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Binary clutter inequalities for integer programs

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Abstract. We introduce a new class of valid inequalities for general integer linear programs, called *binary clutter* (BC) inequalities. They include the $\{0, \frac{1}{2}\}$-cuts of Caprara and Fischetti as a special case and have some interesting connections to binary matroids, binary clutters and Gomory corner polyhedra.

We show that the separation problem for BC-cuts is strongly $\mathcal{NP}$-hard in general, but polynomially solvable in certain special cases. As a by-product we also obtain new conditions under which $\{0, \frac{1}{2}\}$-cuts can be separated in polynomial time.

These ideas are then illustrated using the *Traveling Salesman Problem* (TSP) as an example. This leads to an interesting link between the TSP and two apparently unrelated problems, the *T-join* and *max-cut* problems.

Key words. Integer programming – cutting planes – matroid theory – binary clutters – traveling salesman problem

1. Introduction

Consider an *Integer Linear Program* (ILP) of the form

$$\max\{cx : Ax \leq b, x \in Z^n_+\}, \tag{1}$$

where $A = (a_{ij})$ is an $m \times n$ integer matrix and b is an m-dimensional integer vector. Associated with such an ILP are the following two polyhedra:

- $P = \{x \in \mathbb{R}^n_+ : Ax \leq b\}$ (the feasible region of the LP relaxation), and
- $P_I = \text{conv}\{x \in Z^n_+ : Ax \leq b\}$ (the convex hull of feasible integer solutions).

In this paper we assume that $P \neq P_I$ and also, without loss of generality, that each row of $(A|b)$ contains at least one odd coefficient.

A cutting plane (or *cut* for short) is a linear inequality which is valid for P_I, but not for P. A large number of techniques are known for deriving cuts. We do not have space to summarize them all, but instead refer the reader to Nemhauser & Wolsey [25], Aardal & Weismantel [2] and Cornuéjols & Li [12]. However, the following class of cuts will be referred to in this paper. A $\{0, \frac{1}{2}\}$-*cut* is (see Caprara & Fischetti [5]) a cut of the form $\lfloor \lambda A \rfloor x \leq \lfloor \lambda b \rfloor$, where $\lambda \in \{0, \frac{1}{2}\}^m$ is such that $\lambda b \notin Z$. The $\{0, \frac{1}{2}\}$-cuts induce *facets*, or at least faces of high dimension, for the polyhedra associated with several important combinatorial optimization problems, see Caprara & Fischetti [5] and Caprara, Fischetti & Letchford [6].

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In this paper we define a new class of cutting planes, called *binary clutter inequalities*, or *BC-cuts* for short. As we will show, they include the $\{0, \frac{1}{2}\}$-cuts as a special case. Moreover they have some interesting connections to the *group relaxations* and *corner polyhedra* of Gomory [15, 16], as well as to *binary matroids* and *binary clutters*.

The structure of the paper is as follows. In Section 2 we define the BC-cuts and show the connection with the work of Gomory. We also show that the BC-cuts are a proper generalization of the $\{0, \frac{1}{2}\}$-cuts. In Section 3 we review some known results on binary matroids and binary clutters. In Section 4 we use these results to analyse the BC-cuts. We show that the BC-cuts have unbounded *Chvátal rank* and that the associated *separation problem* is strongly $\mathcal{NP}$-hard. We then give special cases under which the separation problem is polynomially-solvable. As a by-product we also obtain new conditions under which $\{0, \frac{1}{2}\}$-cuts can be separated in polynomial time, which are much more general than those given by Caprara and Fischetti [5]. The framework of binary clutters is then used in Section 5 to analyse some recent advances in the separation of *comb inequalities* for the Traveling Salesman Problem (TSP). This leads to an interesting connection between the TSP and two apparently unrelated problems, the *T-join* and *max-cut* problems. Finally, concluding remarks are made in Section 6.

2. Binary clutter inequalities

2.1. Definition

The well-known *linear programming relaxation* of the ILP (1) is given by $\max\{cx : x \in P\}$. We will denote by z_{LP} the associated upper bound. In [15, 16], Gomory defined a rather different kind of relaxation, the so-called *group relaxation*. We now present the main ideas underlying this relaxation, following the presentation of Nemhauser & Wolsey [25].

We begin by adding slack variables to the system of inequalities defining P_I, to obtain the related polyhedron

$$P_I^s = \text{conv}\,\{(x, s) \in Z_+^{n+m} : Ax + s = b\}.$$

Now let $K \in Z^{m\times m}$ be an arbitrary non-singular square matrix and let $u \in \mathbb{R}^m$ be an arbitrary vector. The problem:

$$\max\{cx + u(b - Ax - s) : Ax + s + Kw = b, (x, s) \in Z_+^{n+m}, w \in Z^m\} \qquad (2)$$

is a relaxation of the original ILP (1). It is bounded from above if and only if u represents a feasible solution to the dual of the LP relaxation. Moreover, if we set u equal to an *optimal* solution to the dual, the objective function in (2) reduces to $z_{LP} - \bar{c}x - us$, where $\bar{c} \in \mathbb{R}_+^n$ is the vector of reduced costs. Since $u \geq 0$, for any choice of K this relaxation is guaranteed to give an upper bound which dominates the standard LP upper bound z_{LP}.

The Gomory group relaxation as commonly understood is obtained by setting K equal to the submatrix of $(A|I)$ defined by an optimal basis in the solution to the LP relaxation. However, this may not always be a good choice from a practical point of view: the only known algorithms for solving the group relaxation have a running time which is proportional to the determinant of K (see Chen & Zionts [8]), and in practice the determinant of the basis submatrix is often very large. Indeed, it is not difficult to show (for example by reduction from the *multi-dimensional knapsack* problem) that the standard group relaxation is strongly $\mathcal{NP}$-hard in general.

In this paper we focus on a rather different group relaxation. We suggest setting $K := 2I$ in (2), where I is an $m \times m$ identity matrix. With this choice of K and with u defined as above, (2) reduces to:

$$\max\{z_{LP} - \bar{c}x - us : Ax + s \equiv b \bmod 2, (x, s) \in Z_+^{n+m}\}, \tag{3}$$

where the congruence modulo 2 is taken to hold for each row. Note that x and s are guaranteed to be binary in any optimal solution to (3). Hence the associated group is the product of cyclic groups of two elements. This is often called a *binary group*, see Gastou & Johnson [14].

At first sight this seems like a rather poor choice of K: the determinant is 2^m and therefore we cannot use standard algorithms to solve (3). Nevertheless, as we will see, there are important cases where the binary group relaxation can be solved in polynomial time. Moreover, important insights can be gained from a consideration of the associated integer polyhedron:

$$P_2^s = \text{conv}\{(x, s) \in Z_+^{n+m} : Ax + s \equiv b \bmod 2\}.$$

This polyhedron is a special kind of *corner polyhedron*, see [14–16]. It is defined independently of the objective function vector c and the multiplier vector u. From now on we ignore c and u.

Now we come to the important point of this section: Any valid inequality for P_2^s yields a valid inequality for P_I, as expressed in the following proposition:

Proposition 1. *If the inequality $\alpha x + \beta s \geq \gamma$ is valid for P_2^s, then the inequality $(\beta A - \alpha)x \leq \beta b - \gamma$ is valid for P_I.*

Proof. By definition, P_I^s is contained in P_2^s. Therefore the inequality $\alpha x + \beta s \geq \gamma$ is valid for P_I^s. Using the mapping $s = b - Ax$ this is equivalent to the inequality $(\beta A - \alpha)x \leq \beta b - \gamma$. Since P_I is the projection of P_I^s onto the space of the x variables, and since this inequality does not involve slack variables, it is also valid for P_I. □

We call these inequalities *binary clutter inequalities* (BC-cuts for short). The reason will be made clear in the next section.

