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Exploring the Relationship Between Max-Cut and Stable Set Relaxations

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Abstract. The *max-cut* and *stable set* problems are two fundamental $\mathcal{NP}$-hard problems in combinatorial optimization. It has been known for a long time that any instance of the stable set problem can be easily transformed into a max-cut instance. Moreover, Laurent, Poljak, Rendl and others have shown that any convex set containing the cut polytope yields, via a suitable projection, a convex set containing the stable set polytope. We review this work, and then extend it in the following ways. We show that the rounded version of certain 'positive semidefinite' inequalities for the cut polytope imply, via the same projection, a surprisingly large variety of strong valid inequalities for the stable set polytope. These include the *clique*, *odd hole*, *odd antihole*, *web* and *antiweb* inequalities, and various inequalities obtained from these via sequential lifting. We also examine a less general class of inequalities for the cut polytope, which we call *odd clique* inequalities, and show that they are, in general, much less useful for generating valid inequalities for the stable set polytope.

As well as being of theoretical interest, these results have algorithmic implications. In particular, we obtain as a by-product a polynomial-time *separation algorithm* for a class of inequalities which includes all web inequalities.

Key words. Max-cut problem – Stable set problem – Polyhedral combinatorics

1. Introduction

This paper is concerned with the *max-cut* and *stable set* problems, which are two well-known, fundamental, strongly $\mathcal{NP}$-hard combinatorial optimization problems. The formal definitions of these problems are as follows. Let $G = (V, E)$ be an undirected graph, where V is the vertex set and E is the edge set. A set of edges $F \subset E$ is called an *edge cutset*, or simply *cut*, if $F = \{\{i, j\} \in E : i \in S, j \in V \setminus S\}$ for some suitable set $S \subset V$. Given a weight $w_e \in \mathbb{R}$ for each edge $e \in E$, the max-cut problem calls for a cut of maximum weight. A vertex set $S \subseteq V$ is called *stable* if no two members of S are adjacent. Given a weight $w_i \in \mathbb{R}_+$ for each vertex $i \in V$, the stable set problem calls for a stable set S with maximum weight.

From the point of view of approximability, these two problems are quite different: for non-negative edge weights, the max-cut problem can be approximated to within a factor of 0.878 in polynomial time (Goemans & Williamson [8]). The stable set problem, on the other hand, cannot be approximated to within any constant unless $\mathcal{P} = \mathcal{NP}$ (Håstad [11]). However, in other senses they are very similar. They have both been intensely

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studied from a polyhedral viewpoint, and, for both problems, there exist natural semidefinite relaxations which can be solved to arbitrary precision in polynomial time (see [8] and also Grötschel, Lovász & Schrijver [9]). In fact, the connection goes deeper than this: there is a simple transformation of the stable set problem into the max-cut problem, and Laurent, Poljak & Rendl [16] showed that valid inequalities for the cut polytope yield, via a suitable projection, valid inequalities for the stable set polytope.

Our work is an extension of the results in Laurent, Poljak and Rendl [16] and also some results of Lovász & Schrijver [18], [19], [21]. Our main results are as follows:

- The classical *positive semidefinite* (psd) inequalities for the cut polytope, in conjunction with the well-known *triangle* inequalities, imply (via projection) the *web* inequalities of Trotter [23] (which include the *clique*, *odd hole* and *odd antihole* inequalities of Padberg [20] as special cases).
- This implies the existence of a polynomial-time exact *separation algorithm* for a class of inequalities which includes all web inequalities.
- A stronger version of the psd inequalities, obtained by rounding the right hand side, yields by the same projection not only the web inequalities, but also the *antiweb* inequalities of Trotter [23], and also various inequalities obtained from web and antiweb inequalities via sequential lifting.

We also examine a less general class of inequalities for the cut polytope, due to Barahona & Mahjoub [2], which we call *odd clique* inequalities. We show that, although they induce facets of the cut polytope, they are much less useful for generating valid inequalities for the stable set polytope.

The structure of the paper is as follows. In Section 2, we review the relevant literature, with special attention being paid to the results of Laurent, Poljak and Rendl [16] and Lovász & Schrijver [19]. In Section 3, we examine the web and antiweb inequalities. Then, in Section 4, we consider the issue of *lifting*. Finally, in Section 5, we make some conclusions. The main conclusion is that the rounded psd inequalities should make extremely good cutting planes in an exact algorithm for the stable set problem, but that the odd clique inequalities are unlikely to be useful.

2. Review of Known Results

2.1. The max-cut problem

For the sake of simplicity and brevity, we assume in this subsection that the graph G is complete. We are concerned with various polyhedra associated with the max-cut problem (see the excellent book by Deza & Laurent [5] for a survey). Let y_e for each edge e be a binary variable taking the value 1 if and only if e lies in the cut. It is well known that a given $y \in \{0, 1\}^{|E|}$ is the incidence vector of a cut if and only if:

$$y_{ij} + y_{ik} + y_{jk} \leq 2 \quad \forall \{i, j, k\} \subset V, \tag{1}$$

$$y_{ij} - y_{ik} - y_{jk} \leq 0 \quad \forall \{i, j, k\} \subset V. \tag{2}$$

The inequalities (1) and (2) are called *triangle* inequalities. It should be noted that the triangle inequalities imply the *trivial bounds* $0 \leq y_{ij} \leq 1$ for all $\{i, j\} \in E$. The polytope defined by the triangle inequalities is sometimes called the *metric polytope*.

The *cut polytope* on n vertices, which we will denote by C_n, is the convex hull in $\mathbb{R}_+^{\binom{n}{2}}$ of the incidence vectors of cuts, i.e.,

$$C_n = \text{conv}\left\{y \in \{0, 1\}^{\binom{n}{2}} : (1), (2) \text{ hold}\right\}.$$

The triangle inequalities induce facets of C_n, so C_n is contained in the metric polytope. Containment is strict for $n \geq 5$.

