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Projection results for vehicle routing

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Abstract. A variety of integer programming formulations have been proposed for Vehicle Routing Problems (VRPs), including the so-called *two-* and *three-index* formulations, the *set partitioning* formulation, and various formulations based on extra variables representing the flow of one or more commodities. Until now, there has not been a systematic study of how these formulations relate to each other. An exception is a paper of Luis Gouveia, which shows that a one-commodity flow formulation of Gavish and Graves yields, by projection, certain 'multistar' inequalities in the two-index space.

We give a survey of formulations for the *capacitated* VRP, and then present various results of a similar flavour to those of Gouveia. In particular, we show that:

– the three-index formulation, augmented by certain families of valid inequalities, gives the same lower bound as the two-index formulation, augmented by certain simpler families of valid inequalities,
– the two-commodity flow formulation of Baldacci et al. gives the same lower bound and the same multistar inequalities as the one-commodity Gavish and Graves formulation,
– a certain non-standard multi-commodity flow formulation, with one commodity per customer, implies by projection certain 'hypotour-like' inequalities in the two-index space,
– the set partitioning formulation implies by projection both multistar and hypotour-like inequalities in the two-index space.

We also briefly look at some other variants of the VRP, such as the *VRP with time windows*, and derive multistar-like inequalities for them. We also present polynomial-time separation algorithms for some of the new inequalities.

Key words. Vehicle routing – Projection – Integer programming

1. Introduction

Vehicle Routing Problems (VRPs) are a fundamental class of problems in Operations Research, with hundreds of published papers on both theory and applications. A variety of models exist and various heuristic and exact optimization algorithms have been proposed. Yet, at present only moderately-sized instances (say, up to 100 customers) can be solved to optimality in a reasonable amount of time.

In this paper, we will be mainly concerned with the simplest vehicle routing problem, in which only capacity constraints are imposed, the so-called *Capacitated Vehicle Routing Problem* (CVRP). Many formulations have been proposed for the CVRP (see Toth and Vigo [38] for surveys). These include the so-called *two-index vehicle flow* formulation, in which there is a variable for every arc of the graph, the *three-index vehicle flow* formulation, in which there is a variable for every arc-vehicle combination, *commodity*

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flow formulations, in which additional flow variables are added to the two- or three-index formulations, and *set partitioning* formulations, in which there is one variable for every feasible vehicle route. At present, the most successful algorithms for the CVRP are based on either two-index vehicle flow or set partitioning formulations, or a combination of the two (Lysgaard et al. [32], Fukasawa et al. [16]).

Until now, very little work has been done on how these different formulations relate to one another. In particular, one may ask whether the linear programming (LP) relaxation of one formulation dominates that of another, or whether two LP relaxations give the same lower bound. One useful method for resolving such questions, *polyhedral projection*, has been applied to other combinatorial optimization problems, such as the *Traveling Salesman* and *Steiner Tree* problems (see for example Padberg and Sung [35] and Goemans and Myung [19], respectively). For the CVRP, however, little has been done. We are aware only of the 1995 paper of Gouveia [20], in which he showed that a one-commodity flow formulation due to Gavish and Graves [17] implied a class of valid inequalities in the standard two-index space which have now come to be known as *generalized large multistar* (GLM) inequalities.

In this paper, we give many results of a similar flavour to that of Gouveia. We take several extended formulations of the CVRP and, for each one, derive inequalities in the two-index space which are implied by the extra flow or set partitioning variables. In particular, we show that:

- The three-index vehicle flow formulation, augmented by certain families of valid inequalities, gives the same lower bound as the two-index vehicle flow formulation, augmented by certain simpler families of valid inequalities.
- The Gavish and Graves one-commodity flow formulation and the two-commodity flow formulation of Baldacci, Mingozzi and Hadjiconstantinou give the same lower bound. Both of them yield by projection the GLM inequalities in the two-index space.
- A certain non-standard multi-commodity flow formulation, with one commodity per customer, implies by projection certain 'hypotour-like' inequalities in the two-index space.
- The set partitioning formulation implies by projection the so-called *knapsack large multistar* (KLM) inequalities of Letchford et al. [31], which generalize the GLM inequalities. They also imply the hypotour-like inequalities.
- If we allow non-elementary routes in the set partitioning formulation, then we obtain by projection a class of inequalities which is intermediate between the KLM and GLM inequalities.

We also give polynomial-time *separation algorithms* (i.e., algorithms for detecting when an inequality is violated) for some of the inequalities discussed. These can be used to avoid the use of extra variables and work in the standard two-index space. Finally, we also briefly examine some other VRPs, such as the *VRP with time windows* (VRPTW) and the *VRP with pickups and deliveries* (VRPPD).

The structure of the paper is as follows. In Section 2, we formally define the CVRP, review the two- and three-index vehicle flow formulations, and present some simple new results for the latter formulation. In Section 3, the longest, we look at extended formulations of the CVRP based on commodity flows, and their projections. In Section 4, we address the set partitioning formulation of the CVRP, and its projection. In Section 5, we turn our attention to the VRPTW. We consider a certain commodity flow formulation of the VRPTW and its projection. In Section 6, we briefly look at flow formulations of some other variants of the VRP, like the VRPPD. Finally, we give some concluding comments in Section 7.

The following conventions will be used throughout the paper. For simplicity, we will work with the directed version, in which the cost of travel in one direction can differ from that of travel in the opposite direction. Our results can be easily adapted to the undirected version. We assume that the problems are defined on a complete directed graph G, with vertex set $V = \{0, \dots, n\}$ and arc set A. Vertex 0 represents the depot; the other vertices represent customers. The set of customer vertices is denoted by $V_c := V \setminus \{0\}$. The cost of travel from vertex i to vertex j is denoted by c_{ij}. A fleet of identical vehicles is available, which all depart from, and return to, the depot. Typically, the number of vehicles available is assumed to be unlimited, but there may be an upper bound, say m, on the number of available vehicles. Sometimes, m is set to the minimum possible subject to the specified constraints. In each case, a solution is a set of feasible vehicle routes of minimum total cost, where each vehicle used leaves from and returns to the depot and each customer is visited exactly once (that is, *split deliveries* are not allowed).

2. The CVRP: vehicle flow formulations

In the CVRP, each vehicle has capacity Q (> 0), each customer i has a demand q_i (> 0), and no vehicle can serve a set of customers whose demand exceeds Q. A large number of formulations have been proposed for the CVRP. In this section we examine *vehicle flow* formulations in which variables represent the movements of vehicles from one vertex to another, but do not keep track of the load delivered at each point. These include the *two-index* vehice flow formulation, in which the variables are indexed according to the start and end vertices only, and the *three-index* vehicle flow formulation, in which the variables have a third index which distinguishes one vehicle from another.

2.1. The two-index formulation

We now present the two-index vehicle flow formulation, which is normally attributed to Laporte and Nobert [27, 28]. For all $(i, j) \in A$, define a binary variable x_{ij}, taking the value 1 if some vehicle travels from i to j, 0 otherwise. For any $S \subset V$, let $\delta^+(S)$ (respectively, $\delta^-(S)$) denote the set of arcs (i, j) with $i \in S$, $j \in V \setminus S$ (respectively, with $i \in V \setminus S$, $j \in S$). Given some $F \subset A$, let $x(F)$ denote $\sum_{(i,j)\in F} x_{ij}$. Finally, for any set of customers $S \subset V_c$, let $q(S) = \sum_{i\in S} q_i$. Note that $\lceil q(S)/Q \rceil$ is a lower bound on the number of vehicles needed to service the customers in S. Then the formulation is:

$$\text{Minimise} \sum_{(i,j)\in A} c_{ij} x_{ij} \tag{1}$$

subject to

$$x(\delta^+(\{i\})) = 1 \qquad (i \in V_c) \tag{2}$$

$$x(\delta^-(\{i\})) = 1 \qquad (i \in V_c) \tag{3}$$

$$x(\delta^+(\{0\})) = x(\delta^-(\{0\})) \le m \tag{4}$$

$$x(\delta^+(S)) \ge \lceil q(S)/Q \rceil \qquad (S \subseteq V_c) \tag{5}$$

$$x_{ij} \in \{0, 1\} \qquad ((i, j) \in A). \tag{6}$$

The *out-degree equations* (2) and the *in-degree equations* (3) ensure that vertices are visited exactly once. The constraint (4) ensures that every vehicle that leaves the depot returns. The constraints (5), called *rounded capacity* (RC) inequalities, prevent the existence of infeasible routes, and also have the side-effect of preventing subtours. Finally, (6) are the integrality conditions on the x-variables.