Example 1. Consider the polyhedron:

$$P = \{x \in \mathbb{R}_+^3 : x_1 + x_2 \leq 1, x_1 + x_3 \leq 1, x_2 + x_3 \leq 1\}.$$

We obtain:

$$\begin{aligned} P_2^s = \text{conv}\{x \in Z_+^3, s \in Z_+^3 : \; & x_1 + x_2 + s_1 \equiv 1 \bmod 2, \\ & x_1 + x_3 + s_2 \equiv 1 \bmod 2, \\ & x_2 + x_3 + s_3 \equiv 1 \bmod 2\}. \end{aligned}$$

Summing together the three congruences we obtain: $s_1 + s_2 + s_3 \equiv 1 \bmod 2$. Since the slacks are non-negative, we immediately see that the inequality $s_1 + s_2 + s_3 \geq 1$ is valid for P_2^s and P_I^s. Therefore the BC-cut $x_1 + x_2 + x_3 \leq 1$ is valid for P_I.

Note that the BC-cut in this case is merely a $\{0, \frac{1}{2}\}$-cut, obtained by setting $\lambda = \{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\}$. This is not really surprising, because both BC-cuts and $\{0, \frac{1}{2}\}$-cuts are derived using 'modulo 2' arguments. What *is* surprising, however, is that BC-cuts are a proper *generalization* of the $\{0, \frac{1}{2}\}$-cuts, as shown in the next subsection.

2.2. *Relation to the* $\{0, \frac{1}{2}\}$*-cuts*

In this subsection we show that the $\{0, \frac{1}{2}\}$-cuts are a special case of the BC-cuts. We begin with the following theorem:

Theorem 1. *Every* $\{0, \frac{1}{2}\}$*-cut is a BC-cut.*

Proof. Let λ be the vector of multipliers used in the derivation of a $\{0, \frac{1}{2}\}$-cut. Since the vector 2λ is integral, we can premultiply the congruence system $Ax + s \equiv b \bmod 2$ by 2λ to obtain: $(2\lambda A)x + 2\lambda s \equiv 2\lambda b \bmod 2$. From the definition of a $\{0, \frac{1}{2}\}$-cut, $2\lambda b$ is odd, so this reduces to $(2\lambda A)x + 2\lambda s \equiv 1 \bmod 2$. Now let $\mathcal{O} \subseteq \{1, \ldots, n\}$ be the set of column indices i such that the ith coefficient of $2\lambda A$ is odd. We can re-write the congruence as $\sum_{i \in \mathcal{O}} x_i + 2\lambda s \equiv 1 \bmod 2$. Since all variables are non-negative, we have that $\sum_{i \in \mathcal{O}} x_i + 2\lambda s \geq 1$ is valid for P_2^s. Hence, the BC-cut $(2\lambda A)x - \sum_{i \in \mathcal{O}} x_i \leq 2\lambda b - 1$ is valid for P_I. Dividing by two yields the $\{0, \frac{1}{2}\}$-cut, $\lfloor \lambda A \rfloor x \leq \lfloor \lambda b \rfloor$. □

Now we give an example of a BC-cut which is not a $\{0, \frac{1}{2}\}$-cut:

Example 2. Consider the polyhedron:

$$P = \{x \in Z_+^4 : 2x_1 + x_2 + x_3 + x_4 \leq 4,\ x_2 + x_3 \leq 1,\ x_2 + x_4 \leq 1,\ x_3 + x_4 \leq 1\}.$$

We obtain:

$$\begin{aligned} P_2^s = \text{conv}\{x \in Z_+^4, s \in Z_+^4 : x_2 + x_3 + x_4 + s_1 &\equiv 0 \bmod 2, \\ x_2 + x_3 + s_2 &\equiv 1 \bmod 2, \\ x_2 + x_4 + s_3 &\equiv 1 \bmod 2, \\ x_3 + x_4 + s_4 &\equiv 1 \bmod 2\}. \end{aligned}$$

It is easy to check that the inequality $x_2 + x_3 + x_4 + s_1 + s_2 + s_3 + s_4 \geq 3$ is valid for P_2^s. The resulting BC-cut, $x_1 + x_2 + x_3 + x_4 \leq 2$, induces a facet of P_I, yet is not a $\{0, \frac{1}{2}\}$-cut.

We have proved the following:

Proposition 2. *BC-cuts are a proper generalization of* $\{0, \frac{1}{2}\}$*-cuts.*

The $\{0, \frac{1}{2}\}$-cuts therefore enjoy the strange property of being a special case of three completely different kinds of inequality simultaneously: they are *Chvátal-Gomory cuts* (see [5, 9, 25]); they are *balanced split cuts* (Caprara & Letchford [7]); and they are BC-cuts.

Example 2 can be used to illustrate another important point: the mapping of valid inequalities for P_2^s onto valid inequalities for P_I does not preserve dominance relations.

Example 2 (continued). Let P_I and P_2^s be defined as before. A complete linear description of P_2^s (obtained using a computer) is given by:

$$x_2 + x_3 + s_2 \geq 1 \qquad x_2 + x_4 + s_3 \geq 1 \qquad x_3 + x_4 + s_4 \geq 1,$$

$$x_2 + s_1 + s_4 \geq 1 \qquad x_3 + s_1 + s_3 \geq 1 \qquad x_4 + s_1 + s_2 \geq 1,$$

$$s_2 + s_3 + s_4 \geq 1,$$

$$x_2 + x_3 + x_4 + s_1 + s_2 + s_3 + s_4 \geq 3,$$

together with non-negativity on all variables.

Now we eliminate the slack variables to obtain valid inequalities for P_I. The first three inequalities are vacuous (i.e., they reduce to $0 \leq 0$). The next three yield the BC-cuts $x_1 + x_2 + x_3 \leq 2$, $x_1 + x_2 + x_4 \leq 2$ and $x_1 + x_3 + x_4 \leq 2$. The seventh yields the BC-cut $x_2 + x_3 + x_4 \leq 1$, and the eighth yields the BC-cut which we saw before, $x_1 + x_2 + x_3 + x_4 \leq 2$. The seventh and eighth BC-cuts, which both induce facets of P_I, dominate all the others.

This example also shows that there is no guarantee in general that a facet of P_2^s will yield a BC-cut which is a facet of P_I.

At this point the reader may be wondering what the precise relationship is between the $\{0, \frac{1}{2}\}$-cuts and the BC-cuts. To understand this fully, it turns out that we have to delve into the literature on *binary matroids* and *binary clutters*. This is the topic of the next section.

3. Binary matroids and binary clutters

3.1. Binary matroids

An introduction to the theory of matroids is given in [25]; for a more detailed treatment see Oxley [26] and Truemper [34]. In this paper we will be concerned only with the so-called *binary* matroids, defined as follows. Let M be a 0-1 matrix with p rows and q columns; assume that M has full row rank (and therefore that $p \leq q$) and denote by $E := \{1, \ldots, q\}$ the set of column indices. Let us say that a set $S \subseteq E$ is *independent* if the corresponding columns are linearly independent under modulo 2 arithmetic. The set E and the collection of independent sets form a binary matroid, and the matrix M is said to *represent* the matroid.