Another important class of valid inequalities for C_n are the following:

$$\sum_{1 \leq i < j \leq n} b_i b_j y_{ij} \leq b(V)^2/4, \tag{3}$$

where $b \in \mathbb{R}^n$ is an arbitrary vector and $b(V) := \sum_{i \in V} b_i$. We will call the inequalities (3) *positive semidefinite* (or just *psd*) inequalities, because they arise from a natural positive semidefinite relaxation of the max-cut problem (Laurent & Poljak [14]). Indeed, a vector y satisfies all psd inequalities if and only if the matrix $J - 2Y$ is psd, where J is an $n \times n$ matrix of all-ones, and Y is a matrix with y_{ij} in row i and column j, and zeroes on the diagonal.

Note that a psd inequality with $b_i = b_j = 1$ and $b_k = 0$ for all $k \in V \setminus \{i, j\}$ is a simple upper bound $y_{ij} \leq 1$, and if we change b_j to -1 we obtain the lower bound $y_{ij} \geq 0$.

It should be noted that there are an infinite number of psd inequalities. Indeed, the convex set defined by the psd inequalities, denoted by $\mathcal{J}_n$ in [15], is not a polytope in general. Yet, one can solve the psd relaxation in polynomial time to arbitrary precision, see for example Grötschel, Lovász & Schrijver [9].

There is a sense in which $\mathcal{J}_n$ 'wraps closely' around C_n. Indeed, Laurent & Poljak [14] gave a result which, expressed in our notation, states that

$$C_n \subset \mathcal{J}_n \subset \left\{\frac{1}{2}(1 - \cos(\pi y)) : y \in C_n\right\},$$

where the cos function is defined component-wise. Simple calculations reveal that

$$\max_{0 \leq r \leq 1} \left\{\frac{1 - cos(\pi r)}{2r}\right\} \approx 1.138.$$

This implies the remarkable fact that, when all edge-weights are non-negative, the upper bound given by the psd relaxation is at most 1.138 times the optimum. Or, equivalently, there exists a cut whose weight is at least $1/1.138 \approx 0.878$ times the upper bound. In the paper by Goemans & Williamson mentioned earlier [8], a polynomial-time randomized algorithm was given for finding such a cut.

By contrast, the upper bound given by the triangle inequalities in the case of non-negative edge-weights can be arbitrarily close to 4/3 times the optimum in dense graphs, and to 2 times the optimum in sparse graphs (see for example Avis & Umemoto [1]).

Obviously, if the b_i are integers, the left hand side of (3) is an integer. Therefore, in this case the right hand side can be rounded down to give:

$$\sum_{1 \leq i < j \leq n} b_i b_j y_{ij} \leq \lfloor b(V)^2/4 \rfloor. \tag{4}$$

We will call (4) *rounded psd* inequalities. The rounded psd inequalities with $\sum_{i \in V} |b_i|$ equal to some fixed integer k are often called *k-gonal*, see [5]. It is known (see for example Avis & Umemoto [1]) that the convex set defined by all rounded psd inequalities with k odd (equivalently, with $b(V)$ odd) is strictly contained in $\mathcal{J}_n$. It is not known however whether this smaller convex set is a polytope.

In the special case where $b \in \{-1, 0, +1\}^n$ and k is odd, the rounded psd inequalities reduce to some inequalities introduced earlier by Barahona and Mahjoub [2]. We will call them *odd clique* inequalities, because their *support graph* (the subgraph of G induced by edges with non-zero coefficients) is a clique of odd cardinality. (They should not however be confused with the clique inequalities for the stable set polytope, to be described in the next subsection.) It follows from a result in [2] that the odd clique inequalities induce facets of C_n. When $k = 3$, they reduce to the triangle inequalities.

The rounded psd inequalities generalize or dominate not only the triangle, odd clique and psd inequalities, but also some other well known inequalities such as the *hypermetric* and *negative type* inequalities (see Deza & Laurent [5]). Moreover, Laurent & Poljak [15] introduced an even stronger version of the rounded psd inequalities, the so-called *gap* inequalities. However, we omit the details for the sake of brevity.

It should also be noted that some other authors prefer to work with a version of the cut polytope in which the variables take values in $\{-1, +1\}$ rather than $\{0, 1\}$. Results for one version can easily be mapped onto results for the other, and we prefer to work here with the binary version.

2.2. *The stable set problem*

Now we assume that G is a general graph (not necessarily complete).

The first systematic study of the stable set polytope was performed by Padberg [20]. Let x_i for each vertex i be a binary variable taking the value 1 if and only if i lies in the stable set. It is obvious that a given $x \in \{0, 1\}^n$ is the incidence vector of a stable set if and only if:

$$x_i + x_j \leq 1 \qquad \forall \{i, j\} \in E. \tag{5}$$

The inequalities (5) are sometimes called *edge* inequalities.

The *stable set polytope*, normally denoted by STAB(G) [9], is the convex hull in $\mathbb{R}^n_+$ of the incidence vectors of stable sets, i.e.,

$$\text{STAB}(G) = \text{conv}\left\{x \in \{0, 1\}^n : (5) \text{ hold}\right\}.$$

Padberg [20] observed that the edge inequalities are dominated by *clique* inequalities $\sum_{i \in C} x_i \leq 1$, which are defined for any vertex set $C \subset V$ inducing a clique in G. He showed that the clique inequalities induce facets of STAB(G) whenever C induces a maximal clique. Graphs for which the clique and non-negativity inequalities suffice to define STAB(G) are called *perfect*.

Padberg also introduced the *odd hole* inequalities, which take the form $\sum_{i \in H} x_i \leq \lfloor \frac{|H|}{2} \rfloor$, where $H \subset V$ induces a chordless simple cycle whose cardinality is odd. The polytope defined by non-negativity, edge and odd-hole inequalities is normally denoted by CSTAB(G) [9, 16].

