



# Exploiting sparsity in pricing routines for the capacitated arc routing problem

Adam N. Letchford\*, Amar Oukil

Department of Management Science, Lancaster University, Lancaster LA1 4YW, UK

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## ABSTRACT

The *capacitated arc routing problem* (CARP) is a well-known and fundamental vehicle routing problem. A promising exact solution approach to the CARP is to model it as a set covering problem and solve it via branch-cut-and-price. The bottleneck in this approach is the pricing (column generation) routine. In this paper, we note that most CARP instances arising in practical applications are defined on *sparse graphs*. We show how to exploit this sparsity to yield faster pricing routines. Extensive computational results are given.

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## 1. Introduction

In most vehicle routing problems, the customers are located at the vertices of a network. In *arc routing problems* (ARPs), however, the demands are located along the edges. Although this might seem like a minor change, ARPs have a rather different structure to standard vehicle routing problems and specialised methods have been devised for them—see for example the edited volume by Dror [1] and the references therein.

This paper is concerned with one of the most well-known ARPs, the *capacitated arc routing problem* (CARP), which is defined as follows. We are given an undirected graph  $G = (V, E)$ , a set  $E_R \subseteq E$  of *required edges*, and a specified depot vertex. Each edge has a *traversing cost*  $c_e > 0$ , and each required edge has a *demand*  $q_e > 0$ . An unlimited fleet of identical vehicles, each of capacity  $Q > 0$ , is available. The task is to find a minimum cost set of vehicle routes, each starting and ending at the depot, such that each required edge is serviced exactly once.

The CARP was introduced by Golden and Wong [2], who showed that even finding a  $\frac{3}{2}$ -approximate solution is  $\mathcal{NP}$ -hard. Comprehensive surveys of heuristics, lower bounds and exact algorithms for the CARP, up to the year 2000, can be found in Dror [1]. For surveys of more recent work, see Ahr [3], Wøhlk [4] and Section 2 of this paper.

At present, the most promising approach to solving the CARP to proven optimality is that of Longo et al. [5]. Their algorithm consists of two steps. First, shortest-path computations are used to transform the CARP into its node routing counterpart, the *capacitated vehicle routing problem* (CVRP). Then, the CVRP instance is solved with

the algorithm of Fukasawa et al. [6]. This algorithm is based on the formulation of the CVRP as a set covering problem, which is then solved by branch-cut-and-price.

In our view, it seems odd that transforming the CARP into the CVRP should be necessary to obtain good results. Indeed, most CARPs arising in practice (and most standard test instances) are defined on *sparse graphs*, such as road networks. The transformation used by Longo et al. leads to a CVRP instance defined on a *complete graph*. This seems to be an unnecessary ‘squaring’ of the amount of data. (We elaborate on this point in Section 2.1.)

Our ultimate goal is to develop a branch-cut-and-price algorithm for the CARP that fully exploits graph sparsity. In this paper, however, we concentrate on a key component of such an algorithm: the *pricing* (column generation) routine. We will show that, when  $G$  is sufficiently sparse, pricing can be performed more quickly. Moreover, unlike in the case of the CVRP, we can price over the so-called *elementary routes*, which leads to a significant improvement in the lower bound.

The remainder of the paper is structured as follows. In Section 2, we review the existing exact approaches to the CARP and point out their strengths and weaknesses. In Sections 3 and 4, we describe our new pricing routines, for non-elementary and elementary routes, respectively. In Section 5, we give extensive computational results. Some concluding remarks are made in Section 6.

Throughout the paper we use the following (fairly standard) notation and terminology. For any  $S \subset V$ , we denote by  $E(S)$  (respectively,  $\delta(S)$ ) the set of edges with both end-vertices (exactly one end-vertex) in  $S$ . We also define  $E_R(S) = E_R \cap E(S)$  and  $\delta_R(S) = E_R \cap \delta(S)$ . We let  $V'$  denote the set of non-depot vertices and  $V_R$  denote the set of vertices that are incident on at least one required edge. Finally, *deadheading* an edge  $e \in E$  means traversing  $e$  without servicing it. (Note that a non-required edge can only be traversed if it is deadheaded.)

\* Corresponding author. Fax: +44 1524 844885.

E-mail address: [a.n.letchford@lancaster.ac.uk](mailto:a.n.letchford@lancaster.ac.uk) (A.N. Letchford).

## 2. Existing integer programming approaches

In this section we briefly review the existing integer programming approaches to the CARP.