As with any matroid, the maximal independent sets are called *bases*, and the minimal dependent sets are called *circuits*. Moreover, in the case of binary matroids, the union of one or more disjoint circuits is called a *cycle*. There is a one-to-one correspondence between the cycles of the matroid and the solutions of the system $Mx \equiv 0 \bmod 2$.

In general a binary matroid may be represented by many different 0-1 matrices. Indeed, the matroid is unchanged if we permute the rows of M or add rows to other rows under modulo 2 arithmetic. If we also allow column permutations (renumbering the elements of the ground set E), then we can always find a representation matrix M of the form $(B|I)$, where I is the identity matrix of dimension p, and the p associated column indices form a basis.

A simple example of a binary matroid is the so-called *graphic* matroid of a graph $G = (V, E)$. In a graphic matroid, the independent sets are the forests in G, the bases are the spanning forests, and the circuits and cycles have their ordinary graphical meaning. The rows of $(B|I)$ are the incidence vectors of certain cuts in G, which are sometimes called *fundamental cuts* (with respect to the given spanning forest). The matrix B itself is the *edge-path incidence matrix* of the given spanning forest.

Given any binary matroid with representation matrix $(B|I)$, the *dual* matroid (also binary) is represented by the matrix $(I|B^T)$, where I is now the identity matrix of size $q - p$ (so that the ground set E remains the same). The dual of a graphic matroid is called a *co-graphic* matroid. In a co-graphic matroid, the cycles correspond to cuts in G, and the circuits are the minimal cuts.

We will also need the concept of *minors*. A minor of a binary matroid is obtained by performing either of the two following operations:

- *Deleting* an element $e \in E$. This is done by simply deleting the associated column of the matrix.
- *Contracting* an element $e \in E$. This is done by pivoting modulo 2 so that the associated column is part of the identity matrix, then removing the column e and the row which has a '1' in that column.

Contracting an element corresponds to deleting an element in the dual matroid and vice-versa. In the case of graphic matroids, deletion and contraction have their normal meanings – either an edge is deleted from G, or an edge is contracted (i.e., the edge is removed and its two end-vertices are identified).

Many important classes of binary matroids are characterized by forbidding certain 'fundamental' binary matroids to be minors. Two of these 'fundamental' matroids will be of particular interest in what follows: the Fano matroid (denoted by F_7), with representation matrix

$$\left(\begin{array}{cccc|ccc} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

and the graphic matroid of the complete graph K_5 (denoted by $\mathcal{M}(K_5)$), with representation matrix

$$\left(\begin{array}{cccccc|cccc} 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right).$$

The duals of F_7 and $\mathcal{M}(K_5)$ are denoted by F_7^* and $\mathcal{M}^*(K_5)$ respectively.

We will be concerned with the following optimization problem associated with binary matroids, originally formulated by Barahona & Grötschel [4]:

> **The Maximum Weight Cycle Problem (MAX-CYC):** *Given a binary matroid $\mathcal{M}$ with ground set E, and a* weight $w_e \in \mathbb{R}$ *for each $e \in E$, find a cycle of maximum weight. Equivalently, solve* $\max\{wx : Mx \equiv 0,\ x \in \{0, 1\}^q\}$*, where M is the representation matrix of $\mathcal{M}$.*

As noted in [4], this problem is strongly $\mathcal{NP}$-hard in general. Indeed, when $\mathcal{M}$ is *co-graphic*, MAX-CYC is nothing but the well-known *max-cut problem*. However, there are some well-solved special cases. When $\mathcal{M}$ is *graphic*, MAX-CYC reduces to the *Eulerian subgraph* problem, which can be solved efficiently by matching techniques (Edmonds & Johnson [13]). Moreover, if $\mathcal{M}$ is the co-graphic matroid of a graph which is not contractible to K_5, the associated max-cut problem can be solved by the algorithm of Barahona [3].

These well-solved special cases were generalized by Grötschel and Truemper [20], who proved the following theorem:

Theorem 2 (Grötschel and Truemper [20]). *MAX-CYC can be solved in polynomial time whenever $\mathcal{M}$ satisfies at least one of the following properties:*

- *It has no F_7 or $\mathcal{M}^*(K_5)$ minor;*
- *It has no F_7^* or $\mathcal{M}^*(K_5)$ minor.*

This includes the well-solved cases already mentioned, because graphic matroids and co-graphic matroids on graphs not contractible to K_5 contain none of the three minors mentioned in the theorem (see [3, 4, 34]).

The following problem will also be of relevance in what follows:

The Short Odd Circuit Problem (SOC): *Given a binary matroid $\mathcal{M}$ with ground set E, a weight $w_e \in \mathbb{R}_+$ for each $e \in E$, and a labelling of each element of the ground set as* odd *or* even, *find an odd circuit of minimum weight, or prove that none exists. Equivalently, solve* $\min\{wx : Mx \equiv 0,\ px \equiv 1,\ x \in \{0,1\}^q\}$, *where M is the representation matrix of $\mathcal{M}$ and p is the* parity *vector (i.e., $p_e = 1$ if e is odd and* 0 *if e is even.)*

This problem is again strongly $\mathcal{NP}$-hard in general (Truemper [32]). However, there are again some important well-solved special cases. When $\mathcal{M}$ is *graphic*, the SOC calls for a minimum weight *odd circuit* in a graph, and can be solved by shortest path methods (Grötschel & Pulleyblank [17]). When $\mathcal{M}$ is *co-graphic*, the SOC calls for a minimum weight *odd cut* in a graph, and can be solved by the algorithm of Padberg & Rao [28]. More generally, Grötschel and Truemper [20] proved:

Theorem 3 (Grötschel and Truemper [20]). *SOC can be solved in polynomial time whenever $\mathcal{M}$ satisfies at least one of the following properties:*

- *It has no F_7 minor;*
- *It has no F_7^* minor.*

(In fact, Grötschel and Truemper only showed how to test if an odd circuit of weight less than 1 exists, but this is easily converted into an algorithm for SOC via scaling and binary search.)

Another polynomially-solvable special case was described by Truemper [32]:

Theorem 4 (Truemper [32]). *SOC can be solved in polynomial time if there is only one odd element (say, f) and at least one of the following conditions is satisfied:*

- *$\mathcal{M}$ has no F_7 minor using f,*
- *$\mathcal{M}$ has no F_7^* minor using f.*

That is, $\mathcal{M}$ is permitted to contain an F_7 (or F_7^*) minor, but to obtain any such minor it must be necessary to either delete or contract f.

This theorem generalizes the well-known fact that the shortest (s, t)-path and minimum (s, t)-cut problems can be solved efficiently.

3.2. Binary clutters

Let $M \in \{0, 1\}^{p \times q}$ represent a binary matroid with ground set $E := \{1, \dots, q\}$ and let $d \in \{0, 1\}^p$ be a given vector. Consider the polyhedron

$$P(M, d) = \text{conv}\left\{x \in Z_+^{|E|} : Mx \equiv d \bmod 2\right\}.$$

This is a corner polyhedron of the same form as the polyhedron P_2^s defined in Subsection 2.1. The extreme points of $P(M, d)$ are 0-1 vectors. Associated with any extreme point x^* is the set $\{i \in E : x_i^* = 1\}$, sometimes called the *support* of x^*. The collection of all such sets is called a *binary clutter*. When M represents a graphic matroid, the members of the clutter correspond to T*-joins*. On the other hand, when M represents a co-graphic matroid, the members of the clutter correspond to what are sometimes called *balancing sets*.

Let us call the problem of optimizing a (non-negative) linear function over $P(M, d)$ the *binary clutter problem* (BCP). The following proposition is implicit in [4, 20], but we give an explicit statement and proof here for the sake of clarity.