When $V = H$, the odd hole inequality induces a facet of STAB(G). However, Padberg pointed out that the odd hole inequalities do not induce facets in general. To convert an odd hole inequality into a facet for a general graph, one must compute appropriate coefficients for the other variables, a process called *lifting*. For example, suppose that we add an additional vertex u which is adjacent to every vertex in the odd hole. Then the lifted odd hole inequality

$$\sum_{i \in H} x_i + \left\lfloor \frac{|H|}{2} \right\rfloor x_u \leq \left\lfloor \frac{|H|}{2} \right\rfloor$$

is valid. This is sometimes referred to as an *odd wheel* inequality, see for example [9].

We deal with lifting in more detail in Section 4.

Finally, Padberg also introduced the *odd antihole* inequalities. These take the form $\sum_{i \in A} x_i \leq 2$, where $A \subset V$ induces an odd antihole, i.e., the complement of an odd hole. Again, odd antihole inequalities do not induce facets in general and must be lifted.

The odd hole and antihole inequalities were generalized by Trotter [23]. Let p and q be integers satisfying $p > 2q+1$ and $q > 1$. We use arithmetic modulo p for simplicity. The web $W(p, q)$ is (see Figure 1) a graph whose vertex set is $\{1, \ldots, p\}$ and which contains edges from i to $\{i+q, \ldots, i-q\}$ for all $1 \leq i \leq p$. The antiweb $AW(p, q)$ is (see Figure 1) the complement of the web $W(p, q)$, i.e., vertex i is connected to vertices $i-q+1, \ldots, i+q-1$.

Trotter [23] showed that, when $G = W(p, q)$, the *web* inequality $\sum_{i=1}^{p} x_i \leq q$ is valid for STAB(G), and facet-inducing if and only if p and q are relatively prime. He also showed that, when $G = AW(p, q)$, the *antiweb* inequality $\sum_{i=1}^{p} x_i \leq \lfloor p/q \rfloor$ is valid for STAB(G). Laurent [13] showed that it induces a facet of STAB(G) if and only if p is not a multiple of q. Again, in general graphs which contain a web or antiweb as a node-induced subgraph, the web and antiweb inequalities need to be lifted to make them facet-inducing.

Note that the graphs $AW(p, 2)$ and $W(2q+1, q)$ are odd holes, whereas $W(p, 2)$ and $AW(2q+1, q)$ are odd antiholes. Also, a clique on p vertices can be regarded as a 'degenerate' web $W(p, 1)$. (If p is odd, the clique can also be regarded as a degenerate antiweb $AW(p, \lceil p/2 \rceil)$.)

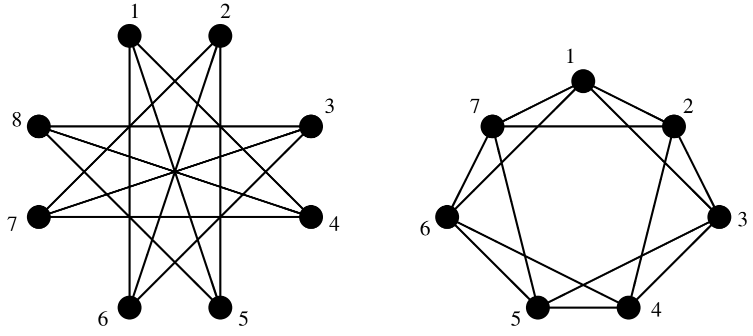

Fig. 1. An (8, 3)-web and a (7, 3)-antiweb

As in the case of max-cut, some rather different valid inequalities for STAB(G) can be derived by semidefinite programming. This idea first appeared in Lovász [18]. Lovász constructed an $n \times n$ matrix in which the entry in row i and column j represents the product $x_i x_j$. We will use the notation $Y' = \{y'_{ij}\}$ for this matrix, because we reserve y for max-cut variables. Then, clearly, if Y' corresponds to a stable set, we have $y'_{ij} = 0$ for all $\{i, j\} \in E$, $y'_{ij} = y'_{ji}$ for all $\{i, j\} \notin E$, and $y'_{ii} = x_i$ for all $i \in V$. Moreover, as explained by Lovász, the matrix Y' - xx^T must be psd.

The convex hull of all matrices Y' satisfying the above three conditions is defined in $\mathbb{R}^{n\times n}$, but if we project it onto x-space (using the identity $y'_{ii} = x_i$) we obtain a convex set in $\mathbb{R}^n_+$, which is sometimes called the (Lovász) *theta body*. The theta body, sometimes denoted by TH(G), is not a polytope in general.

The theta body is known to be defined by the non-negativity inequalities and the so-called *orthonormal representation* (OR) inequalities. The OR inequalities take the form $\sum_{i \in V} (c^T \cdot u_i)^2 x_i \le 1$, where the $u_i \in \mathbb{R}^n$ for $i \in V$ are vectors satisfying $||u_i|| = 1$ for all i, and $u_i \cdot u_j = 0$ for all $\{i, j\} \notin E$, and $c \in \mathbb{R}^n$ is an arbitrary vector with $||c|| = 1$ (Grötschel, Lovász & Schrijver [9]). Remarkably, the OR inequalities generalize the clique inequalities. This implies that, when G is perfect, TH(G) coincides with STAB(G) and is therefore a polytope itself. However, when G is not perfect, TH(G) is no longer a polytope and its structure is still somewhat mysterious (see Shepherd [22]).

Although TH(G) satisfies all clique inequalities, it does not wrap as closely around STAB(G) as $\mathcal{J}_n$ wraps around C_n. Indeed it is known that, in a random graph where every edge appears with a fixed probability, the maximum cardinality stable set almost surely contains $\mathcal{O}(\log n)$ vertices, the upper bound given by optimizing over the clique and non-negativity inequalities is almost surely $\mathcal{O}(n/\log n)$, and the upper bound given by optimizing over TH(G) is almost surely $\mathcal{O}(\sqrt{n})$ (see Juhász [12]). Thus, the OR inequalities can give a significant improvement over the clique inequalities, but not enough to lead to a constant-factor approximation. This is confirmed by the inapproximability result of Hasted [11] mentioned in the introduction.