Many classes of valid inequalities have been presented for the integer polytope associated with the formulation (2)–(6). There is not space to review them all here, and the reader is referred to the survey by Naddef and Rinaldi [34]. We will, however, be interested in the following inequalities:

– The *fractional capacity* (FC) inequalities:

$$x(\delta^+(S)) \ge \frac{q(S)}{Q} \qquad (S \subseteq V_c). \tag{7}$$

– The *subtour elimination* (SE) inequalities:

$$x(\delta^+(S)) \ge 1 \qquad (S \subseteq V_c). \tag{8}$$

– The *generalized large multistar* (GLM) inequalities (see Gouveia [20]):

$$x(\delta^+(S)) \ge \frac{q(S)}{Q} + \frac{1}{Q} \sum_{i \in S} \sum_{j \in V_c \setminus S} q_j (x_{ij} + x_{ji}). \tag{9}$$

– The *knapsack large multistar* (KLM) inequalities (see Letchford et al. [31]):

$$x(\delta^+(S)) \ge \frac{1}{\beta} \sum_{i \in S} \Big(\alpha_i + \sum_{j \in V_c \setminus S} \alpha_j (x_{ij} + x_{ji}) \Big), \tag{10}$$

where $\alpha, \beta > 0$ are such that the inequality $\sum_{i \in V_c} \alpha_i y_i \le \beta$ is valid for the 0-1 knapsack polytope:

$$KP(Q, q) := \text{conv} \left\{ y \in \{0, 1\}^{|V_c|} : \sum_{i \in V_c} q_i y_i \le Q \right\}.$$

– The *hypotour* inequalities (Augerat [3]):

$$x(A \setminus F) \ge 1, \tag{11}$$

where $F \subset A$ is a set of arcs such that there is no feasible CVRP solution which uses only arcs in F.

Obviously, the RC inequalities dominate the FC and SE inequalities, and the GLM inequalities dominate the FC inequalities. It is also not difficult to see that the KLM inequalities include the GLM and SE inequalities as special cases, and that the hypotour inequalities generalize the SE inequalities. In general, there are no other dominance relations.

2.2. *The three-index formulation*

When an upper bound m on the number of vehicles is available, the CVRP can also be formulated as follows. Let x_{ij}^k take the value 1 if vehicle k travels from i to j and 0 otherwise. Also let y_i^k take the value 1 if vehicle k visits vertex i, 0 otherwise. We use the notation $x^k(F)$ to denote $\sum_{(i,j)\in F} x_{ij}^k$ for $F \subseteq A$ and $k \in \{1, \ldots, m\}$. Then, a model for the CVRP is:

$$\text{Minimize} \sum_{(i,j)\in A} c_{ij} \sum_{k=1}^{m} x_{ij}^k \tag{12}$$

subject to

$$x^k(\delta^+(\{i\})) = x^k(\delta^-(\{i\})) = y_i^k \qquad (k \in \{1, \ldots, m\}, i \in V) \tag{13}$$

$$x^k(\delta^+(S)) \geq y_i^k \qquad (k \in \{1, \ldots, m\}, S \subset V_c, i \in S) \tag{14}$$

$$\sum_{i \in V_c} q_i y_i^k \leq Q \qquad (k \in \{1, \ldots, m\}) \tag{15}$$

$$\sum_{k=1}^{m} y_i^k = 1 \qquad (i \in V_c) \tag{16}$$

$$x_{ij}^k \in \{0, 1\} \qquad (k \in \{1, \ldots, m\}, (i, j) \in A) \tag{17}$$

$$y_i^k \in \{0, 1\} \qquad (k \in \{1, \ldots, m\}, i \in V). \tag{18}$$

This formulation, which we call the *three-index* vehicle flow formulation, is based on those of Fisher and Jaikumar [15] and Fischetti et al. [14]. It is possible to eliminate the $(y^1, \ldots, y^m)$-variables, but we keep them for simplicity.

The LP relaxation of (12)–(18) is very weak in general. (See Proposition 2 below.) To strengthen the relaxation, a variety of valid inequalities are available. One set of valid inequalities arises from a consideration of the so-called *multiple knapsack polytope* (Ferreira et al. [13]).

Proposition 1. *The projection of the integer polytope associated to the three-index formulation onto* $(y^1, \ldots, y^m)$*-space is the multiple knapsack polytope:*

$$\textit{conv}\,\{(y^1, \ldots, y^m) \in \{0, 1\}^{n\times m} : (15), (16) \textit{ and } (18) \textit{ hold}\}.$$

Therefore, any valid inequality for the multiple knapsack polytope [13] yields valid inequalities for the three-index formulation that only involve the y-variables. We call such inequalities *multiple knapsack inequalities*. The inequalities (15) and (16) themselves are trivial multiple knapsack inequalities.

The following proposition, however, indicates that multiple knapsack inequalities are likely to be of little use as cutting planes.

Proposition 2. *The lower bound obtained by solving the LP relaxation of the three-index formulation (13)–(18), even when it is augmented by all possible multiple knapsack inequalities, is equal to the lower bound obtained by solving the LP relaxation of the two-index formulation given by the degree equations (2)–(4), subtour elimination inequalities (8), and the trivial bounds* $0 \leq x_{ij} \leq 1$.

Proof. First we show that, given any vector x in the two-index space satisfying (2)–(4), (8) and the trivial bounds, we can construct a vector $(x^1, \dots, x^m, y^1, \dots, y^m)$ in the three-index space, satisfying (13)–(16), non-negativity, and all multiple knapsack inequalities, with the same cost. This is done by setting $x_{ij}^k = x_{ij}/m$ for $(i, j) \in A$ and $k = 1, \dots, m$ and setting $y_i^k = 1/m$ for $i \in V_c$ and $k = 1, \dots, m$. Note that the resulting vector $(y^1, \dots, y^m)$, with all components equal to $1/m$, lies at the centre of the multiple knapsack polytope and therefore satisfies all possible multiple knapsack inequalities.

Similarly, given any vector $(x^1, \dots, x^m, y^1, \dots, y^m)$ in the three-index space satisfying (13)–(16) and non-negativity, we can construct a vector x in the two-index space, satisfying (2)–(4), (8) and the trivial bounds, with the same cost. This is done by setting $x_{ij} = \sum_{k=1}^m x_{ij}^k$ for $(i, j) \in A$. □

Another, more useful class of valid inequalities for the three-index formulation arises from the following trivial observation.

Proposition 3. *If we add the two-index variables to the three-index formulation, via the identities $x_{ij} = \sum_{k=1}^m x_{ij}^k$ for all $(i, j) \in A$, and project the resulting integer polytope into the two-index space, we obtain the integer polytope associated with the two-index formulation.*

Therefore, if $\alpha x \leq \beta$ is any valid inequality for the two-index formulation, the inequality

$$\sum_{(i,j)\in A} \alpha_{ij} \sum_{k=1}^m x_{ij}^k \leq \beta$$

is valid for the three-index formulation. We call such inequalities *aggregated* inequalities.

For example, the *aggregated RC inequalities* take the form

$$\sum_{k=1}^m x^k(\delta^+(S)) \geq \lceil q(S)/Q \rceil; \tag{19}$$

and the *aggregated KLM inequalities* take the form:

$$\sum_{k=1}^m x^k(\delta^+(S)) \geq \frac{1}{\beta} \sum_{i\in S} \Big(\alpha_i + \sum_{j\in V_c\setminus S} \alpha_j \sum_{k=1}^m (x_{ij}^k + x_{ji}^k)\Big). \tag{20}$$

Unlike the multiple knapsack inequalities, the addition of aggregated inequalities to the LP can lead to an increase in the lower bound. Indeed, if we were to add *all possible* aggregated inequalities to the LP relaxation of (13)–(18), the resulting lower bound would be equal to the cost of the optimal integer solution. This follows since, if $(x^1, \dots, x^m, y^1, \dots, y^m)$ in three-index space satisfies all aggregated inequalities, then by setting $x_{ij} = \sum_{k=1}^m x_{ij}^k$ we obtain a feasible solution x in two-index space, with the same cost, which satisfies all valid two-index inequalities.