Proposition 3. *MAX-CYC and BCP are equivalent.*

Proof. Let an instance of MAX-CYC be given by M and w. Let $F = \{e \in E : w_e > 0\}$. Then the optimal objective value for MAX-CYC is equal to

$$\sum_{e \in F} w_e - \min\left\{\sum_{e \in E} |w_e| x_e : Mx \equiv d,\ x \in Z_+^q\right\},$$

where the right hand side vector d is obtained by summing together the columns in F (modulo 2).

Now let an instance of BCP be given by M, w and d. We can test if BCP is feasible, and find a feasible solution if one exists, by Gaussian elimination. Suppose that a feasible solution, say $\tilde{x}$, exists. Let F be the support of $\tilde{x}$. Then the optimal objective value for BCP is equal to:

$$\sum_{e \in F} w_e - \max\left\{\sum_{e \in F} w_e x_e - \sum_{e \notin F} w_e x_e : Mx \equiv 0,\ x \in \{0, 1\}^n\right\}.$$

□

Thus, BCP is strongly $\mathcal{NP}$-hard in general, because finding a minimum weight balancing set is as hard as the max-cut problem. Yet, Proposition 3 and Theorem 2 yield polynomially-solvable cases of BCP. Moreover, by elementary row operations modulo 2 we can always re-write the system of congruences $Mx \equiv d$ in the (odd circuit) form $M'x \equiv 0$, $px \equiv 1$, where M' now represents a *different* binary matroid. This yields further polynomially solvable cases via Theorems 3 and 4.

Before proceeding, we need the concept of *blocking* clutters. Suppose that M has been expressed in the standard form $(B|I)$. Then the extreme points of the polyhedron

$$\hat{P}(M, d) = \text{conv}\left\{x \in Z_+^{|E|} : (I|B^T)x \equiv 0 \bmod 2,\ (0|d^T)x \equiv 1 \bmod 2\right\},$$

viewed as sets, also form a binary clutter, called the *blocking clutter*. It can be shown that, if $S \subset E$ is any member of the original clutter and $T \subset E$ is any member of the blocking clutter, then $S \cap T \neq \emptyset$. (In fact, $|S \cap T|$ is always odd.) This relationship is symmetrical, i.e., the blocking clutter of the blocking clutter is the original clutter. Therefore one can speak of a *blocking pair* of clutters. For example, for any graph $G = (V, E)$ and any set $T \subseteq V$ with $|T|$ even, the clutter of T-joins and the clutter of odd cuts form a blocking pair, and so do the clutter of balancing sets and the clutter of odd circuits.

The blocking clutter can be used to formulate any BCP as a *Set Covering Problem* (SCP). If $S \subset E$ is a member of the blocking clutter, i.e., if S is the support of an extreme point of $\hat{P}(M, d)$, then the *blocking* inequality $x(S) \geq 1$ is valid for $P(M, d)$. Moreover, every non-dominated valid inequality for $P(M, d)$ of the form $x(S) \geq 1$ can be derived in this way. Hence BCP can be formulated as the SCP:

$$\min\{cx : x(S) \geq 1 \ \forall S \text{ in the blocking clutter}, x \in \{0, 1\}^q\}.$$

Solving the LP relaxation of this SCP yields a lower bound for BCP. Note however that the number of sets in the blocking clutter can grow exponentially with q, so the SCP has in general an exponential number of constraints. Indeed, the LP relaxation of the SCP is also hard to solve in general. However, Theorem 3 and the ellipsoid method [18] imply that the LP relaxation can be solved in polynomial time if the matroid represented by M is F_7-free or F_7^*-free.

A strange paradox, originally pointed out by Gastou & Johnson [14], is that the BCP may be polynomially solvable even when it is $\mathcal{NP}$-hard to solve the above-mentioned LP relaxation. In particular, the odd circuit problem on a graph is polynomially solvable, yet optimizing over $\hat{P}(M, d)$ amounts to a max-cut problem in this case. This is a (rare) example of a natural integer programming problem which is easier than its LP relaxation.

A natural question is, when is the polyhedron $P(M, d)$ completely described by the non-negativity and blocking inequalities? In such cases the binary clutter is said to have the *weak max-flow min-cut* property [29], or, more briefly, to be *ideal* [11, 21]. A famous unresolved conjecture of Seymour [29, 31] gives necessary and sufficient conditions for a binary clutter to be ideal, in terms of so-called *signed minors*. We do not have space to go into this here, but we will need the basic idea of signed minors for later. Essentially, signed minors are the binary clutter analogue of matroid minors, defined by suitably-modified versions of deletion and contraction. From a polyhedral viewpoint, deleting e amounts to taking the face of $P(M, d)$ defined by $x_e = 0$, whereas contracting e amounts to projecting $P(M, d)$ onto the hyperplane defined by $x_e = 0$.

Results of Edmonds & Johnson [13] imply that Seymour's conjecture is true when the matroid represented by M is *graphic*. Guenin [21] recently proved it to be also true when the matroid is *co-graphic*. Further progress on this conjecture can be found in Cornuéjols & Guenin [11].

4. Implications

4.1. BC-cuts versus $\{0, \frac{1}{2}\}$*-cuts*

The results discussed in the previous section should now make clear the precise relationship between the BC-cuts and the $\{0, \frac{1}{2}\}$-cuts. Recall that

$$P_2^s = \text{conv}\{(x, s) \in Z_+^{n+m} : Ax + s \equiv b \bmod 2\}.$$

Theorem 5. *The blocking inequalities for P_2^s, when expressed in terms of the original variables, are the (non-dominated)* $\{0, \frac{1}{2}\}$*-cuts.*

Proof. The blocking clutter is given by the extreme points of the following polyhedron: $\hat{P}_2^s := \text{conv}\{(x, s) \in Z_+^{n+m} : x + A^T s \equiv 0 \bmod 2, b^T s \equiv 1 \bmod 2\}$. Let $(\hat{x}, \hat{s})$ be an extreme point of $\hat{P}_2^s$. The resulting blocking inequality is $\hat{x}x + \hat{s}s \geq 1$, or equivalently,

$$(\hat{s}A - \hat{x})x \leq \hat{s}b - 1. \tag{4}$$

From the definition of $\hat{P}_2^s$ we know that the components of $\hat{s}A - \hat{x}$ are even integers, and $\hat{s}b$ is odd. Therefore (4) is equivalent to:

$$\lfloor \hat{s}A/2 \rfloor x \leq \lfloor \hat{s}b/2 \rfloor.$$

This is clearly a $\{0, \frac{1}{2}\}$-cut with multiplier vector $\lambda = \hat{s}/2$. Conversely, given any $\{0, \frac{1}{2}\}$-cut with multiplier vector λ, there is a $0 - 1$ point $(\hat{x}, \hat{s})$ in $\hat{P}_2^s$, with $\hat{s} := 2\lambda$, and $\hat{x}_i$ set accordingly. (If $(\hat{x}, \hat{s})$ is not an extreme point of $\hat{P}_2^s$, then the $\{0, \frac{1}{2}\}$-cut is dominated by the inequalities defining P and other $\{0, \frac{1}{2}\}$-cuts.) □

This immediately yields the following corollary:

Corollary 1. *If the binary clutter associated with P_2^s is ideal, then the BC-cuts and the* $\{0, \frac{1}{2}\}$*-cuts are equivalent.*

This explains the choice of Example 2 in Subsection 2.2: The associated binary clutter is not ideal (it contains the so-called *odd* F_7^* as a signed minor), and this enabled us to find a BC-cut which is not a $\{0, \frac{1}{2}\}$-cut.