The relaxation of Lovász [18] was strengthened by Schrijver [21] by adding the non-negativity inequalities $y'_{ij} \ge 0$ for all $\{i, j\} \notin E$. It is not known what inequalities this stronger relaxation yields when projected onto x-space. Lovász & Schrijver [19] defined an even stronger relaxation by adding also the inequalities

$$y'_{ij} \le x_i \quad \forall \{i, j\} \subset V, \tag{6}$$

$$y'_{ik} + y'_{jk} \le x_k \quad \forall \{i, j\} \in E, k \ne i, j, \tag{7}$$

$$x_i + x_j + x_k \le 1 + y'_{ik} + y'_{jk} \quad \forall \{i, j\} \in E, k \ne i, j. \tag{8}$$

They showed that the projection of this stronger relaxation onto x space, the so-called N_+-relaxation, satisfies all OR, odd hole and odd antihole inequalities, and also the odd wheel inequalities.

The importance of these extended formulations is that one can solve the associated optimization problems to arbitrary precision in polynomial time. This implies that one can separate in polynomial time over a class of inequalities which includes all OR, odd hole, odd antihole, and odd wheel inequalities (see [9] for a detailed explanation of separation). This is an extremely powerful result. Yet, we will extend it in Section 3 to include all web inequalities.

2.3. Connections between the cut and stable set polytopes

It has been known for some time that there is a strong relationship between the max-cut and stable set problems. Indeed, it is part of the folklore that any instance of the stable set problem can be easily transformed into an instance of the max-cut problem. For the sake of clarity, here is a description.

Let $G = (V, E)$ and $w \in \mathbb{R}^n_+$ be the input data for the stable set instance. Construct a new graph $G'(V', E')$ by adding an additional node 0 which is adjacent to every vertex in G. Give each edge of G a weight of M, where M is a large positive number. Then, for $i = 1, \dots, n$, give the edge $\{0, i\}$ a weight of $w_i - Md_i$, where d_i is the degree of i in G. Now consider an arbitrary set of vertices $S \subset V$ and let $\delta(S)$ (respectively, $E(S)$) be the set of edges in G with exactly one end-vertex (both end-vertices) in S. The weight of the cut in G' from S to $V' \setminus S$ is

$$M|\delta(S)| + \sum_{i \in S} w_i - M \sum_{i \in S} d_i,$$

which is easily shown to be equal to $\sum_{i \in S} w_i - 2M|E(S)|$. Thus, if M is large enough, the solution to the max-cut problem in G' will choose a set S which maximises $\sum_{i \in S} w_i$, subject to the condition that $E(S) = \emptyset$, i.e., that S is stable in G. A suitable value for M is $\max_{i \in V} w_i$.

The effect of the penalty term M is to prevent i and j both lying on the opposite shore of the cut to 0 whenever $\{i, j\} \in E$.

This transformation also has important consequences for the associated polyhedra (see Laurent, Poljak and Rendl [16] for a clear exposition). Let us assume that the graph G' has been made complete, which is easily done by adding edges of zero weight. Now, for $0 \le i < j \le n$, let y_{ij} be a binary variable taking the value 1 if the edge $\{i, j\}$ lies in the cut. Then, the convex hull of the feasible y vectors is a cut polytope on $n + 1$ vertices. Note that y_{0i} is analogous to x_i, in the sense that placing the edge $\{0, i\}$ in the cut corresponds to putting i into the stable set. Moreover, if y is a cut vector, $y_{ij} = y_{0i} + y_{0j} - 2y_{0i}y_{0j}$ holds for any $1 \le i < j \le n$. Therefore, y_{ij} is analogous to the quantity $x_i + x_j - 2x_i x_j$. Indeed, the penalty term forces the triangle inequality $y_{ij} \le y_{0i} + y_{0j}$ to hold as an equality whenever $\{i, j\} \in E$, which is analogous to enforcing $x_i x_j = 0$.

From the above observations, it is clear that the stable set polytope STAB(G) can be obtained by taking the face of the cut polytope C_n defined by the equations $y_{ij} = y_{0i} + y_{0j}$ for all $\{i, j\} \in E$, and then projecting this face into x-space using the identity $y_{0i} = x_i$ for all $i \in V$. Thus, we can take any convex set (respectively, polytope) containing C_n and, by projecting it in this way, obtain a convex set (polytope) which contains STAB(G). Laurent, Poljak and Rendl [16] show that under this mapping:

- The metric polytope (defined by the triangle inequalities) yields CSTAB(G) (defined by the non-negativity, edge and odd hole inequalities).
- $\mathcal{J}_n$ (defined by the psd inequalities) yields TH(G) (defined by the non-negativity and OR inequalities).
- C_n yields STAB(G).

Further results along this line can be derived from the work of Lovász and Schrijver [19]. Indeed, as mentioned above, our variables y_{ij} are analogous to the quantity $x_i + x_j - 2x_i x_j$, whereas the variables y'_{ij} of Lovász and Schrijver represent $x_i x_j$. Therefore, the two extended formulations are related via the identities

- $y'_{ij} = (x_i + x_j - y_{ij})/2$ and
- $y_{ij} = x_i + x_j - 2y'_{ij}$.

Therefore, the condition $y'_{ij} = 0$ for all $\{i, j\} \in E$ imposed in [18], [19], [21] is equivalent to the equations $y_{ij} = y_{0i} + y_{0j}$ for all $\{i, j\} \in E$ mentioned above. It is also not difficult to check that the linear inequalities used to define the N_+-relaxation in [19] correspond to a *subset* of triangle inequalities for the cut polytope on $n + 1$ vertices. Namely, they include all triangle inequalities except the ones involving triples $\{i, j, k\}$ inducing a stable set in G. Therefore, we have

- The psd inequalities and the subset of the triangle inequalities mentioned above imply all OR, odd hole, odd antihole and odd wheel inequalities.

We examine the status of the web and antiweb inequalities in the next section.

3. Web and Antiweb Inequalities

In this section, we show that the psd and triangle inequalities together imply all web inequalities (which as we have seen include all clique, odd hole and odd antihole inequalities). We also show that the *rounded* psd inequalities imply all antiweb inequalities. We also briefly examine which of these web and antiweb inequalities are implied by the odd clique inequalities.