Of course, we do not have a characterization of all aggregated inequalities, because this would mean having a complete linear description of the integer polytope associated to the two-index formulation. Such a characterization is unlikely to exist unless $\mathcal{NP} =$ co-$\mathcal{NP}$ (Papadimitriou and Yanakakis [36]).

A third family of valid inequalities arises from a consideration of feasible routes for a *single vehicle*.

Proposition 4. *If we relax the equations (16) in, say, a Lagrangean fashion, the three-index formulation decomposes into m independent identical subproblems. Each subproblem corresponds to finding a* capacitated circuit *(i.e., a circuit passing through a set of vertices whose total demand does not exceed Q) passing through the depot.*

Therefore, any valid inequality for the *capacitated circuit polytope* yields valid inequalities for the three-index formulation which involve only a single $k \in \{1, \dots, m\}$. We call these *one-vehicle* inequalities. The constraints (13)–(15) themselves, and the bounds $0 \le x^k_{ij} \le 1$ and $0 \le y^k_i \le 1$, are trivial examples of one-vehicle inequalities. Other inequalities for the capacitated circuit polytope can be found in Bauer et al. [6] and Bixby et al. [7].

As a non-trivial example, the following *one-vehicle KLM* inequalities

$$x^k(\delta^+(S)) \ge \frac{1}{\beta} \sum_{i \in S} \big(\alpha_i y^k_i + \sum_{j \in V_c \setminus S} \alpha_j (x^k_{ij} + x^k_{ji})\big) \tag{21}$$

are one-vehicle inequalities. (They are easily shown to be valid for all $k \in \{1, \dots, m\}$.)

Note that the one-vehicle KLM inequalities (21) include the inequalities (14) as a special case. They also dominate the aggregated KLM inequalities (20). On the other hand, the aggregated RC inequalities (19) are not in general dominated by one-vehicle inequalities.

The next theorem shows that, in a certain sense, it is preferable to use aggregated inequalities rather than multiple knapsack and one-vehicle inequalities.

Theorem 1. *Let $\mathcal{F}_1$ be any class of aggregated inequalities for the three-index formulation and let $\mathcal{F}_2$ be any class of one-vehicle inequalities. Suppose we have separation algorithms for both $\mathcal{F}_1$ and $\mathcal{F}_2$. Then there exists a class $\mathcal{F}'_2$ of aggregated inequalities, and a separation algorithm for $\mathcal{F}'_2$, such that the lower bound obtained using the inequalities in $\mathcal{F}_1$ and $\mathcal{F}'_2$ is equal to the lower bound obtained using the inequalities in $\mathcal{F}_1$ and $\mathcal{F}_2$, together with all multiple knapsack inequalities.*

Proof. We construct the family $\mathcal{F}'_2$ as follows. Given any inequality in $\mathcal{F}_2$ of the form

$$\sum_{(i,j) \in A} \alpha_{ij} x^k_{ij} \ge \sum_{i \in V_c} \beta_i y^k_i + \gamma,$$

we put the aggregated inequality

$$\sum_{k=1}^{m} \sum_{(i,j) \in A} \alpha_{ij} x^k_{ij} \ge \sum_{i \in V_c} \beta_i + m\gamma$$

into $\mathcal{F}'_2$. It is obvious that, if a point $(x^1, \dots, x^m, y^1, \dots, y^m)$ in three-index space satisfies all inequalities in $\mathcal{F}_2$, together with all multiple knapsack inequalities, then it also satisfies the inequalities in $\mathcal{F}_2'$. Moreover, if a point $(x^1, \ldots, x^m, y^1, \ldots, y^m)$ in three-index space satisfies the inequalities in $\mathcal{F}_1$ and $\mathcal{F}_2'$, we can construct another point $(\tilde{x}^1, \ldots, \tilde{x}^m, \tilde{y}^1, \ldots, \tilde{y}^m)$, with the same objective value, by setting $\tilde{x}_{ij}^k = \sum_{r=1}^m x_{ij}^r/m$ for all i, j, k, and $\tilde{y}_i^k = \sum_{r=1}^m y_i^r/m = 1/m$ for all i, k. Note that $(\tilde{x}^1, \ldots, \tilde{x}^m, \tilde{y}^1, \ldots, \tilde{y}^m)$ satisfies all inequalities in $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_2'$. Also note that $(\tilde{y}^1, \ldots, \tilde{y}^m)$ lies in the centre of the multiple knapsack polytope and therefore $(\tilde{x}^1, \ldots, \tilde{x}^m, \tilde{y}^1, \ldots, \tilde{y}^m)$ also satisfies all possible multiple knapsack inequalities.

It is easy to see that we can solve the separation problem for $\mathcal{F}_2'$ as follows. Given a point $(x^1, \ldots, x^m, y^1, \ldots, y^m)$ to be separated, we construct the point $(\tilde{x}^1, \ldots, \tilde{x}^m, \tilde{y}^1, \ldots, \tilde{y}^m)$ as before, and give it as input to the separation algorithm for $\mathcal{F}_2$. □

Theorem 1 indicates that, given any cutting plane algorithm based on the three-index vehicle flow formulation, one can construct a cutting plane algorithm based on the two-index vehicle flow formulation which gives lower bounds of the same quality. Moreover, the two-index version, having far fewer variables, is likely to run much faster in practice. Our conclusion is that the two-index formulation is preferable to the three-index one. (This only holds when, as we are assuming in this paper, the vehicles are identical. If the vehicle fleet is mixed, then a three-index formulation may be necessary.)

We end this subsection by noting that there are also other valid inequalities for the three-index formulation which appear to be non-redundant, but which do not lie in any of the classes discussed. (That is, they are neither multiple knapsack, nor aggregated, nor one-vehicle type.) Here is a simple example.

Proposition 5. *The following generalized capacity inequalities are valid for the three-index formulation:*

$$\sum_{k \in K} x^k(\delta^+(S)) \geq \lceil q(S)/Q \rceil - m + |K|, \tag{22}$$

for any $S \subset V_c$ and any $K \subseteq \{1, \ldots, m\}$ such that $|K| > m - \lceil q(S)/Q \rceil$.

Proof. Since at least $\lceil q(S)/Q \rceil$ vehicles must enter S, at most $m - \lceil q(S)/Q \rceil$ vehicles can *not* enter S. Thus, if we select any $|K|$ vehicles with $|K| > m - \lceil q(S)/Q \rceil$, at least $\lceil q(S)/Q \rceil - m + |K|$ of them must enter S. □

The inequalities (22) are similar to a class of valid inequalities for the *Capacitated Arc Routing Problem* presented by Eglese and Letchford [12]. They reduce to aggregated RC inequalities when $|K| = m$. In other cases they do not lie in any of the above classes. Moreover, when $\lceil q(S)/Q \rceil = m$ and $|K| = 1$, we obtain the inequalities $x^k(\delta^+(S)) \geq 1$ for all k. Although these involve only a single vehicle index k, they are not 'one-vehicle' inequalities in the sense given above. (An arbitrary capacitated circuit need not visit any vertex in S. The reason each vehicle needs to visit at least one vertex in S is the presence of the upper bound m.)

3. The CVRP: commodity flow formulations

The two and three-index vehicle flow models have an exponential number of constraints to enforce connectivity. Several alternative models impose this requirement by using a set of continuous variables representing the flow of one or more ‘commodities’ between the depot and the customers. We summarize here some of these models and analyze the projection of these variables in the two-index vehicle flow space.

3.1. The one-commodity flow formulation

This formulation was presented by Gavish and Graves [17], and subsequently explored by Gouveia [20]. It is obtained by replacing the constraints (5) in the two-index vehicle flow formulation by the following variables and constraints.

