-time algorithm for solving the following problem:

The totally tight separation problem (TT-SEP): Given some $x^* \in P = \{x \in \mathbb{R}^n : Ax \leqslant b\}$, find a TT Chvátal–Gomory cut, or prove that none exists.

Of course, if x^* is a (fractional) extreme point of P, then TT-SEP is trivial: it suffices to generate a Gomory cut from the associated simplex tableau. Moreover, if x^* lies in the *strict interior* of P, then TT-SEP is again trivial: $s_j^* > 0$ for $j = 1, \dots, m$ and no TT cut exists. We are really concerned with the non-trivial case, when x^* is not a vertex of P, but lies on a face of P. The key to resolving TT-SEP in this case is the following integer version of Farkas' lemma, taken from Edmonds and Giles [7] (see also [14], Chapters 4 and 5).

Theorem 1 (Edmonds and Giles, 1977). *For a given system of linear equations of the form $Ax = b$, precisely one of the following statements is true:*

- *There is an integer solution to $Ax = b$.*
- *There is some rational vector y such that $y^T A$ is integral, but $y^T b$ is not. Moreover, either an x, or a y can be found in polynomial time.*

Now let $A'x \leqslant b'$ denote the subsystem of $Ax \leqslant b$ which is tight at x^*. (If there are no tight inequalities, there can be no TT cuts.)

Lemma 1. *A TT cut exists if and only if there is some rational vector y such that $y^T A'$ it is integral, but $y^T b'$ is not.*

Proof. By definition, a TT cut exists if and only if there is some $y' \geqslant 0$ such that $(y')^T A'$ is integral, but $(y')^T b'$ is not. But a suitable y' immediately yields a suitable y, and one can obtain a suitable y' from a suitable y by taking fractional parts. □

Geometrically speaking, the situation is as follows. The minimal face of P containing x^* is $F := \{x \in P : A'x = b'\}$. A TT cut exists if and only if there are no integer points in the affine hull of F. Moreover, a TT cut, if one exists, will cut off F in its entirety.

Theorem 2. *TT-SEP can be solved in polynomial time.*

Proof. Given A, b and x^*, it takes $O(nm)$ time to identify A' and b'. (If x^* has been found by a simplex-based cutting plane algorithm, the slack variables are available and only $O(m)$ time is needed.) Then, by Theorem 1, obtaining a suitable y, or proving that none exists, can be performed in polynomial time. Finally, converting y to y' as required in Lemma 1 takes $O(m)$ time.

The precise details of the implementation are as follows. First, note that, in the case of degeneracy, the rows of (A', b') may be linearly dependent. In this case, not all of the rows are needed. What is needed is a minimal, linearly independent set of equations defining the affine hull of F. Such a set of rows can be found by Gaussian elimination, which was shown to run in polynomial time by Edmonds [6].

Thus, from now on we assume that the rows of (A', b') are linearly independent. Since b' is in the columnspace of A', this implies that also the rows of A' are linearly independent.

Next, we need the concept of *Hermite normal form* (see, e.g. [13,14]).

Definition. A square nonsingular integer matrix is a *Hermite matrix* if it satisfies the following conditions:

- $h_{ij} = 0$ for $i < j$ (i.e., H is lower triangular),
- $h_{ii} > 0$ for $i = 1, \ldots, p$,
- $h_{ij} \leqslant 0$ and $|h_{ij}| < h_{ii}$ for $i > j$.

Definition. A square integer matrix is called *unimodular* if its determinant is equal to 1 or -1.

Lemma 2. *Given an r by n integer matrix C which is of full row rank, there is an n by n unimodular matrix U and a unique r by r Hermite matrix H such that*:

- $CU = (H, 0)$,
- $H^{-1}C$ *is integral.*

The matrix $(H, 0)$ is called the *Hermite normal form* (HNF) of C and can be found in polynomial time, see [13,14]. So suppose that $(H, 0)$ is the HNF of A'. If $H^{-1}b'$ is integral, then $x := U\binom{H^{-1}b'}{0}$ is an integral solution to $A'x = b$ and no TT cut exists. If, on the other hand, the ith component of $H^{-1}b'$ is fractional, then the ith row of H^{-1} provides a y, as required in Lemma 1, from which a TT cut can be derived. Indeed, any row of $H^{-1}A'$ for which $H^{-1}b'$ is fractional yields a TT cut and, by taking integer multiples of these rows it is possible to produce a 'group' of TT cuts (see [9,4]).

To close, we mention a few other key references. Further applications of HNF to the generation of cutting planes are given in Hung and Rom [12] and Bockmayr and Eisenbrand [1]. Fast algorithms for HNF computation can be found in Storjohann and Labahn [15] and Havas et al. [11].

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References

[1] A. Bockmayr, F. Eisenbrand, Cutting planes and the elementary closure in fixed dimension, Math. Oper. Res. 26 (2001) 304–312.

[2] A. Caprara, M. Fischetti, $\{0, \frac{1}{2}\}$-Chvátal–Gomory cuts, Math. Programming 74 (1996) 221–235.

[3] A. Caprara, M. Fischetti, A.N. Letchford, On the separation of maximally violated mod-k cuts, Math. Programming 87 (2000) 37–56.

[4] S. Ceria, G. Cornuéjols, M. Dawande, Combining and strengthening Gomory cuts, in: E. Balas, J. Clausen (Eds.), Proceedings of the Fourth International IPCO Conference, Lecture Notes in Computer Science, Vol. 920, Springer, New York, 1995.

[5] V. Chvátal, Edmonds polytopes and a hierarchy of combinatorial problems, Discrete Math. 4 (1973) 305–337.

[6] J. Edmonds, Systems of distinct representatives and linear algebra, J. Res. Nat. Bur. Standards B 71 (1967) 233–240.

[7] J. Edmonds, R. Giles, A min–max relation for submodular functions on graphs, Ann. Discrete Math. 1 (1977) 185–204.

[8] F. Eisenbrand, On the membership problem for the elementary closure of a polyhedron, Combinatorica 19 (1999) 297–300.

[9] R.E. Gomory, An algorithm for integer solutions to linear programs, in: R.L. Graves, P. Wolfe (Eds.), Recent Advances in Mathematical Programming, McGraw-Hill, New York, 1963.

[10] M. Grötschel, L. Lovász, A.J. Schrijver, Geometric Algorithms and Combinatorial Optimization, Springer, Berlin, 1988.

[11] G. Havas, B.S. Majewski, K.R. Matthews, Extended GCD and Hermite normal form algorithms via lattice basis reduction, Experimental Math. 7 (1998) 125–136.

[12] M.S. Hung, W.O. Rom, An application of the Hermite normal form in integer programming, Linear Algebra Appl. 140 (1990) 163–179.

[13] G.L. Nemhauser, L.A. Wolsey, Integer and Combinatorial Optimization, Wiley, New York, 1988.

[14] A. Schrijver, Theory of Linear and Integer Programming, Wiley, New York, 1986.

[15] A. Storjohann, G. Labahn, Asymptotically fast computation of Hermite normal forms of integer matrices, in: Y.N. Lakshman (Ed.), Proceedings of ISSAC 96, ACM Press, New York, 1996, pp. 259–266.