It is important to note, however, that the $\{0, \frac{1}{2}\}$-cuts and BC-cuts may be equivalent even when the binary clutter relaxation is *not* ideal. As mentioned in Subsection 2.2, the mapping of valid inequalities for P_2^s onto BC-cuts does not preserve dominance relations. Here is a trivial example:

Example 3. Consider the polyhedron:

$$P = \{x \in Z_+^3 : x_1 + x_2 + x_3 \leq 4, \ x_1 + x_2 \leq 3, \ x_1 + x_3 \leq 3, \ x_2 + x_3 \leq 3\}.$$

Note that $P = P_I$ in this case. We obtain:

$$\begin{aligned} P_2^s := \text{conv}\{x \in Z_+^3, s \in Z_+^4 : x_1 + x_2 + x_3 + s_1 &\equiv 0 \bmod 2, \\ x_1 + x_2 + s_2 &\equiv 1 \bmod 2, \\ x_1 + x_3 + s_3 &\equiv 1 \bmod 2, \\ x_2 + x_3 + s_4 &\equiv 1 \bmod 2\}. \end{aligned}$$

The associated binary clutter is the odd F_7^*, and the inequality $x_1 + x_2 + x_3 + s_1 + s_2 + s_3 + s_4 \geq 3$ induces a facet of P_2^s. Yet the resulting BC-cut, $x_1 + x_2 + x_3 \leq 5$, is not even supporting for P_I.

At this point we would like to mention some important differences between BC-cuts and $\{0, \frac{1}{2}\}$-cuts. First we consider the issue of *Chvátal rank* (see Chvátal [9]). Only one application of integer rounding is needed to derive the $\{0, \frac{1}{2}\}$-cuts from the inequalities defining P, and therefore their rank is equal to 1. The BC-cuts, on the other hand, may have rank greater than 1 (i.e., repeated application of the integer rounding argument may be necessary to derive them from the inequalities defining P). Here is an example.

Example 4. Consider the polyhedron:

$$P = \{x \in Z_+^4 : x_i + x_j \le 1 \; (1 \le i < j \le 4)\}.$$

We obtain:

$$\begin{aligned} P_2^s := \mathrm{conv}\{x \in Z_+^4, s \in Z_+^6 : \; & x_1 + x_2 + s_1 \equiv 1 \bmod 2, \\ & x_1 + x_3 + s_2 \equiv 1 \bmod 2, \\ & x_1 + x_4 + s_3 \equiv 1 \bmod 2, \\ & x_2 + x_3 + s_4 \equiv 1 \bmod 2, \\ & x_2 + x_4 + s_5 \equiv 1 \bmod 2, \\ & x_3 + x_4 + s_6 \equiv 1 \bmod 2\,\}. \end{aligned}$$

It can be checked by brute-force enumeration that the inequality $\sum_{i=1}^4 x_i + \sum_{i=1}^6 s_i \ge 4$ is valid for P_2^s. The resulting BC-cut, $x_1 + x_2 + x_3 + x_4 \le 1$, is easily seen to have Chvátal rank 2.

In fact, the BC-cuts are in general *much* stronger than the $\{0, \frac{1}{2}\}$-cuts, as seen in the following proposition.

Proposition 4. *If the polyhedron P_I is a* set packing polytope *of the form* $\{x \in \{0, 1\}^n : Ax \le 1\}$*, where A is a 0-1 matrix, then the BC-cuts give a complete description of P_I.*

Proof. In this case, we have

$$P_I^s = \mathrm{conv}\,\{(x, s) \in Z_+^{n+m} : Ax + s = 1\} \text{ and}$$

$$P_2^s = \mathrm{conv}\{(x, s) \in Z_+^{n+m} : Ax + s \equiv 1 \bmod 2\}.$$

It is clear that P_I^s is a *face* of P_2^s. Hence, a complete linear description of P_2^s yields a complete description of P_I^s. □

This proposition, together with the results of Section 3, implies several new polynomially-solvable cases of the Set Packing Problem. However, we do not explore this further here for the sake of space. The main conclusions we wish to draw from it are the following:

Corollary 2. *The BC-cuts have unbounded Chvátal rank.*

Proof. Set packing polytopes were shown to have unbounded Chvátal rank in Chvátal, Cook & Hartmann [10]. □

Corollary 3. *The recognition problem for BC-cuts (i.e., given an inequality, decide whether it is a BC-cut), is co-$\mathcal{NP}$-complete.*

Proof. The problem is in *co-$\mathcal{NP}$* because, to show that an inequality is *not* a BC-cut, it suffices to present a solution to the binary clutter relaxation which violates it. The problem of deciding whether an inequality is valid for the set packing polytope is *co-$\mathcal{NP}$*-complete, see [25]. The result follows from Proposition 4. □

The recognition problem for $\{0, \frac{1}{2}\}$-cuts, on the other hand, is clearly in $\mathcal{NP}$. (To show that an inequality is a $\{0, \frac{1}{2}\}$-cut, it suffices to give a suitable multiplier vector λ.) It is not obviously in *co-$\mathcal{NP}$* and may well be $\mathcal{NP}$-complete.

4.2. Solving the binary group relaxation

The results mentioned in Section 3 imply several conditions under which the binary group relaxation (3) is solvable in polynomial time. In order to specify these conditions, we will need the following definition, taken from Caprara & Fischetti [5]:

Definition (Caprara & Fischetti [20]). The *mod-2 support* of an integer matrix A is the matrix obtained by replacing each entry in A by its parity (0 if even, 1 if odd).

We will denote the mod-2 support of A by $\bar{A}$. The mod-2 support of b will be defined similarly and denoted by $\bar{b}$.

Theorem 6. *Let $P := \{x \in \mathbb{R}^n_+ : Ax \leq b\}$. The binary group relaxation can be solved in polynomial time if the matroid represented by $(\bar{A}|I)$ satisfies at least one of the following conditions:*

- *It has no F_7 or $\mathcal{M}^*(K_5)$ minor,*
- *It has no F_7^* or $\mathcal{M}^*(K_5)$ minor.*

Proof. Follows from Theorem 2 and Proposition 3. □

Theorem 7. *Let $P := \{x \in \mathbb{R}^n_+ : Ax \leq b\}$. Let $\mathcal{M}$ be the matroid represented by the augmented matrix $(\bar{A}|I|\bar{b})$, and let f be the index of the right-most column. The binary group relaxation can be solved in polynomial time if $\mathcal{M}$ satisfies at least one of the following conditions:*

- *It has no F_7 minor using f,*
- *It has no F_7^* minor using f.*

Proof. Theorem 4 shows that the Short Odd Circuit problem (SOC) for $\mathcal{M}$ and f can be solved in polynomial time under the same conditions. It is readily checked that the SOC is in this case equivalent to optimizing over P_2^s. □

We can also apply Theorem 3 in this context. To do this, it is necessary to re-write the congruence system $Ax + s \equiv b \bmod 2$ in odd circuit form. That is, using elementary row operations modulo 2, we write the system in the form $Cx + Ds \equiv 0 \bmod 2$, $tx + ws \equiv 1 \bmod 2$, where C and D are 0-1 matrices and t and w are 0-1 vectors.