Define variables f_{ij} for all $i, j \in V$, with the following interpretation: if $x_{ij} = 0$, then f_{ij} is set to 0. However, if $x_{ij} = 1$, then f_{ij} represents the amount of load delivered by the vehicle when it leaves vertex i. As before, we use $f(F)$ instead of $\sum_{(i,j)\in F} f_{ij}$ for all $F \subseteq A$. Then, add the constraints:

$$f(\delta^+(\{i\})) = f(\delta^-(\{i\})) + q_i \qquad (i \in V_c) \tag{23}$$

$$0 \le f_{ij} \le Q x_{ij} \qquad ((i, j) \in A). \tag{24}$$

The constraints (23) ensure that q_i units of flow are delivered at vertex i. With the convention $q_0 := -\sum_{i\in V_c} q_i$, the equality also holds when $i = 0$, and it means that $-q_0$ units of flow are collected at the depot. The constraints (24) are bounds on the f-variables. Even if it is not necessary, one can impose $f(\delta^+(\{0\})) = 0$. For other VRP variants this inequality could be non-redundant.

Gouveia [20] showed that, if we project the polyhedron associated to the LP relaxation of the above formulation onto the space defined by the x-variables, then we obtain the fractional capacity (FC) inequalities mentioned in Subsection 2.1. Gouveia [20] also showed that, if instead of the bounds (24), we use the stronger bounds

$$\max\{0, q_i\} x_{ij} \le f_{ij} \le (Q - \max\{0, q_j\}) x_{ij} \qquad ((i, j) \in A), \tag{25}$$

we obtain after projection the GLM inequalities (9).

To aid clarity, we give a proof of this result in a more general way that will be used later to derive new inequalities for other VRP variants.

Theorem 2. *Let $\underline{q}_{ij}$ and $\overline{q}_{ij}$ be known lower and upper bounds, respectively, on the load of the vehicle when traversing the arc (i, j). Suppose the flow bounds (24) are replaced by stronger bounds*

$$\underline{q}_{ij} x_{ij} \le f_{ij} \le \overline{q}_{ij} x_{ij} \qquad ((i, j) \in A). \tag{26}$$

Then the inequalities (23) and (26) imply the following multistar-like inequalities:

$$\sum_{i\in S} \sum_{j\in V\setminus S} (\overline{q}_{ij} x_{ij} - \underline{q}_{ji} x_{ji}) \ge q(S) \qquad (S \subset V). \tag{27}$$

Proof. Sum the inequalities (23) over all $i \in S$ and simplify, to obtain

$$f(\delta^+(S)) = f(\delta^-(S)) + q(S).$$

Using the inequalities (26) we obtain

$$\sum_{i\in S}\sum_{j\in V\setminus S}\overline{q}_{ij}x_{ij} \ge \sum_{i\in V\setminus S}\sum_{j\in S}\underline{q}_{ij}x_{ij} + q(S).$$

□

The classical max-flow/min-cut theorem implies that no other inequalities can be obtained by this projection.

Obviously, if $\underline{q}_{ij} > \overline{q}_{ij}$ then we can delete arc (i, j) from A (or, equivalently, fix $x_{ij} = 0$).

An example of an application of Theorem 2 is the following. In some practical applications, the number of vehicles available is fixed at the minimum possible, which is usually equal to $m = \lceil q(V_c)/Q\rceil$. In this case, each vehicle must deliver a load of at least $q_{\min} = q(V_c) - (m-1)Q$. Thus, we can set $\underline{q}_{i0} := \max\{q_i, q_{\min}\}$ for $i = 1, \dots, n$.

Obviously, the inequalities (27) are redundant when constraints of the form (23) and (26) are present. However, if we wish to avoid the use of the f-variables, and work in the x-space, then inequalities like (27) can be useful (see, for example, Letchford et al. [31] using the GLM inequalities).

To use the inequalities (27) as cutting planes in the x-space, we require a *separation algorithm*, i.e., a procedure for detecting when the inequalities are violated by a given fractional LP solution x^* (Grötschel et al. [21]). It follows from Gouveia's projection argument, and the ellipsoid method, that a polynomial-time exact separation algorithm exists for the inequalities (27). Indeed, an inequality (27) is valid if and only if there is a flow vector f satisfying (23) and

$$\underline{q}_{ij}x^*_{ij} \le f_{ij} \le \overline{q}_{ij}x^*_{ij} \qquad ((i, j) \in A).$$

This is a feasible flow problem with lower and upper bounds, which can be reduced to a max-flow/min-cut problem (see Ahuja et al. [1]). Here is an explicit construction.

Theorem 3. *The separation problem for the multistar-like inequalities (27) can be solved via a max-flow/min-cut problem.*

Proof. First, we re-write the inequality (27) as:

$$\sum_{i\in S}\sum_{j\in V\setminus S}(\overline{q}_{ij} - \underline{q}_{ij})x_{ij} + \sum_{i\in V\setminus S}\sum_{j\in V\setminus\{i\}}(\underline{q}_{ji}x_{ji} - \underline{q}_{ij}x_{ij}) \ge q(S).$$

Then we re-write the inequality so that the left-hand-side coefficients are non-negative and the right-hand-side value is independent of S:

$$\sum_{i\in S}\sum_{j\in V\setminus S}(\overline{q}_{ij} - \underline{q}_{ij})x_{ij} + \sum_{i\in V\setminus S}\sum_{j\in V\setminus\{i\}}\underline{q}_{ji}x_{ji} + \sum_{i\in S}\sum_{j\in V\setminus\{i\}}\underline{q}_{ij}x_{ij} + \sum_{i\in V\setminus S}q_i$$
$$\ge \sum_{i\in V}q_i + \sum_{i\in V}\sum_{j\in V\setminus\{i\}}\underline{q}_{ij}x_{ij}. \qquad (28)$$

Now construct a graph $G' = (V', A')$ with $V' = V \cup \{u, u'\}$ and $A' = A \cup \{(u, i) : i \in V\} \cup \{(i, u') : i \in V\}$. Give each $(i, j) \in A$ a weight of

$$(\overline{q}_{ij} - \underline{q}_{ij})x^*_{ij}.$$

Then increase the weight of arcs (u, i) by

$$\sum_{j \in V \setminus \{i\}} \underline{q}_{ij} x^*_{ij}.$$

Finally, give each arc (i, u') the weight

$$q_i + \sum_{j \in V \setminus \{i\}} \underline{q}_{ji} x^*_{ji}.$$

It can be readily checked that a (u, u')-cut in G' has weight equal to the left-hand side of (28), where S is the set of vertices on the same shore as u'. Thus, a multistar-like inequality is violated if and only if the minimum weight (u, u')-cut in G' has a value less than the right hand side of (28). This solves the separation problem. □

This theorem generalizes results on the FC inequalities by Naddef and Rinaldi [34], and on the GLM inequalities by Blasum and Hochstättler [8] and by Letchford et al. [31].

Theorem 4. *There is a one-to-one correspondence between (fractional) solutions of the LP-relaxation of model (1)–(6) with (5) replaced by (9), and the (fractional) solutions of the LP-relaxation of model (1)–(6) with (5) replaced by (23) and (25).*

Proof. Given a (fractional) solution x^* of the two-index model, we aim to find a flow f^* such that (x^*, f^*) is a (fractional) solution of the one-commodity flow formulation, and viceversa. This is a consequence of the max-flow/min-cut problem on a network where $\underline{q}_{ij} x^*_{ij}$ and $\overline{q}_{ij} x^*_{ij}$ are the lower and upper arc capacities, respectively, and a vertex i is a sink if $q_i < 0$ or a source if $q_i > 0$. □

3.2. The two-commodity flow formulation

Baldacci et al. [5] present a two-commodity flow formulation for the undirected VRP. To show it here, it is convenient to represent the depot by two vertices, 0 and $n + 1$. An upper bound m on the number of vehicles is assumed. One flow will start at vertex 0 with $q(V_c)$ units, delivering q_i units when visiting customer i, and arriving at vertex $n + 1$ with 0 units. Another flow will start at vertex $n + 1$ with mQ units, delivering q_i units when visiting customer i, and arriving at vertex 0 with $mQ - q(V_c)$ units. The two flows through each arc (i, j) can be represented by one continuous variable f_{ij}, as done in the previous one-commodity flow formulation. The idea is that, if the vehicle travels from i to j, then f_{ij} gives the vehicle load along the arc and f_{ji} gives the vehicle residual capacity. That is, $f_{ij} + f_{ji} = Q$.