In what follows, we use *lhs* (respectively, *rhs*) to mean ‘left hand side’ (respectively, ‘right hand side’). We also say that a psd (or rounded psd) inequality is *global* if $b_i = 1$ for all $i \in V$ (and b_0 is an arbitrary integer). We also use the notation $V^0 := V \cup \{0\}$. To aid clarity, we also write x_i instead of y_{0i} throughout.

3.1. Web inequalities

The main result in this subsection is the following theorem.

Theorem 1. *The psd and triangle inequalities, together, imply the web inequalities.*

Proof. Let $G = W(p, q)$. First, we consider the case in which q is odd. We construct the following global psd inequality with $b_0 = 2q - p$:

$$\sum_{1 \le i < j \le p} y_{ij} + (2q - p) \sum_{1 \le i \le p} x_i \le q^2. \tag{9}$$

Now, let $h = (q - 1)/2$. We sum together hp triangle inequalities of the form:

$$y_{i,i+q} - y_{i,i+j} - y_{i+j,i+q} \le 0, \tag{10}$$

for $i = 1, \ldots, p$ and $j = 1, \ldots, h$, giving:

$$h \sum_{i=1}^{p} y_{i,i+q} - \sum_{i=1}^{p} \sum_{j=1}^{q-1} y_{i,i+j} \le 0 \tag{11}$$

Summing together (9) and (11) we obtain

$$h \sum_{i=1}^{p} y_{i,i+q} + \sum_{\{i,j\} \in E} y_{ij} + (2q - p) \sum_{i=1}^{p} x_i \le q^2.$$

Now, using the identities $x_i + x_j = y_{ij}$ for every edge $\{i, j\} \in E$, we obtain

$$q \sum_{i=1}^{p} x_i \le q^2.$$

Finally, dividing by q, we obtained $\sum_{i=1}^{p} x_i \le q$ as required.

For the case in which q is even, we construct the same global psd inequality (9). Let $h = q/2$. Then, summing the inequalities (10), for $i = 1, \ldots, p$ and $j = 1, \ldots, h-1$, plus half of the inequalities (10), for $i = 1, \ldots, p$ and $j = h$, we obtain again the inequality (11). So, proceeding as above, we obtain the result. □

We note that the triangle inequalities used in the proof of Theorem 1 appear in the definition of the N_+-relaxation of Lovász & Schrijver [19]. This yields the following two corollaries:

Corollary 1. *The N_+-relaxation of Lovász & Schrijver [19] satisfies all web inequalities (in addition to all OR and odd wheel inequalities).*

Corollary 2. *There exists a polynomial-time separation algorithm for a class of inequalities which includes all OR, odd wheel and web inequalities (and therefore all edge, clique, odd hole and odd antihole inequalities).*

We have also obtained some partial results about the odd clique inequalities. It is not difficult to show that the odd clique inequalities imply the $W(p, q)$ inequalities with $p = 2q + 1$ (the odd holes) and $p = 2q + 2$, and that they do not imply any web inequalities with $p > 4q$. Using a computer program, we found that the odd clique inequalities also imply a few other web inequalities, such as the $W(10, 3)$, $W(13, 4)$ and $W(13, 5)$ inequalities. We did not succeed in finding a full characterization, but in any case the odd clique inequalities are less productive than the general rounded psd inequalities.

3.2. Antiweb inequalities

The case of antiwebs is more complicated. It appears that the antiweb inequalities are *not* in general implied by the psd and triangle inequalities. Instead, it appears to be necessary to use the *rounded* psd inequalities, as explained in the following lemma and theorem.

Lemma 1. *Let G be the antiweb $AW(p,q)$. For any $1 \le s \le p-q$, the following inequality is implied by the rounded psd inequalities:*

$$2(q-1)\sum_{i=1}^{p} x_i - \sum_{i=1}^{p} \sum_{j=i+s}^{i+q-1+s} y_{ij} \le 0. \tag{12}$$

Proof. For $t = 1, \dots, p$, we construct a rounded psd inequality K_{st} by setting $b_i^{st} = 1$ for every $i = t, t+1, \dots, t+q-1$, $b_i^{st} = -1$ for $i = t+q-1+s$, $b_0^{st} = 2-q$ and $b_i^{st} = 0$ otherwise. Then, K_{st} has the following form:

$$(2-q) \sum_{i=t}^{t+q-1} x_i + (q-2)x_{t+q-1+s} + \sum_{i=t}^{t+q-2} \sum_{j=i+1}^{t+q-1} y_{ij} - \sum_{i=t}^{t+q-1} y_{i,t+q-1+s} \le 0. \tag{13}$$

Using the identities $x_i + x_j = y_{ij}$ for $t \le i < j \le t+q-1$, we see that the third term in (13) is equal to $(q-1)\sum_{i=t}^{t+q-1} x_i$. Therefore, (13) reduces to:

$$\sum_{i=t}^{t+q-1} x_i + (q-2)x_{t+q-1+s} - \sum_{i=t}^{t+q-1} y_{i,t+q-1+s} \le 0.$$

Summing together all p such inequalities, and simplifying, we obtain the desired inequality (12). □

Theorem 2. *The rounded psd inequalities imply all $AW(p,q)$ inequalities.*

Proof. Let $G = AW(p,q)$. In this proof, we let $\alpha = \lfloor p/q \rfloor$ and let r be such that $p = \alpha q + r$.
We begin by constructing the following global rounded psd inequality with $b_0 = 2\alpha - p + 1$:

$$(2\alpha - p + 1) \sum_{1 \le i \le p} x_i + \sum_{1 \le i < j \le p} y_{ij} \le \alpha(\alpha+1).$$

Using the identities $x_i + x_j = y_{ij}$ for every edge $\{i,j\} \in E$, this can be re-written as:

$$(2\alpha + 2q - p - 1) \sum_{1 \le i \le p} x_i + \sum_{\{i,j\} \notin E} y_{ij} \le \alpha(\alpha+1). \tag{14}$$