Theorem 8. *Let C and D be defined as above. The binary group relaxation can be solved in solved in polynomial time if the matroid represented by* $(C|D)$ *satisfies at least one of the following conditions:*

- *It has no* F_7 *minor,*
- *It has no* F_7^* *minor.*

Proof. Under the conditions stated, minimising a linear function over P_2^s amounts to finding a Short Odd Circuit in the matroid mentioned. By Theorem 3, this can be performed in polynomial time. □

4.3. Separation

The *separation problem* associated with a given family of cuts is (see Grötschel, Lovász & Schrijver [18]) the problem of detecting when a cut in that family is violated by a given vector $x^* \in P$. Proposition 4 implies that the separation of BC-cuts is strongly $\mathcal{NP}$-hard. On the other hand, the well-known polynomial equivalence of separation and optimization [18] implies the following:

Proposition 5. *The separation problem for BC-cuts can be solved in polynomial time under any of the conditions specified in Theorems 6–8.*

Proof. Under the conditions stated, one can optimize a linear function over P_2^s in polynomial time. Hence, via the ellipsoid method, one can separate from P_2^s in polynomial time. Given any vector $x^* \in P$, we can construct the associated slack vector $s^* = b - Ax^*$ in polynomial time. Then there is an inequality separating (x^*, s^*) from P_2^s if and only if there is a BC-cut violated by x^*. □

This result relies on the equivalence of separation and optimization, and therefore on the *ellipsoid method* [18]. Although theoretically polynomial, the ellipsoid method is slow in practice and therefore it would be desirable to find combinatorial algorithms for BC-separation. Indeed, as we will see below (Theorem 12), a combinatorial separation algorithm is already known when the conditions of Theorem 7 hold.

Now we turn our attention to the separation of the $\{0, \frac{1}{2}\}$-cuts themselves. Prior to the writing of this paper, the most general results known were the following:

Theorem 9 (Caprara & Fischetti [5])**.** *Let* $P := \{x \in \mathbb{R}^n_+ : Ax \leq b\}$*. The separation problem for* $\{0, \frac{1}{2}\}$*-cuts is strongly* $\mathcal{NP}$*-hard, but it can be solved in polynomial time if:*

(i) $\bar{A}$ *is the edge-path incidence matrix of a tree,*
(ii) the transpose of $\bar{A}$ *is the edge-path incidence matrix of a tree.*

Theorem 10 (Caprara, Fischetti & Letchford [6])**.** *The maximum amount by which a* $\{0, \frac{1}{2}\}$*-cut can be violated under the assumption that* $x^* \in P$ *is* $\frac{1}{2}$*. Moreover, one can detect in polynomial time whether such a maximally violated* $\{0, \frac{1}{2}\}$*-cut exists.*

From the results mentioned in Section 3, it should be clear that conditions (i) and (ii) in Theorem 9 are equivalent to:

(i') the matroid represented by $(\bar{A}|I)$ is graphic,
(ii') the matroid represented by $(\bar{A}|I)$ is co-graphic.

Using the results of Grötschel & Truemper [20], we can generalize Theorem 9 substantially:

Theorem 11. *Let* $P := \{x \in \mathbb{R}^n_+ : Ax \leq b\}$ *and let* $\mathcal{M}$ *be the matroid represented by* $(\bar{A}|I)$*. The separation problem for* $\{0, \frac{1}{2}\}$*-cuts can be solved in polynomial time if* $\mathcal{M}$ *is* F_7*-free or* F_7^**-free.*

Proof. Let (x^*, s^*) be the point to be separated. From Theorem 5, the $\{0, \frac{1}{2}\}$-cuts are equivalent to the blocking inequalities for P_2^s. Hence, separation of $\{0, \frac{1}{2}\}$-cuts is equivalent to:

$$\min \left\{ x^* x + s^* s : Ix + A^T s \equiv 0 \bmod 2,\ b^T x \equiv 1 \bmod 2,\ (x, s) \in Z_+^{n+m} \right\}.$$

This is a Short Odd Circuit problem in the dual of $\mathcal{M}$. From Theorem 3, this can be solved efficiently if the dual matroid is F_7-free or F_7^*-free. Matroid duality then shows the same for $\mathcal{M}$. □

Comparing Theorems 11 and 6, we see that the condition for BC-cut separation to be easy is more restrictive than the condition for $\{0, \frac{1}{2}\}$-cut separation. Moreover, the separation algorithm of Theorem 11 is combinatorial, whereas the separation algorithm of Theorem 6 and Proposition 5 relies on the ellipsoid method. Thus a considerable price must be paid to separate the stronger cuts.

Next we consider how to apply Theorem 7 to the separation of $\{0, \frac{1}{2}\}$-cuts. Perhaps surprisingly, we have the following theorem.

Theorem 12. *Let* P*,* $\mathcal{M}$ *and* f *be as defined in Theorem 7. Then under the conditions stated in Theorem 7, the BC-cuts and* $\{0, \frac{1}{2}\}$*-cuts are equivalent and there is a combinatorial (non-ellipsoidal) algorithm for the separation problem.*

Proof. Under the conditions stated, the matroid has the *max-flow min-cut* property [29], and the binary clutter is therefore ideal. Thus, by Theorem 5 and Corollary 1, the BC-cuts and $\{0, \frac{1}{2}\}$-cuts are equivalent. Truemper [32] showed that in this case the separation of the blocking inequalities amounts to finding a short odd *co-circuit* of $\mathcal{M}$ which uses f. Truemper [32] also gave a combinatorial polynomial time algorithm for this problem. □

Finally, we consider the analogue of Theorem 8. In this case we again need to assume that the congruence system $Ax + s \equiv b \bmod 2$ has been written in the odd circuit form $Cx + Ds \equiv 0 \bmod 2$, $tx + ws \equiv 1 \bmod 2$.

Theorem 13. *Let* C *and* D *be defined as above. The separation problem for* $\{0, \frac{1}{2}\}$*-cuts can be solved in polynomial time if the matroid represented by* $(C|D)$ *satisfies at least one of the following conditions:*

- *It has no* F_7 *or* $\mathcal{M}(K_5)$ *minor,*
- *It has no* F_7^* *or* $\mathcal{M}(K_5)$ *minor.*

Proof. Under the conditions stated, the dual matroid is $\mathcal{M}^*(K_5)$-free and either F_7- or F_7^*-free. Hence one can find a minimum weight member of the blocking clutter in polynomial time, which solves the $\{0, \frac{1}{2}\}$-cut separation problem. □

Here the paradox mentioned in Subsection 3.2 becomes apparent: even though the BC-cuts are a generalization of the $\{0, \frac{1}{2}\}$-cuts, the conditions in Theorem 8 are *less* restrictive than those in Theorem 13! Again, however, the separation algorithm of Theorem 13 is combinatorial, whereas the separation algorithm of Theorem 8 and Proposition 5 relies on the ellipsoid method.

We finish this section by noting that, even when the system $Ax \leq b$ does not satisfy any of the conditions stated in these theorems, it may be possible to separate BC-cuts (or $\{0, \frac{1}{2}\}$-cuts) efficiently. Indeed, let $x^* \in P$ be the point to be separated, and let $s^* \in \mathbb{R}^m_+$ be the associated vector of slack variables. The following results can be proved by elementary geometrical arguments:

Proposition 6. *If $x_i^* = 0$, then $(x^*, s^*) \in P_2^s$ if and only if (x^*, s^*) lies in the face of P_2^s defined by the equation $x_i = 0$.*

Proposition 7. *If $x_i^* \geq 1$, then $(x^*, s^*) \in P_2^s$ if and only if the projection of (x^*, s^*) onto the hyperplane defined by the equation $x_i = 0$ lies in the projection of P_2^s onto the same hyperplane.*

This implies that x_i can be *deleted* from the binary clutter when $x_i^* = 0$, and *contracted* when $x_i^* \geq 1$, without affecting the separation problem. Exactly the same argument applies to slack variables. After deleting and contracting these elements, the binary clutter may satisfy the conditions of one of the above theorems, and therefore separation may become easy.