The set of constraints for a directed VRP are the following:

$$x(\delta^+(\{i\})) = 1 \qquad (i \in V_c) \tag{29}$$
$$x(\delta^-(\{i\})) = 1 \qquad (i \in V_c) \tag{30}$$
$$x(\delta^+(\{0\})) = x(\delta^-(\{n+1\})) \tag{31}$$
$$x(\delta^-(\{0\})) = x(\delta^+(\{n+1\})) = 0 \tag{32}$$
$$f(\delta^+(\{i\})) = f(\delta^-(\{i\})) + 2q_i \qquad (i \in V_c) \tag{33}$$
$$f(\delta^+(\{0\})) = q(V) \tag{34}$$
$$f(\delta^-(\{0\})) = mQ - q(V) \tag{35}$$
$$f(\delta^+(\{n+1\})) = mQ \tag{36}$$
$$f(\delta^-(\{n+1\})) = 0 \tag{37}$$
$$f_{ij} + f_{ji} = Q(x_{ij} + x_{ji}) \qquad ((i,j) \in A) \tag{38}$$
$$0 \le f_{ij} \qquad ((i,j) \in A) \tag{39}$$
$$x_{ij} \in \{0,1\} \qquad ((i,j) \in A). \tag{40}$$

By replacing (39) by

$$q_i x_{ij} \le f_{ij} \qquad ((i,j) \in A), \tag{41}$$

then it also follows that $f_{ij} \le (Q - q_j)x_{ij}$.

The next theorem shows that there is no benefit obtained by using two commodities instead of one, which is in accordance with similar results known for the TSP (see, e.g., Langevin et al. [25]).

Theorem 5. *The lower bound obtained from the LP relaxation of the two-commodity flow formulation with (39) (respectively, with (41)) coincides with the lower bound obtained from the LP relaxation of the one-commodity flow formulation with (24) (respectively, with (25)).*

Proof. Let LP-F1 and LP-F2 be the linear programming relaxation of the one-commodity flow formulation and of the two-commodity flow formulation, respectively. We will now prove that there is a one-to-one correspondence between (fractional) solutions of LP-F1 and LP-F2, each pair of solutions with the same objective value.

On one side, given a (fractional) solution vector (x^*_{ij}, f^*_{ij}) of LP-F2, it is possible to build a (fractional) solution vector of LP-F1 by simply increasing f^*_{ij} with $x^*_{ji}Q - f^*_{ji}$. Theorem 2 shows that (41) implies that vector x^* satisfies the GLM inequalities.

On the other side, given a (fractional) solution vector (x^*_{ij}, f^*_{ij}) of LP-F1, it is possible to build a (fractional) solution vector of LP-F2 by simply extending this vector with f^* defined by a solution of equations (33)–(38) and inequalities (41). Vector f^* represents two flows, one going out from 0 with $q(V)$ units and going into each vertex $i \in V_c$ to deliver q_i units. The existence of this flow is guarantee by the max-flow/min-cut theorem. Thus $f^*_{ij} \le Qx^*_{ij}$. Additionally, each value f^*_{ij} can be augmented to satisfy (38), using the fact that in an optimal flow $f^*_{ij} = 0$ when $f^*_{ji} > 0$. □

3.3. A three-index one-commodity flow formulation

It is possible to produce a three-index version of the one-commodity flow formulation of Gavish and Graves [17]. This is done by taking the formulation (12)–(18), and replacing (14) and (15) with:

$$f^k(\delta^+(\{i\})) - f^k(\delta^-(\{i\})) = q_i y_i^k \qquad (k = 1, \ldots, m;\, i \in V_c) \tag{42}$$

$$0 \le f_{ij}^k \le Q x_{ij}^k \qquad (k = 1, \ldots, m;\, (i, j) \in A). \tag{43}$$

Using a similar method to that used in Theorem 2, it can be shown that, if this is done, we obtain by projection onto the three-index space the following *one-vehicle FC* inequalities:

$$x^k(\delta^+(S)) \ge \frac{1}{Q} \sum_{i \in S} q_i y_i^k,$$

which are valid for all $S \subseteq V_c$ and all $k \in \{1, \ldots, m\}$. Notice that the one-vehicle FC inequalities are a disaggregated version of the FC inequalities.

By analogy with the two-index case, the bounds $0 \le f_{ij}^k \le Q x_{ij}^k$ can be replaced with the stronger bounds $q_i x_{ij}^k \le f_{ij}^k \le (Q - q_j) x_{ij}^k$. If this is done, we obtain by projection the stronger *one-vehicle GLM* inequalities:

$$x^k(\delta^+(S)) \ge \frac{1}{Q} \sum_{i \in S} \Big(q_i y_i^k + \sum_{j \in V_c \setminus S} q_j (x_{ij}^k + x_{ji}^k) \Big). \tag{44}$$

Notice that the one-vehicle GLM inequalities are a disaggregated version of the GLM inequalities, and are also a special case of the one-vehicle KLM inequalities (21).

Theorem 1 shows that the LP relaxation of this three-index one-commodity flow formulation gives the same bound as we obtain from the Gavish and Graves formulation. Therefore, although this formulation may be of theoretical interest, there seems to be little practical use for it. (However, as we mentioned near the end of Subsection 2.2, three-index formulations can be useful when the vehicle fleet is mixed.)

3.4. Multi-commodity flow formulations

Interesting *multi-commodity flow* formulations arise if we take the two-index vehicle flow formulation (1)–(6), and add extra variables representing *one commodity per customer*. One such formulation, based on another unpublished paper by Gavish and Graves [18], was explored by Laporte and Nobert [28] and Baldacci et al. [5]. In this formulation, a flow variable f_{jl}^i is defined for all $i \in V_c$ and for all $(j, l) \in A$, representing the load on the vehicle *destined for* vertex i when the vehicle travels from j to l. The formulation is then obtained by removing constraints (5) and replacing them with:

$$
\begin{aligned}
f^i(\delta^-(\{i\})) - f^i(\delta^+(\{i\})) &= q_i && (i \in V_c)\\
f^i(\delta^+(\{0\})) - f^i(\delta^-(\{0\})) &= q_i && (i \in V_c)\\
f^i(\delta^+(\{j\})) - f^i(\delta^-(\{j\})) &= 0 && (i \in V_c;\ j \in V_c \setminus \{i\})\\
f^i_{ji} &= q_i x_{ji} && (i \in V_c;\ j \neq i)\\
0 \le f^i_{jl} &\le q_i x_{jl} && (i \in V_c;\ (j,l) \in A)\\
\sum_{j \in V_c \setminus \{i\}} f^j(\delta^+(\{i\})) &\le Q - q_i && (i \in V_c).
\end{aligned}
$$

We do not have a full characterisation of the inequalities in the two-index space which are obtained from this formulation by projection. However, as noted by Laporte and Nobert [28], this formulation implies not only the GLM inequalities (9), but also the SE inequalities (8). Thus, this formulation is provably stronger than the one- and two-commodity formulations presented above, but at the expense of a large number of variables and constraints.

We will now present a slightly different version of this formulation, which has some interesting connections to the *hypotour* inequalities mentioned in Subsection 2.1. To motivate this, we need to say something about how hypotour inequalities are used in practice. Let x^* be a fractional solution to the two-index vehicle flow formulation, and consider the *support graph* $G^* = (V, A^*)$, where $A^* := \{(u,v) : x^*_{uv} > 0\}$. Suppose that, for some i, there does not exist a directed circuit in G^*, from the depot to i and back, visiting a set of vertices with total demand not exceeding Q. Then $x(A \setminus A^*) \ge 1$ is a violated hypotour inequality. (This observation was used in Augerat et al. [4], Blasum and Hochstättler [8] and Lysgaard et al. [32] to devise heuristics for the separation of hypotour inequalities.)