Next, we sum together $\alpha - 1$ inequalities of the form (12), with s spaced at regular intervals of size q. Specifically, we set s equal to $\lceil (q+r)/2 \rceil$, $\lceil (q+r)/2 \rceil + q, \dots, \lceil (q+r)/2 \rceil + (\alpha-2)q$. Since the last term of this sequence, $\lceil (q+r)/2 \rceil + (\alpha-2)q$, is easily shown to be equal to $p - q - \lfloor (q+r)/2 \rfloor$, the resulting inequality takes the form:

$$2(q-1)(\alpha-1)\sum_{i=1}^{p} x_i - \sum_{i=1}^{p} \sum_{j=i+\lceil (q+r)/2 \rceil}^{i+p-\lfloor (q+r)/2 \rfloor - 1} y_{ij} \le 0. \tag{15}$$

Now we consider the term involving the y variables in (15). When $r < q - 2$, it is equivalent to:

$$2\sum_{e \notin E} y_e + \sum_{i=1}^{p} \sum_{j=i+\lceil (q+r)/2 \rceil}^{i+q-1} y_{ij} + \sum_{i=1}^{p} \sum_{j=i+\lfloor (q+r)/2 \rfloor + 1}^{i+q-1} y_{ij}.$$

Given the identities $x_i + x_j = y_{ij}$ for every edge $\{i, j\}$ with $i + \lfloor (q+r)/2 \rfloor \le j < i+q$, this reduces to:

$$2(q - r - 1)\sum_{t=1}^{p} x_t + 2\sum_{e \notin E} y_e. \tag{16}$$

It is easy to check that the term reduces to (16) also when $r \in \{q-2, q-1\}$. Therefore, (15) is equivalent to

$$2\left((q-1)(\alpha-1) - q + r + 1\right)\sum_{t=1}^{p} x_t - 2\sum_{\{i,j\} \notin E} y_{ij} \le 0. \tag{17}$$

Summing the inequality (14) and half the inequality (17), we obtain

$$(\alpha + 1)\sum_{t=1}^{p} x_t \le \alpha(\alpha + 1).$$

Finally, dividing by $\alpha + 1$, we obtain $\sum_{t=1}^{p} x_t \le \alpha$ as required. □

Theorems 1 and 2 together show that, remarkably, the rounded psd inequalities imply all clique, web and antiweb inequalities. The following theorem shows that the odd clique inequalities, on the other hand, only imply a small subset of the antiweb inequalities:

Theorem 3. *The odd clique inequalities imply all* $AW(p, q)$ *inequalities with* $q \in \{2, 3\}$ *and* $p \equiv 1 \bmod q$.

Proof. We already know that the triangle inequalities imply the odd hole inequalities. Since triangle inequalities are odd clique inequalities, and the odd hole inequalities are the antiweb inequalities with $q = 2$, this solves the case $q = 2$.

Now we consider the case $q = 3$ and $p = 3\alpha+1$. We define α odd clique inequalities, for $i = 0, \ldots, \alpha-1$, using the coefficients $b_j = 1$ for $j = 3i+1, \ldots, 3i+4$, $b_0 = -1$ and $b_j = 0$ otherwise. We also consider $\alpha - 1$ triangle inequalities, for $i = 1, \ldots, \alpha-1$, of the form $y_{1,3i+4} \ge y_{i,3i+1} + y_{3i+1,3i+4}$. Summing these inequalities together, we obtain

$$\sum_{i=1}^{p} y_{i,i+1} + \sum_{0 \le i < \alpha} (y_{3i+1,3i+3} + y_{3i+2,3i+4}) - \sum_{i=1}^{p} x_i - \sum_{1 \le i < \alpha} x_{3i+1} \le 2\alpha.$$

Using the identities $y_{ij} = x_i + x_j$ to eliminate y variables we obtain

$$2\sum_{i=1}^{p} x_i \le 2\alpha.$$

Dividing by two yields the antiweb inequality. □

We conjecture that this condition is also necessary, with the exception of some degenerate (non-facet-inducing) cases. Note that when $q = 2$, the antiweb reduces to an odd hole. Also, $AW(7, 3)$, the 7-antihole, is isomorphic to $W(7, 2)$. Therefore, these cases were already dealt with in the previous subsection. However, there exist non-trivial antiwebs satisfying $q = 3$ and $p \equiv 1 \bmod 3$, e.g., $AW(10, 3)$. In any case, the odd clique inequalities again are less productive than the rounded psd inequalities.

4. Sequential Lifting

As mentioned above, the idea of sequential lifting goes back to Padberg [20]. Let $G = (V, E)$ be a graph, let $S \subset V$ be given and let $\sum_{i \in S} \alpha_i x_i \leq \beta$ be a valid inequality for STAB(G) (with α, β positive). It may be that the inequality can be strengthened by increasing the lhs coefficients for the variables in $V \setminus S$. In practice, lifting is usually done sequentially, i.e., we order the members of $V \setminus S$ arbitrarily and then increase one coefficient at a time.

It turns out that, under certain conditions, the lifted inequality is implied by the rounded psd inequalities whenever the original inequality is. We give two examples in the following two subsections.

4.1. Vertex replication

The simplest form of sequential lifting goes back to Lovász [17] and Fulkerson [7]. Let $G = (V, E)$ be a graph. For a given vertex i, we denote by $n(i)$ the set of *neighbours* of i. Now let u be an arbitrary vertex. By *replicating* u we mean the construction of a new graph $\tilde{G}$, obtained by adding an extra vertex ($\tilde{u}$, say) which is adjacent to both u and to every vertex in $n(u)$.

Now let $\alpha x \leq \beta$ be a valid inequality for STAB(G). We construct a valid inequality $\tilde{\alpha} x \leq \beta$ for STAB($\tilde{G}$) by setting $\tilde{\alpha}_i = \alpha_i$ for all $i \in V$ and $\tilde{\alpha}_{\tilde{u}} = \alpha_u$. Again, we say that the inequality $\tilde{\alpha} x \leq \beta$ has been obtained from $\alpha x \leq \beta$ by replication. It is easy to prove, and has been noted by several authors, that the replicated inequality is facet-inducing if and only if the original inequality is. Moreover, replication can be applied iteratively.