### 2.1. The node routing approach

In the *node routing* approach, the given CARP instance is first transformed into an instance of the standard CVRP. The transformation involves the computation of all-pairs shortest paths in  $G$ . The resulting CVRP instance can then be solved by any CVRP algorithm. This approach was first suggested by Pearn et al. [7], and explored further by Letchford [8], Dror and Langevin [9], Longo et al. [5] and Baldacci and Maniezzo [10]. It benefits from the fact that some excellent exact algorithms exist for the CVRP, such as the branch-and-cut algorithm of Lysgaard et al. [11] and the branch-cut-and-price algorithm of Fukasawa et al. [6].

Although the node routing approach has given rather good results so far, it does have some drawbacks. For example:

1. The transformation leads to a CVRP instance defined on a complete graph with  $2|E_R| + 1$  vertices (or even  $3|E_R| + 1$  in the case of Pearn et al. [7]). If the original graph  $G$  is sparse, yet  $|E_R|$  is large, this leads to a huge increase in memory requirements.
2. The graph  $G$  may have some kind of special structure that could be exploited algorithmically. (E.g., if  $G$  is a tree, then one can adapt the approach of Labbé et al. [12] to the CARP.) Any structure of this kind is lost in the transformation.
3. The approach can suffer seriously from *symmetry*. Suppose, for example, that the graph  $G$  is a star, all edges are required, and only one vehicle is needed. When any of the approaches that we describe in the following three subsections are used, there is only one optimal solution: deadhead each edge exactly once. The CVRP instance resulting from the transformation, on the other hand, has  $|E|!$  alternative optimal solutions.

In our opinion, approaches that can avoid this transformation and work directly on the original graph  $G$  merit serious consideration.

### 2.2. The two-index approach

Suppose that an upper bound, say  $m$ , is known on the number of vehicles to be used in the optimal solution. Under these circumstances, it is possible to formulate the CARP using  $\mathcal{O}(m|E|)$  variables, which looks desirable when  $m$  is small and  $G$  is sparse. To our knowledge, this approach was first suggested by Welz [13]. However, Belenguer and Benavent [14] were the first to explore the approach in depth.

In the approach of Belenguer and Benavent [14], there is a binary variable  $x_{ek}$  for each  $e \in E_R$  and for  $k = 1, \dots, m$ , taking the value 1 if and only if the edge  $e$  is serviced by that vehicle. There is also a general integer variable  $y_{ek}$  for each  $e \in E$  and for  $k = 1, \dots, m$ , representing the number of times the edge  $e$  is deadheaded by the given vehicle.

Valid inequalities and separation algorithms for the two-index formulation have been devised by Belenguer and Benavent [14], Letchford [8] and Ahr [3]. Computational results presented in [3,14] indicate that the two-index approach works very well when  $m$  is small (less than five). However, when  $m$  is larger, it becomes unattractive. This is due in part to the large number of variables, but also to the presence of symmetry: there always exist at least  $m!$  optimal solutions. Ghiani et al. [15] recently proposed some ‘symmetry-breaking’ constraints, in an attempt to alleviate this latter problem.

### 2.3. The one-index approach

In what we call the *one-index* approach, there is only one variable for each edge. This approach was originally proposed for a sparse variant of the CVRP by Fleischmann [16]. It was adapted to the CARP independently by Letchford [8] and Belenguer and Benavent [17].

In the case of the CARP, the variables are defined as follows. For each  $e \in E$ , let  $z_e$  be a general integer variable representing the total number of times  $e$  is deadheaded, summed over all vehicles. These variables can be regarded as *aggregated* versions of the  $y_{ek}$  variables in the two-index approach, in the sense that  $z_e = \sum_{k=1}^m y_{ek}$ .

Although there is no known way to formulate the CARP as an integer program using only these aggregated variables, it is easy to derive valid inequalities that must be satisfied by feasible solutions. Letchford [8] and Belenguer and Benavent [17] independently proposed the following *capacity* inequalities:

$$\sum_{e \in \delta(S)} z_e \geq 2k(S) - |\delta_R(S)| \quad (S \subset V),$$

where  $k(S)$  is any lower bound on the number of vehicles that must enter  $S$ . They also proposed the following *R-odd cut* inequalities:

$$\sum_{e \in \delta(S)} z_e \geq 1 \quad (S \subset V' : |\delta_R(S)| \text{ odd}).$$

Belenguer and Benavent [17] also presented a third class of inequalities, called *disjoint path* inequalities. Since the description of these inequalities is rather complicated, we do not go into details.

Effective exact and heuristic separation algorithms for the capacity, *R-odd cut* and *disjoint path* inequalities can be found in Belenguer and Benavent [17] and Ahr [3]. Their computational results show that the one-index approach gives excellent lower bounds extremely quickly. Unfortunately, one cannot build a full exact algorithm for the CARP using the  $z$  variables alone, because it is strongly  $\mathcal{NP}$ -hard to decide whether a vector  $z \in \mathbb{Z}_+^{|E|}$  represents a feasible solution to the CARP.