Thus, for any fixed $i \in V_c$, the support graph A^* should contain a capacitated circuit passing through the depot and i. This leads us to use auxiliary flow variables to represent these circuits. Specifically, let f^i_{jl} take the value 1 if a vehicle traverses (j,l) on the way from the depot to i, and the value 0 otherwise. Then, let g^i_{jl} take the value 1 if a vehicle traverses (j,l) on the way from i to the depot, and the value 0 otherwise. Then, we take the formulation (1)–(6), and add the following constraints:

$$
\begin{aligned}
f^i(\delta^+(\{0\})) - f^i(\delta^-(\{0\})) &= 1 && (i \in V_c)\\
f^i(\delta^+(\{j\})) - f^i(\delta^-(\{j\})) &= 0 && (i \in V_c;\ j \in V_c \setminus \{i\})\\
f^i_{ji} &= x_{ji} && (i \in V_c;\ j \neq i)\\
g^i(\delta^-(\{0\})) - g^i(\delta^+(\{0\})) &= 1 && (i \in V_c)\\
g^i(\delta^+(\{j\})) - g^i(\delta^-(\{j\})) &= 0 && (i \in V_c;\ j \in V_c \setminus \{i\})\\
g^i_{ij} &= x_{ij} && (i \in V_c;\ j \neq i)\\
f^i_{jl} + g^i_{jl} &\le x_{jl} && (i \in V_c;\ (j,l) \in A)\\
\sum_{j \in V_c \setminus \{i\}} q_j \left(f^i(\delta^+(\{j\})) + g^i(\delta^-(\{j\}))\right) &\le Q - q_i && (i \in V_c)\\
f^i_{jl}, g^i_{jl} &\ge 0 && (i \in V_c;\ (j,l) \in A).
\end{aligned}
$$

Note that, if we discard the g variables and associated constraints, and scale the remaining equations appropriately, we obtain the original multi-commodity flow formulation. Thus, the LP relaxation of this new multi-commodity flow formulation is at least as strong as that of the original. In particular, the new formulation also implies all GLM and SE inequalities in the two-index space. Now, however, if for some i there does not exist a capacitated circuit in G^* passing through the depot and i, then there will not exist a feasible assignment of values to the f- and g-variables. Therefore, the presence of the f- and g- variables has the effect of preventing 'structural faults' in the support graph, just as the hypotour inequalities do.

We have not been able to determine whether the formulation implies some or all hypotour inequalities by projection, but it does seem at least that some 'hypotour-like' inequalities are implied. This is an interesting topic for future research. In any case, it can be shown that the LP relaxation of this new multi-commodity flow formulation is not in general dominated by the LP relaxation of the two-index vehicle flow formulation (2)–(6).

4. The CVRP: set partitioning formulations

We will also need to understand the following *set partitioning* formulation, which is valid for many variants of the VRP. The basic idea here is to define one variable for every possible (feasible) route which a single vehicle can take. Let Ω denote the set of possible routes, and let z_r for each $r \in \Omega$ be a binary variable taking the value 1 if that route is used, and the value 0 otherwise. Define the constants a_{ir} taking the value 1 if customer i is served by route r, 0 otherwise. Finally let c_r denote the cost of route r. Then the formulation is:

$$\text{Minimise} \sum_{r \in \Omega} c_r z_r \tag{45}$$

subject to

$$\sum_{r \in \Omega} z_r \leq m \tag{46}$$

$$\sum_{r \in \Omega} a_{ir} z_r = 1 \qquad (i \in V_c) \tag{47}$$

$$z_r \in \{0, 1\} \qquad (r \in \Omega). \tag{48}$$

It is not difficult to show that this formulation can be obtained by applying Dantzig-Wolfe decomposition to the three-index vehicle flow formulation, and then simplifying by exploiting symmetry.

Obviously, the number of variables in the set partitioning formulation can grow exponentially with the size of the problem, making column generation necessary. Unfortunately, the column generation (pricing) subproblem is strongly $\mathcal{NP}$-hard. Therefore, in practice, it is common to enlarge the set Ω by allowing so-called *non-elementary* routes, which are routes in which the vehicle is permitted to visit customers more than once. Although non-elementary routes are infeasible, this relaxation has the advantage that the pricing subproblem becomes solvable in pseudo-polynomial time (by dynamic programming). Moreover, the overall integer programming formulation remains valid, because the constraint (47) ensures that the variables representing non-elementary routes will be automatically eliminated when z is binary. (See Cordeau et al. [9] and Desrosiers et al. [11] for more details.)

The standard set partitioning formulation is based on the z-variables only and does not involve the x-variables of the two-index formulation. However, as pointed out by Fukusawa et al. [16], it is easy to add constraints linking the x-variables with the z-variables. For each arc (i, j) and each route r, let b_{ijr} denote the number of times arc (i, j) is traversed in route r. Then we can formulate the CVRP as:

$$\text{Minimise} \sum_{(i,j)\in A} c_{ij}x_{ij} \tag{49}$$

subject to

$$x(\delta^+(\{i\})) = 1 \qquad (i \in V_c) \tag{50}$$

$$x_{ij} = \sum_{r\in\Omega} b_{ijr}z_r \qquad ((i, j) \in A) \tag{51}$$

$$\sum_{r\in\Omega} z_r \leq m \tag{52}$$

$$z_r \in \{0, 1\} \qquad (r \in \Omega). \tag{53}$$

Note that the equations (47) are not needed here, as they are implied by (50) and (51). Also, integrality conditions on the x-variables are redundant.

We now consider two variants of this formulation in more detail: one in which only elementary routes are allowed, and one in which non-elementary routes are permitted.

4.1. Elementary routes

Let (x^*, z^*) be a feasible solution to the LP relaxation of the above formulation, in the case in which only elementary (feasible) routes are permitted. Intuitively, we can decompose x^* into a linear combination of capacitated circuits, each passing through the depot. It follows from the theory of Dantzig-Wolfe decomposition that the lower bound obtained from this LP relaxation is equal to the lower bound obtained from the LP relaxation of the three-index vehicle flow formulation after all *one-vehicle* inequalities have been added (see Subsection 2.2). It is possible to re-state this in terms of the two-index vehicle flow formulation, as follows.

Definition 1. *A valid inequality for the two-index vehicle flow formulation of the form*

$$\sum_{(i,j)\in A} \alpha_{ij}x_{ij} \geq \beta$$

is said to be decomposable *if there exists a one-vehicle inequality for the three-index vehicle flow formulation of the form:*

$$\sum_{(i,j)\in A} \alpha_{ij}x^k_{ij} \geq \sum_{i\in V_c} \beta_i y^k_i$$

with $\sum_{i \in V_c} \beta_i = \beta$. *(We assume that the upper bound* m *is large, e.g.,* $m = n$.*)*

Proposition 6. *The degree equations, the bounds* $0 \le x_{ij} \le 1$ *and the KLM inequalities are decomposable.*

Proof. The bounds are trivial. For the degree equations, the desired one-vehicle inequalities are the equations (13). For the KLM inequalities, the desired one-vehicle inequalities (21) have already been given above (Subsection 2.2). □

Proposition 7. *A vector* (x^*, z^*) *satisfies the inequalities (50), (51) and the bounds* $0 \le z_r \le 1$ *if and only if the vector* x^* *satisfies all decomposable inequalities.*

Proof. By definition, the vector x^* can be written as a conical combination $\sum_{k=1}^{n} \lambda^k x^k$, with $\lambda^k \ge 0$ for all k. (By Caratheodory's theorem, n components suffice.) Here, each vector x^k represents the incidence vector of a feasible route for a single vehicle. But this means that for such a vector x^k, there exists a suitable vector y^k such that (x^k, y^k) satisfies all one-vehicle inequalities and the assignment constraints (16). Thus, x^* satisfies all decomposable inequalities. □

Corollary 1. *The lower bound obtained from the LP relaxation of the set partitioning formulation is at least as good as the one obtained by using degree equations, bounds and KLM inequalities in the two-index vehicle flow formulation.*

To appreciate the strength of the set partitioning relaxation, recall that the KLM inequalities include the GLM and SE inequalities as special cases, and that no efficient separation algorithm is known at present for the KLM inequalities.

The following result shows that the LP relaxation of the set partitioning formulation is also at least as strong as the LP relaxations of the multi-commodity flow formulations mentioned in Subsection 3.4.

Theorem 6. *If a vector* (x^*, z^*) *satisfies the inequalities (50), (51) and the bounds* $0 \le z_r \le 1$, *then there exists a vector* f^* *such that* (x^*, f^*) *is a feasible solution to the LP relaxation of the first multicommodity flow formulation, and there exist vectors* f^* *and* g^* *such that* (x^*, f^*, g^*) *is a feasible solution to the LP relaxation of the second multicommodity flow formulation.*

Proof. For any vertex $i \in V_c$, any arc (j, l), and any route $r \in \Omega$, let b_{ijlr} (respectively, b'_{ijlr}) be a binary constant which takes the value 1 if and only if, in route r,

- vertex i is visited and
- the arc (j, l) is traversed on the way from the depot to i (respectively, on the way from i to the depot).