For example, the graph in Figure 3 is obtained from the 5-hole by replicating vertex 2. The original odd hole inequality $\sum_{i=1}^{5} x_i \leq 2$ becomes $\sum_{i=1}^{6} x_i \leq 2$.

Interestingly, the rounded psd inequalities can be used to accomplish replication, as expressed in the following theorem.

Theorem 4. *Let G be a graph and let $\alpha x \leq \beta$ be a valid inequality for STAB(G). If $\alpha x \leq \beta$ is implied by the rounded psd inequalities, then the replicated inequality $\tilde{\alpha} x \leq \beta$ for STAB($\tilde{G}$) is also implied by the rounded psd inequalities.*

Proof. Let us suppose that the inequality $\alpha x \leq \beta$ is derived from r k-gonal inequalities, and, for $h = 1, \dots, r$, let $b_0^h, \dots, b_n^h$ be the coefficients used to generate the hth rounded psd inequality. That is, the hth inequality takes the form

$$K_h := \sum_{1 \leq i < j \leq n} b_i^h b_j^h y_{ij} + \sum_{i \in V} b_0^h b_i^h x_i \leq \left\lfloor \frac{(b^h(V^0))^2}{4} \right\rfloor .$$

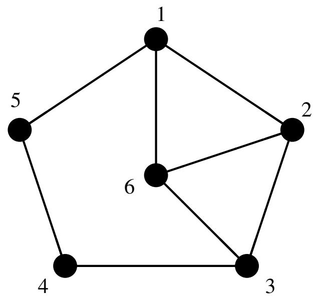

Fig. 2. Replicating one vertex in a 5-hole

Also let $\lambda_h > 0$ for $h = 1, \ldots, r$ be the multiplier given to K_h in the derivation. Note that the coefficients of y_{ij} must cancel each other out when $\{i, j\}$ is not present in the graph, since there is no equation available to relate y_{ij} to x_i and x_j in this case. More formally, we have:

$$\sum_{h=1}^{r} \lambda_h b_i^h b_j^h = 0, \qquad (\forall \{i, j\} \notin E)$$

and, in particular, we have

$$\sum_{h=1}^{r} \lambda_h b_i^h b_u^h = 0, \qquad (\forall i \in V \setminus (\{u\} \cup n(u))). \tag{18}$$

Also, from the equations $y_{ij} = x_i + x_j$ for all $\{i, j\} \in E$, we have by assumption that

$$\sum_{h=1}^{r} \lambda_h \sum_{j \in n(i) \cup \{0\}} b_i^h b_j^h = \alpha_i, \qquad (\forall i \in V)$$

and, in particular,

$$\sum_{h=1}^{r} \lambda_h \sum_{j \in n(u) \cup \{0\}} b_u^h b_j^h = \alpha_u. \tag{19}$$

Now, we 'expand' each of the individual rounded psd inequalities involved, as follows. The modified version $\tilde{K}_h$ is given by setting $\tilde{b}_i^h := b_i^h$ for all $i \in V$, $\tilde{b}_{\tilde{u}}^h = b_u^h$, and $\tilde{b}_0^h = b_0^h - b_u^h$. Now, we observe that

$$\tilde{b}^h(\tilde{V}^0) = \tilde{b}^h(V) + \tilde{b}_0^h + \tilde{b}_{\tilde{u}}^h = b^h(V) + (b_0^h - b_u^h) + b_u^h = b^h(V^0)$$

and therefore the rhs of $\tilde{K}_h$ is identical to the rhs of K_h. The lhs of $\tilde{K}_h$, on the other hand, is equal to the lhs of K_h plus

$$\sum_{i \in V} b_i^h b_u^h (y_{i\tilde{u}} - x_i) + b_u^h (b_0^h - b_u^h) x_{\tilde{u}}.$$

So, if we use $\tilde{K}_1, \ldots, \tilde{K}_r$ in place of $K_1, \ldots, K_r$, the total increase in the lhs is

$$\sum_{h=1}^{r} \lambda_h \sum_{i \in V} b_i^h b_u^h (y_{i\tilde{u}} - x_i) + \sum_{h=1}^{r} \lambda_h b_u^h (b_0^h - b_u^h) x_{\tilde{u}}. \tag{20}$$

From (18), and the equations $y_{i\tilde{u}} = x_i + x_{\tilde{u}}$ for all $i \in n(u) \cup \{u\}$, this increase is equal to

$$\sum_{h=1}^{r} \lambda_h \sum_{i \in n(u) \cup \{0\}} b_i^h b_u^h x_{\tilde{u}}.$$

Finally, from (19), this is equivalent to adding $\alpha_u x_{\tilde{u}}$ to the lhs. □

We note that the proof of Theorem 4 can be easily modified to work for non-rounded psd inequalities as well as rounded ones. Since the non-rounded psd inequalities imply the OR inequalities, this means that a replicated OR inequality is again an OR inequality. However, an analogous result does not hold for the odd clique inequalities. That is, a replicated inequality need not be implied by the odd clique inequalities even when the original inequality is.

4.2. Sequential lifting via stable sets

We now discuss a slightly more sophisticated version of sequential lifting. Let $G = (V, E)$ be a graph and let S be a stable set in G. We say that the graph $\tilde{G}$ is obtained from G by *lifting on S* if $\tilde{G}$ is obtained by adding an extra vertex (u, say) which is adjacent to every member of S and also to every vertex in $n(i)$ for all $i \in S$.

Now let $\alpha x \leq \beta$ be a valid inequality for STAB(G). We construct a valid inequality $\tilde{\alpha} x \leq \beta$ for STAB($\tilde{G}$) by setting $\tilde{\alpha}_i = \alpha_i$ for all $i \in V$ and $\tilde{\alpha}_u = \sum_{i \in S} \alpha_i$. We say that the inequality $\tilde{\alpha} x \leq \beta$ has been obtained from $\alpha x \leq \beta$ by lifting on S.