There are two minor complications, however, which must be addressed if one wishes to use Propositions 6 and 7 in practice. First, after deleting variables which are currently at zero, we may obtain the *empty clutter*. That is, the reduced system of congruences may become inconsistent. Fortunately, however, it is easy to find a violated inequality when this happens: the sum of the deleted variables must be at least 1. Note that this inequality is of blocking type, and therefore the corresponding BC-cut is a $\{0, \frac{1}{2}\}$-cut. (In fact it turns out to be a *maximally violated* $\{0, \frac{1}{2}\}$-cut.)

The second complication is concerned with *lifting*. Suppose that we have found a violated inequality, say $\alpha x + \beta s \geq 1$, for the reduced binary clutter. In order to convert this into a valid inequality for the original binary clutter, one needs to compute suitable left hand side coefficients for the *deleted* variables (though not for the *contracted* ones). One can of course set the left hand side coefficients to 1, but the resulting cut may be weak. To get a stronger cut, some form of sequential or simultaneous *lifting* will be necessary (see [25] and also [14]). Lifting can be performed efficiently when the inequality is of blocking type, but it seems likely that it is strongly $\mathcal{NP}$-hard in general.

We will use Propositions 6 and 7 to good effect in the next section.

5. Application to the TSP

The famous *Travelling Salesman Problem* (TSP) is the $\mathcal{NP}$-hard problem of finding a minimum cost Hamiltonian cycle (or *tour*) in a graph. Here we are concerned with the *symmetric* version, the STSP, in which the graph $G = (V, E)$ is undirected. The associated integer polytope is:

STSP(G) := conv{$x \in Z_+^{|E|}$:

$$x(\delta(\{i\})) = 2 \qquad \forall i \in V, \tag{5}$$

$$x(E(S)) \leq |S| - 1 \qquad \forall S \subset V : 2 \leq |S| \leq |V| - 2\}, \tag{6}$$

where $\delta(S)$ (respectively, $E(S)$) denotes the set of edges in G with exactly one end-vertex (respectively, both end vertices) in S. Equations (5) are called *degree equations* and the inequalities (6) are called *subtour elimination constraints* (SECs).

An excellent survey on the STSP and the associated polyhedra is given by Naddef [24]. Here we are concerned with the (facet-inducing) *comb* inequalities of Grötschel & Padberg [19]. Let $t \geq 3$ be an odd integer. Let $H \subset V$ and $T_j \subset V$ for $j = 1, \dots, t$ be such that $T_j \cap H \neq \emptyset$ and $T_j \setminus H \neq \emptyset$ for $j = 1, \dots, t$, and also let the T_j be vertex-disjoint. The comb inequality is:

$$x(E(H)) + \sum_{j=1}^{t} x(E(T_j)) \leq |H| + \sum_{j=1}^{t} |T_j| - \lceil 3t/2 \rceil. \tag{7}$$

The set H is called the *handle* of the comb and the T_j are called *teeth*.

Padberg & Grötschel [27] showed that the comb inequalities with $|T_j \cap H| = |T_j \setminus H| = 1$, the so-called *two-matching* inequalities, could be separated in polynomial time using the ideas of Padberg & Rao [28]. They also *conjectured* that there exists a polynomial-time separation algorithm for comb inequalities in general. This conjecture is still unsettled. However, considerable progress has been made. For a full description of exact and heuristic separation algorithms for comb inequalities and variants, see Naddef [24] and Letchford & Lodi [23]. Here we mention only a few of these results, namely those which are directly related to binary matroids and clutters.

To the knowledge of the author, Applegate et al. [1] were the first to realise that comb inequalities were related to congruences modulo 2. In our terminology, they realised that the comb inequalities could be derived as $\{0, \frac{1}{2}\}$-cuts from the degree equations and SECs. Based on this observation, they devised a heuristic for separating what we would now call maximally violated comb inequalities, based on solving a certain system of congruences. Unfortunately, not all solutions to their congruence system represented comb inequalities. They therefore implemented a heuristic for finding solutions to the system that actually did represent comb inequalities. We will comment further on this below.

Next, Caprara et al. [6] formally defined the maximally violated $\{0, \frac{1}{2}\}$-cuts and gave the separation algorithm which we mentioned in Subsection 4.3. They also showed that there are (many) facet-inducing inequalities for STSP(G) which, though not comb inequalities, are $\{0, \frac{1}{2}\}$-cuts.

Shortly after this, Letchford [22] defined the *domino-parity* (DP) inequalities. These inequalities are intermediate in generality between the comb inequalities and the $\{0, \frac{1}{2}\}$-cuts themselves. They are defined as follows:

Definition. *A domino-parity (DP) inequality is an inequality which can be derived as a $\{0, \frac{1}{2}\}$-cut from the degree equations (5) and inequalities of the form:*

$$2x(E(S_1) \cup E(S_2)) + x(E(S_1 : S_2)) \leq 2|S_1 \cup S_2| - 3, \tag{8}$$

where S_1 and S_2 are disjoint non-empty subsets of V such that $V \setminus (S_1 \cup S_2) \neq \emptyset$, and $E(S_1 : S_2)$ denotes the set of edges with one end-node in S_1 and the other in S_2.

Note that the inequality (8) is the sum of the SECs on S_1, S_2 and $S_1 \cup S_2$. Hence, these inequalities are weaker than SECs. We will call them *tooth* inequalities. Note that a variable x_e receives an odd coefficient in a tooth inequality if and only if $e \in E(S_1 : S_2)$. The edge set $E(S_1 : S_2)$ is called a *semicut*, see [22] and the references contained therein.

The key result in [22], then, is that the DP inequalities can be separated in polynomial time whenever the *support graph* is planar. The support graph is the subgraph of G induced by the edges with $x_e^* > 0$.

Very recently, another important step forward was made. In Letchford & Lodi [23], a polynomial-time separation algorithm was given for the so-called *simple* comb inequalities. A comb inequality is said to be *simple* if, for each tooth T_j, either $|T_j \cap H| = 1$ or $|T_j \setminus H| = 1$ (or both). This generalizes the earlier result on two-matching inequalities.

We will show that the results in [1], [22] and [23] are essentially corollaries of the results given in the previous section. To this end, we will need a suitable binary clutter relaxation of the STSP. The most natural approach is to include the degree equations and SECs in our initial LP relaxation, but we have been unable to derive any useful results in this way. It has proven more fruitful to use the degree equations and the *tooth inequalities*. Note that, even with this choice, the set of BC-cuts will include all DP inequalities, and therefore all comb inequalities.

So let $Cx = d$ denote the degree equations, let $Ax \leq b$ denote the tooth inequalities, and let s denote the slack variables associated with the tooth inequalities. Since every degree equation has an even right hand side, and every tooth inequality has an odd right hand side, our binary clutter polyhedron is:

$$P_2^s := \text{conv}\{x \in Z_+^{|E|}, s \in Z_+^m : Cx \equiv 0 \bmod 2,\ Ax + s \equiv 1 \bmod 2\},$$

where m now represents the number of tooth inequalities.

Obviously, there is an exponential number of possible tooth inequalities. However, by Proposition 7, we only need to consider tooth inequalities whose slacks are less than 1. This number is polynomially bounded under the assumption that $x^* \in P$, because there are only $\mathcal{O}(|V|^2)$ candidates for S_1 and S_2 in the tooth inequality (8), see Lemma 2 of [23]. Therefore our matrix A has a polynomial number of rows.