Note that, due to (50) and (51), we have $\sum_{r \in \Omega} \sum_{j \in V \setminus \{i\}} b_{ijir} z_r^* = 1$ and $\sum_{r \in \Omega} \sum_{j \in V \setminus \{i\}} b'_{iijr} z_r^* = 1$. The desired vector f^* for the first formulation is obtained by setting $(f_{jl}^i)^* = q_i \sum_{r \in \Omega} b_{ijlr} z_r^*$ for all $i \in V_c$ and all $(j, l) \in A$. The desired vectors f^* and g^* for the second formulation are obtained by setting $(f_{jl}^i)^* = \sum_{r \in \Omega} b_{ijlr} z_r^*$ and $(g_{jl}^i)^* = \sum_{r \in \Omega} b'_{ijlr} z_r^*$ for all $i \in V_c$ and all $(j, l) \in A$. □

So, the projection of the set partitioning relaxation onto x-space also satisfies the 'hypotour-like' inequalities mentioned in Subsection 3.4, though not necessarily the hypotour inequalities themselves.

4.2. Non-elementary routes

As mentioned above, in practice one allows non-elementary routes to make the pricing tractable in the column generation approach. This weakening of the formulation in the (x, z)-space means that we lose some of the KLM inequalities in the x-space, as expressed in the following proposition:

Proposition 8. *When non-elementary routes are permitted, the only KLM inequalities which are implied by the (x, z) formulation are those in which $\alpha y \leq \beta$ is valid for the* general integer *knapsack polytope*

$$conv\, \{y \in Z^n_+ : \sum_{i \in V_c} q_i y_i \leq Q\}.$$

Proof. Suppose we modify the three-index formulation so that customers may be visited more than once. Then the one-vehicle KLM inequalities (21) remain valid, but only when the knapsack polytope is modified as stated. The result then again follows from decomposition theory. We skip the details for brevity. □

These less general KLM inequalities still include the GLM inequalities as a special case, but no longer include the SE inequalities. However, it is worth noting that no explicit separation algorithm is known for even these special KLM inequalities.

The next proposition shows that we also lose some other inequalities by allowing non-elementary routes:

Proposition 9. *When non-elementary routes are allowed, the LP relaxation of the set partitioning formulation can be weaker than the LP relaxations of the multi-commodity flow formulations mentioned in Subsection 3.4.*

Proof. Consider a VRP instance with only two customer vertices, in which $q_1 = q_2 = 1$, but Q is an arbitrary positive even integer. The following fractional solution is feasible for the LP relaxation of (50)–(53): $x^*_{01} = x^*_{20} = 2/Q$, $x^*_{12} = 1$, $x^*_{21} = (Q-2)/Q$, $x^*_{10} = x^*_{02} = 0$; $z^*_r = 2/Q$ for a route starting at 0, going to 1, then going from 1 to 2 and back $(Q-2)/2$ times, then going to 2 and to 0; $z^*_r = 0$ for all other routes. However, since this solution violates the subtour elimination inequality $x(\delta^+(\{1, 2\}) \geq 1$, such an x-vector could not be obtained in an LP relaxation of either of the multi-commodity flow formulations. □

This potential weakness of the set partitioning relaxation with non-elementary routes has led some authors to forbid non-elementary routes containing 'short cycles'. This complicates the pricing subproblem, but leads to improved lower bounds (see Irnich and Villeneuve [23], Fukasawa et al. [16]).

On the other hand, even when non-elementary routes are permitted, it is possible for the LP relaxation of the set partitioning formulation to be stronger than that of the two-index vehicle flow formulation (2)–(6). This occurs, for example, when the objective function is equal to the left-hand side of a non-redundant KLM inequality.

In any case, the above considerations may offer some intuitive explanation for the success of the set partitioning approach when the vehicle capacity is small and the number of vehicles is large (Fukasawa et al. [16]). In this case, the decomposable inequalities (such as the KLM and ‘hypotour-like’ inequalities) seem to become more important, and the use of column generation enables one to handle at least some of them ‘implicitly’, even though we do not have a good characterisation of them.

5. The VRP with time windows

In the vehicle routing problem with time windows (VRPTW), it takes a time $t_{ij} \geq 0$ to traverse arc (i, j). When $i \in V_c$, the quantity t_{ij} includes any time required to service i. For each $i \in V_c$, service must begin within the time window $[e_i, l_i]$, where $0 \leq e_i \leq l_i \leq \infty$. We will also allow that each vehicle be required to leave the depot at time e_0 or afterwards, and arrives back at the depot by time l_0 or earlier. A vehicle is permitted to wait at a vertex, either before or after serving a customer.

It is standard to use set partitioning formulations, usually allowing non-elementary routes, to solve the VRPTW (see Cordeau et al. [9] or Desrosiers et al. [11]). However, it is possible to work in the two-index space by adding constraints forbidding *infeasible paths* (as in Ascheuer et al. [2] and Fischetti et al. [14]). Yet, in this paper we will look at a certain *two-commodity flow* formulation, based loosely on a formulation of a constrained TSP due to Maffioli and Sciomachen [33]. Unlike the situation mentioned in Subsection 3.2, there is indeed a benefit obtained by using two commodities when time windows are present.

In what follows we assume for simplicity that there are no capacity constraints. (These can be handled as in the previous sections.) We also assume that the VRPTW instance has been preprocessed, to reduce the width of the time windows. Techniques for doing this can be found in Desrosiers et al. [11]. Once this is done, we may also be able to increase travel times and eliminate arcs as expressed in the following two propositions:

Proposition 10. *Let $l_i + t_{ij} < e_j$ for some pair (i, j). Then the optimal solution is unchanged if we increase t_{ij} to $e_j - l_i$.*

Proposition 11. *All arcs (i, j) such that $e_i + t_{ij} > l_j$ can be eliminated from the problem.*

We will make use of these two propositions. Here is the formulation. For all $(i, j) \in A$, define a binary variable x_{ij}, taking the value 1 if some vehicle travels from i to j, and the value 0 otherwise.

Define variables u_{ij} and v_{ij} for all $i, j \in V$, with the following interpretation: if $x_{ij} = 0$, then u_{ij} and v_{ij} are set to 0. However, if $x_{ij} = 1$, then u_{ij} represents the time at which the vehicle begins service at i, and v_{ij} represents the time when the vehicle begins service at j. We set u_{0j} to zero for all j. Finally, given some $F \subset A$, let $x(F)$ denote $\sum_{(i,j)\in F} x_{ij}$, and similarly for $u(F)$ and $v(F)$. The MILP formulation is:

$$\text{Minimise} \sum_{(i,j)\in A} c_{ij} x_{ij} \tag{54}$$

subject to

$$x(\delta^+(\{i\})) = 1 \qquad (i \in V_c) \tag{55}$$
$$x(\delta^-(\{i\})) = 1 \qquad (i \in V_c) \tag{56}$$
$$x(\delta^+(\{0\})) = x(\delta^-(\{0\})) \le m \tag{57}$$
$$u(\delta^+(\{i\})) = v(\delta^-(\{i\})) \qquad (i \in V_c) \tag{58}$$
$$v_{ij} \ge u_{ij} + t_{ij}x_{ij} \qquad (i, j \in V_c; i \ne j) \tag{59}$$
$$u_{ij} \ge e_i x_{ij} \qquad (i \in V_c;\ j \in V;\ i \ne j) \tag{60}$$
$$v_{ij} \le l_j x_{ij} \qquad (i \in V;\ j \in V_c;\ i \ne j) \tag{61}$$
$$u_{ij} \le \min\{l_i, l_j - t_{ij}\}x_{ij} \qquad (i \in V_c;\ j \in V;\ i \ne j) \tag{62}$$
$$v_{ij} \ge \max\{e_j, e_i + t_{ij}\}x_{ij} \qquad (i \in V;\ j \in V;\ i \ne j) \tag{63}$$
$$x_{ij} \in \{0, 1\} \qquad ((i, j) \in A). \tag{64}$$

Here, we use the convention $e_0 := 0$.

The objective function is self-explanatory. The equations (55)–(57) are the same as (2)–(4) in the case of the CVRP. The equations (58) link the u- and v-variables. Constraints (59) ensure that serving a customer and travelling between customers uses up the correct amount of time. The following four sets of constraints ensure that the time windows are obeyed. The coefficients have been made as tight as possible. The constraints also collectively forbid subtours. Finally, (64) are the integrality conditions.