It is not difficult to show, again following the ideas in Padberg [20], that the lifted inequality is facet-inducing if and only if the original inequality is. (Moreover, replication is the trivial special case obtained when $|S| = 1$.) By iterative application of this lifting operation, we can get a huge variety of interesting inequalities, including for example the odd wheel inequalities. The next theorem shows that we can use rounded psd inequalities to accomplish this more general lifting.

Theorem 5. *Let G be a graph, let S be a stable set in G and let $\alpha x \leq \beta$ be a valid inequality for STAB(G). If $\alpha x \leq \beta$ is implied by the rounded psd inequalities, then the lifted inequality $\tilde{\alpha} x \leq \beta$ for STAB($\tilde{G}$) obtained by lifting on S is also implied by the rounded psd inequalities.*

Proof. We use the notation K_h, λ_h and b_i^h from the proof of Theorem 4. We also define $n(S) := \bigcup_{i \in S} n(i)$. As in that proof, we have:

$$\sum_{h=1}^{r} \lambda_h b_i^h b_j^h = 0, \qquad (\forall \{i, j\} \notin E)$$

and, in particular, we have:

$$\sum_{h=1}^{r} \lambda_h b_i^h b_j^h = 0, \qquad (\forall i \in S,\ j \in V \setminus (\{i\} \cup n(i))). \tag{21}$$

We also again have

$$\sum_{h=1}^{r} \lambda_h \sum_{j \in n(i) \cup \{0\}} b_i^h b_j^h = \alpha_i, \qquad (\forall i \in V) \tag{22}$$

and in particular this holds for every $i \in S$. Now, in this case the modified version $\tilde{K}_h$ is given by setting $\tilde{b}_i^h := b_i^h$ for all $i \in V$, $\tilde{b}_u^h = \sum_{i \in S} b_i^h$, and $\tilde{b}_0^h = b_0^h - \sum_{i \in S} b_i^h$. As before, the rhs of $\tilde{K}_h$ is identical to the rhs of K_h. The lhs of $\tilde{K}_h$, on the other hand, is equal to the lhs of K_h plus

$$\sum_{j \in S} \sum_{i \in V} b_i^h b_j^h (y_{iu} - x_i) + \sum_{j \in S} b_j^h \left(b_0^h - \sum_{i \in S} b_i^h \right) x_u.$$

So, if we use $\tilde{K}_1, \ldots, \tilde{K}_r$ in place of $K_1, \ldots, K_r$, the total increase in the lhs is

$$\sum_{j \in S} \sum_{h=1}^{r} \lambda_h \sum_{i \in V} b_i^h b_j^h (y_{iu} - x_i) + \sum_{j \in S} \sum_{h=1}^{r} \lambda_h b_j^h \left(b_0^h - \sum_{i \in S} b_i^h \right) x_u. \tag{23}$$

From (21), and the fact that S is a stable set, this increase is equal to

$$\sum_{j \in S} \sum_{h=1}^{r} \lambda_h \sum_{i \in n(j) \cup \{0, j\}} b_i^h b_j^h (y_{iu} - x_i) - \sum_{j \in S} \sum_{h=1}^{r} \lambda_h (b_j^h)^2 x_u. \tag{24}$$

Now, using the equations $y_{iu} = x_i + x_u$ for all $i \in S \cup n(S)$, the increase is equal to

$$\sum_{j \in S} \sum_{h=1}^{r} \lambda_h \left(\sum_{i \in n(j) \cup \{0\}} b_i^h b_j^h \right) x_u.$$

Finally, from (22), this is equivalent to adding $\sum_{i \in S} \alpha_i x_u$ to the lhs. □

Again, a similar result can be shown to hold for non-rounded psd inequalities. That is, when this lifting operation is applied to an OR inequality, it yields another OR inequality. However, an analogous result does not hold for the odd clique inequalities. It can be shown that the odd clique inequalities imply the odd wheel inequalities, but in general the odd clique inequalities are of little use for lifting.

5. Conclusions

Our main goal in this paper has been to review and extend the work done by Laurent, Poljak and Rendl [16] on the relationships between various relaxations of the max-cut and stable set problems. Essentially, Laurent, Poljak and Rendl showed that the psd (respectively, triangle) inequalities in the max-cut space imply the non-negativity and OR (respectively, the non-negativity, edge and odd hole) inequalities in the stable set space. We have shown that the psd and triangle inequalities *in combination* imply, in addition, the web and odd wheel inequalities. This result yields, to our knowledge, the first proof that the separation problem for (a generalization of) the web inequalities can be solved in polynomial time.

We have also shown that the *rounded* psd inequalities imply, in addition to the above inequalities, all antiweb inequalities, and a variety of lifted versions of these inequalities. The odd clique inequalities for the cut polytope, on the other hand, have proven to be of limited use for generating valid inequalities for the stable set polytope.

It is worth noting that the psd and rounded psd inequalities which we have used in all our proofs have a special structure. Indeed, in each case we have $b_i \in \{0, -1, +1\}$ for all $i \in V$, and b_0 is an arbitrary integer. (Monique Laurent pointed out to us that the rounded psd inequalities with this structure are all switched hypermetric inequalities.) It is remarkable that this highly structured class of inequalities gives such powerful results for the stable set problem. It would be interesting to know whether these rounded psd inequalities imply any other known valid inequalities for the stable set polytope.

From a computational point of view, it would be interesting to implement and test cutting plane algorithms for the max-cut and stable set problems based on rounded psd inequalities. Encouraging results, obtained using only psd and triangle inequalities, have been reported recently by Fischer et al. [6] (for max-cut) and Gruber & Rendl [10] (for stable set). In principle, cutting plane algorithms based on the more general rounded psd inequalities could produce even better bounds. However, one would need an effective method, either exact or heuristic, to solve the separation problem for the rounded psd inequalities. From the above observation, in the case of the stable set problem it might be worthwhile tailoring the separation method to the case in which $b_i \in \{0, -1, +1\}$ for all $i \in V$. A separation heuristic for a subclass of these inequalities appears in De Simone & Rinaldi [4].

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