It is instructive to write the congruence system in standard form, as follows. Let $T \subset E$ be a spanning tree, let $W = E \setminus T$, and partition the x vector into x_T and x_W accordingly. Then the system can be re-written as

$$A_1 x_W + x_T \equiv 0,\ A_2 x_W + s \equiv 1, \tag{9}$$

where A_1 is the edge-path incidence matrix of the given tree, and $A_2 \in \{0, 1\}^{m \times |W|}$ is constructed as follows. Given any $e \in W$, the graph $G = (V, T \cup \{e\})$ contains a unique (fundamental) cycle, which we will denote by C_e. The matrix A_2 has a '1' in row $j \in \{1, \dots, m\}$ and column $e \in W$ if and only if the intersection of the cycle C_e with the jth semicut has odd cardinality.

The first set of congruences in (9) essentially states that every Hamiltonian tour touches every cut an even number of times. The second set places parity restrictions on the slack vector. The blocking clutter, on the other hand, is defined by the system

$$A_1^T x_T + x_W + A_2^T s \equiv 0,\ \sum_{j=1}^{m} s_j \equiv 1. \tag{10}$$

Thus, to separate DP-inequalities we must find a minimum weight solution to (10). This makes it clear that the number of teeth must be odd.

The main question now is whether these binary clutters meet any of the conditions for polynomial solvability given in Subsection 4.3. Unfortunately this does not seem to be the case in general. Indeed, even for $n = 6$ the associated matroids may contain both F_7 and F_7^* as a minor. This leads us to look for some well-solved special cases.

The first case of interest is when we restrict attention to *maximally violated* DP inequalities (i.e., DP inequalities which are violated by $\frac{1}{2}$.) To detect such inequalities, it suffices to delete all columns from the system (10) corresponding to variables which are positive at (x^*, s^*). Interestingly, it turns out that the reduced system is equivalent to the one used by Applegate et al. [1]. (The details are omitted for the sake of brevity.) Thus, every solution to the Applegate et al. congruence system corresponds to a maximally violated DP inequality. Hence, the comb separation heuristic in [1] can also be used as an exact algorithm for detecting maximally violated DP inequalities.

The second case of interest is when all teeth are *simple*, i.e., when we only use tooth inequalities (8) in which either $|S_1| = 1$ or $|S_2| = 1$ (or both). As shown in [23], *provided* we restrict attention to simple tooth inequalities with slack less than one-half, the constraint matrix $\binom{C}{A}$ is an EPT matrix. This result is interpreted in the framework of binary matroids and clutters in the following proposition.

Proposition 8. *Let $Cx = d$ represent the degree equations and let $Ax \leq b$ represent the simple tooth inequalities with slack less than $\frac{1}{2}$. Then:*

- *The matroid represented by $\left(\begin{smallmatrix} C \\ A \end{smallmatrix} \middle| I\right)$ is graphic.*
- *The binary clutter relaxation is a T-join problem in a suitable graph.*
- *The clutter is ideal.*
- *All BC-cuts are DP-inequalities.*
- *The separation problem for DP-inequalities can be solved by computing a minimum weight odd cut in the same graph.*

Thus the framework of binary matroids also gives an intuitive explanation of the separation algorithm given in [23].

The third case of interest is when the *support graph* is planar. As shown in [22], in this case the DP inequalities can be separated by computing a minimum weight odd *circuit* in a certain labelled weighted graph. (This graph is not necessarily planar.) For this case we have the following proposition, the proof of which we again omit for the sake of brevity.

Proposition 9. *Let $Cx = d$ represent the degree equations, and let $Ax \leq b$ represent the tooth inequalities with slack less than* 1. *If the support graph is planar, and all variables with value zero have been deleted in accordance with Proposition 6, then:*

- *The matroid represented by $\left(\begin{smallmatrix} C & 0 \\ A & I \end{smallmatrix}\right)$ is co-graphic.*
- *The binary clutter relaxation is a balancing set problem (equivalently, a max-cut problem) in a suitable labelled weighted graph.*
- *The separation problem for DP-inequalities can be solved by computing a minimum weight odd circuit in the same graph.*

In this case, it is not clear whether the clutter is ideal or not, so there might be BC-cuts which are not DP-inequalities. Hence, it is conceivable that some of the known valid

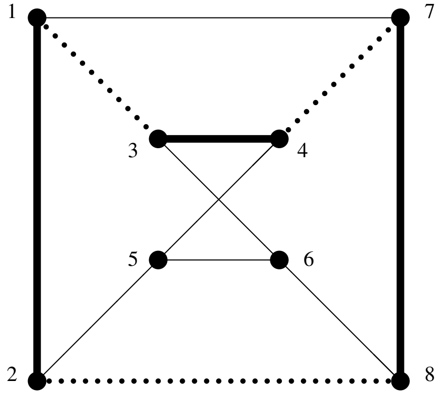

Fig. 1. A fractional point on 8 vertices.

inequalities and separation algorithms for the max-cut problem could yield new valid inequalities and separation algorithms for the STSP with planar support. We do not explore this here, but leave it as a direction for future research.

We close this section by showing that Proposition 7 can also be useful in the context of the STSP. We display in Figure 1 a fractional point for the STSP on 8 vertices which satisfies the degree equations, the SECs and the simple comb inequalities. Bold, plain and dotted lines represent edges e with $x^*_e = 1, 2/3$ and $1/3$, respectively. Note that the support graph is non-planar. Moreover, there is no maximally violated $\{0, \frac{1}{2}\}$-cut. Hence, there is no use in applying the separation algorithms given in [1, 6, 22, 23]. Yet, there *is* a violated comb inequality: it has $H = \{1, 3, 4, 7\}$, $T_1 = \{1, 2\}$, $T_2 = \{3, 4, 5, 6\}$ and $T_3 = \{7, 8\}$.

We now show that this comb inequality can be detected easily via Propositions 6 and 7. Note that the TSP on 8 vertices has 28 variables. Moreover, any spanning tree contains 7 edges. Therefore the matrices A_1 and A_2 in (9) each have 21 columns. Thus, after deleting the 16 variables x_e with $x^*_e = 0$ and contracting the 3 variables x_e with $x^*_e = 1$, the matrices A_1 and A_2 in (9) each have only 2 columns. The associated matroid is both graphic and co-graphic, which means that separation of BC-cuts is trivial. Moreover, the clutter is ideal, and therefore the BC-cuts are equivalent to the DP inequalities. In fact, there is only one violated BC-cut, namely, the comb inequality mentioned above.

This example suggests that, even when the support graph is *non-planar*, it may be possible to separate DP-inequalities efficiently, provided that there is a sufficiently large number of 1-edges. This issue too is left for future research.

6. Conclusion

Using a non-standard Gomory group relaxation, we can associate a binary clutter with any ILP. The resulting cutting planes, the BC-cuts, are a proper generalization of the $\{0, \frac{1}{2}\}$-cuts of Caprara & Fischetti [5]. This observation enables us to exploit several ideas from the binary clutter literature, especially algorithms for optimization and separation, to yield new results for ILPs in general. This has been illustrated using the STSP as an example.

The hope of the author is that this will provide researchers with further motivation to study binary clutters. In particular, it would be interesting to explore in more depth the precise boundary between 'easy' and 'hard' cases of the Binary Clutter Problem. It would also be worthwhile trying to derive more general conditions under which the DP inequalities for the STSP can be separated in polynomial time. Finally, binary clutter relaxations of other combinatorial problems could be explored.

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