Note that the optimal objective value of the LP relaxation of the formulation (55)–(64) increases as the time windows decrease in width. Thus, the preprocessing procedures mentioned above are likely to be beneficial for algorithms based on this formulation. (This is true even for algorithms based on column generation — see Desrosiers et al. [11].)

We now consider the projection of the LP relaxation onto x-space, leading to a result similar to Theorem 2. (For other valid inequalities in the x-space, see for example Ascheuer et al. [2].) To do this, let $A(S)$ denote the set of arcs with both end-vertices in S, i.e., $A(S) = \{(i, j) \in A : i, j \in S\}$.

Theorem 7. *For any $S \subseteq V_c$ with $|S| \ge 2$, the inequality*

$$\sum_{i \in S, j \in V \setminus S} \min\{l_i, l_j - t_{ij}\}x_{ij} \ge \sum_{i \in V \setminus S, j \in S} \max\{e_j, e_i + t_{ij}\}x_{ij} + \sum_{(i,j) \in A(S)} t_{ij}x_{ij} \tag{65}$$

is valid for the x-variable formulation of the VRPTW.

Proof. Summing (58) over all $i \in S$ and rearranging yields $y(\delta^+(S)) + y(A(S)) = z(\delta^-(S)) + z(A(S))$. Together with the inequalities (59) for $(i, j) \in A(S)$, this implies:

$$y(\delta^+(S)) \ge z(\delta^-(S)) + \sum_{(i,j) \in A} t_{ij}x_{ij}.$$

The result follows from inequalities (60) to (63). □

We call the inequalities (65) *projection inequalities*, because their validity is proved by projecting the MILP formulation onto the subspace of the x-variables. Intuitively, and informally, they say that for any set S, the sum of the upper time limits for vehicles leaving S must be greater than or equal to the sum of the lower time limits for the vehicles entering S, plus the time the vehicles spend travelling inside S.

Note that the strength of the projection inequalities depends upon the width of the time windows. This is a further incentive for thorough preprocessing. In particular, increasing t_{ij} as suggested by Proposition 10 will make the right-hand side of the projection inequality larger, thus strengthening the inequality.

From the equivalence of separation and optimization, it follows that the projection inequalities can be solved exactly in polynomial time. In fact it again suffices to solve a max-flow/min-cut problem, as the following theorem shows.

Theorem 8. *Let x^* be non-negative and satisfy the in- and out-degree equations. If the VRPTW instance has been preprocessed as expressed in Propositions 10 and 11, then projection inequalities can be separated in polynomial time.*

Proof. Let $\alpha_{ij} = \max\{0, l_j - l_i - t_{ij}\}$ and $\beta_{ij} = \max\{0, e_i - e_j + t_{ij}\}$. Using the in- and out-degree equations, it is possible to rewrite the projection inequality in the form:

$$\sum_{i \in S} \left(l_i - e_i - \sum_{j \in V \setminus S} (\alpha_{ij} x_{ij} + \beta_{ji} x_{ji}) \right) + \sum_{(i,j) \in A(S)} (e_j - l_i - t_{ij}) x_{ij} \geq 0. \quad (66)$$

To find a violated projection inequality, then, we wish to choose a set $S \subseteq V_c$ which minimizes the left-hand side of (66). To this end, we define a 0-1 variable s_i for each $i \in V_c$, taking the value 1 if and only if $i \in S$. It is easy (but tedious) to show that the problem of minimizing the left-hand side of (66) is equivalent to minimising

$$\sum_{i \in V_c} \left(l_i - e_i - \sum_{j \in V_c \setminus \{i\}} (\alpha_{ij} x^*_{ij} + \beta_{ji} x^*_{ji}) \right) s_i + \sum_{i \in V_c} \sum_{j \in V_c \setminus \{i\}} \big(\max\{e_j - t_{ij}, e_i\} - \min\{l_i, l_j - t_{ij}\} \big) x^*_{ij} s_i s_j. \quad (67)$$

By examining all four cases of where the maximum and minimum is achieved in the second summation, it is easy to show that for all pairs (i, j) each quadratic coefficient is non-positive when the stated conditions are met. Thus, minimizing (67) is a boolean quadratic minimization problem with non-positive quadratic coefficients. Problems of this type can be solved via a max-flow/min-cut problem (see Picard and Ratliff [37]). □

6. Other variants of the VRP

Similar formulations and projection results can be given for various other VRPs. We briefly discuss three examples.

6.1. The distance-constrained VRP

In the distance-constrained version (DVRP), each arc has a distance d_{ij}, which need not be equal to c_{ij}, and no vehicle can travel a distance larger than some constant $D > 0$ (see Laporte et al. [29]). This can be viewed as a special case of the VRPTW by setting $t_{ij} = d_{ij}$, $e_i = d_{oi}$ and $l_i = D - d_{i0}$. It can be checked that the projection inequality (65) in this case reduces to

$$Dx(\delta^+(S)) \geq \sum_{i \in S, j \in V \setminus S} (d_{0j} + d_{ji})(x_{ij} + x_{ji}) + \sum_{(i,j) \in A(S)} d_{ij} x_{ij}.$$

A slightly weaker (and symmetric) version of these was presented in Letchford and Eglese [30], under the name of *D-radius inequalities*. Thus, projection inequalities dominate D-radius inequalities.

6.2. The VRP with a single deadline

Similarly, suppose that the only constraint is a deadline for service, i.e., we have the special case of the VRPTW in which $l_i = T$ and $e_i = t_{0i}$ for all $i \in V_c$, but $l_0 = \infty$. In this case the projection inequality (65) reduces to:

$$Tx(\delta^+(S)) \geq \sum_{i \in S, j \in V \setminus S} \big(t_{ij} x_{ij} + (t_{0j} + t_{ji}) x_{ji}\big) + \sum_{(i,j) \in A(S)} t_{ij} x_{ij} + \sum_{i \in S} t_{0i} x_{0i}$$

where, again, $t_{00} := 0$. A slightly weaker (and symmetric) version of these was presented in Letchford and Eglese [30], under the name of *T-radius inequalities*. Thus, projection inequalities dominate T-radius inequalities also.

6.3. The VRP with pickups and deliveries

In the sections on the CVRP, we assumed that all the customers have a positive demand, i.e., that the vehicle delivers q_i units of a product to each customer. In some problems, however, customer demands are allowed to be negative, which means that some customers increase the load of the vehicle when they are visited. These latter customers are termed *pickup customers*, to be distinguished from the others customers, termed *delivery customers*. See Desaulniers et al. [10] for some references, and Hernández and Salazar [22] for a cutting plane approach.

Results in this paper can be adapted to this case. For example, the one-commodity flow formulation given in Subsection 3.1 is still valid, but the inequalities (24) need to be replaced by the stronger bounds:

$$\max\{0, q_i, -q_j\} x_{ij} \leq y_{ij} \leq (Q + \min\{0, q_i, -q_j\}) x_{ij} \qquad ((i,j) \in A).$$

In this way one can derive stronger multistar-like inequalities, see [22] for details. (However, because some demands can be non-positive, subtours disconnected from the depot should be prevented, for example by using SE inequalities (8) or additional flow variables.)

7. Conclusion

We have shown that various one- and two-commodity flow-based formulations, as well as set partitioning formulations, yield various multistar inequalities in the standard two-index space. We have also given efficient separation algorithms for these multistar inequalities. We also explored two multi-commodity flow formulations of the CVRP and showed that they yield certain constraints which are related to the hypotour inequalities. Finally, we examined other problems such as the VRPTW, and found similar multistar inequalities.

The main conclusion from these theoretical results is that three-index and one- and two-commodity flow formulations are unlikely to give the best performance for problems in which all vehicles are identical. The multi-commodity flow formulations presented in Subsection 3.4 look interesting from a theoretical point of view, but, due to their size, it is not clear whether they could be made to work in practice. It seems better to use either *branch-and-cut*, based on the two-index formulation, or *branch-and-price*, based on the set partitioning formulation. In fact, the evidence of Fukasawa et al. [16] on the CVRP and Kohl et al. [24] on the VRPTW, suggests that the best approach is to combine both set partitioning and two-index variables, leading to *branch-cut-and-price* algorithms.

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