### 2.4. A direct set covering approach

Gómez-Cabrero et al. [18] proposed formulating the CARP as a set covering problem directly, without any transformation to the CVRP. We refer the reader to Dror and Langevin [9] and Bramel and Simchi-Levi [19] for introductions to the set covering approach to vehicle routing and associated concepts such as column generation and branch-and-price.

Let  $\Omega$  denote the set of all possible feasible routes for a single vehicle. We define a binary variable  $\lambda_r$  for each  $r \in \Omega$ , taking the value 1 if and only if route  $r$  is used. We also denote by  $c_r$  the cost of route  $r$ . The set covering formulation takes the following standard form:

$$\begin{aligned} \min \quad & \sum_{r \in \Omega} c_r \lambda_r \\ \text{s.t.} \quad & \sum_{r \in \Omega} a_{re} \lambda_r \geq 1 \quad (e \in E_R), \\ & \lambda_r \in \{0, 1\} \quad (r \in \Omega), \end{aligned} \quad (1)$$

where  $a_{re}$  denotes the number of times edge  $e$  is serviced (not merely traversed) by route  $r$ .

Since this formulation has an exponential number of variables, one must solve its LP relaxation via column generation. Unfortunately, the pricing (column generation) subproblem is itself easily shown to be strongly  $\mathcal{NP}$ -hard. Gómez-Cabrero et al. [18] therefore enlarge the set of columns by permitting *non-elementary* routes, i.e., routes in which customers may be serviced more than once. When the problem is relaxed in this way, the pricing subproblem can be solved in pseudo-polynomial time by dynamic programming.

Gómez-Cabrero et al. [18] strengthen the LP relaxation by adding the capacity and  $R$ -odd cut inequalities mentioned above. To link the  $z$  variables with the  $\lambda$  variables, they use the identities:

$$z_e = \sum_{r \in \Omega} a'_{re} \lambda_r \quad (e \in E),$$

where  $a'_{re}$  denotes the number of times route  $r$  deadheads edge  $e$ . (A similar approach was used for the CVRP by Fukasawa et al. [6].)

Since the resulting extended formulation has an exponential number of both variables and constraints, Gómez-Cabrero et al. [18] use both row and column generation (cut-and-price) to solve its LP relaxation. The resulting lower bounds are much better than those obtained by pricing alone, and slightly better than those obtained by cutting alone.

The pricing subroutine of Gómez-Cabrero et al. [18] exploits the sparsity of  $G$  only to some extent. In the next section, we present alternative pricing subroutines that fully exploit sparsity.

### 3. Pricing with non-elementary routes

In this section we give algorithms for pricing, when non-elementary routes are permitted, that exploit graph sparsity. In the first subsection, we consider exact pricing, and in the second, we turn to heuristic pricing.

#### 3.1. Exact pricing

If we transform the CARP into the standard CVRP, as done in Longo et al. [5], then pricing with non-elementary routes can easily be performed in  $\mathcal{O}(Q|E_R|^2)$  time by dynamic programming (see, e.g., Bramel and Simchi-Levi [19]). The pricing algorithm described by Gómez-Cabrero et al. [18] is a little more sophisticated, and can be shown to run in  $\mathcal{O}(Q|V_R|^2)$  time. However, both approaches require all-pairs shortest paths to be computed in  $G$  in a pre-processing stage.

We now show that pricing with non-elementary routes can in fact be performed in  $\mathcal{O}(Q(|E| + |V| \log |V|))$  time without any all-pairs shortest path computations. Like the existing algorithms, the new algorithm is based on dynamic programming. However, we use Dijkstra's single-source shortest path algorithm as a subroutine to speed up certain computations.

To explain our new algorithm, we will need some notation. For each required edge  $e \in E_R$ , let  $\pi_e$  be the dual price of the associated constraint (1) in the optimal solution to the current restricted master LP. For  $i \in V$  and  $q = 0, \dots, Q$ , let  $f(i, q)$  denote the cumulative reduced cost of the best path starting at the depot and ending at vertex  $i$ , having delivered a cumulative load of  $q$ . (If no such path exists,  $f(i, q) = \infty$ .) Now, for  $i \in V_R$  and  $q = 0, \dots, Q$ , let  $g(i, q)$  denote the cumulative reduced cost of the best path starting at the depot and ending at vertex  $i$ , having delivered a cumulative load of  $q$ , subject to the restriction that it has just serviced an edge when it arrives at  $i$ . (Again, if no such path exists,  $g(i, q) = \infty$ .) Note that, by definition,  $g(i, q) \geq f(i, q)$  for all  $i \in V_R$  and for  $q = 0, \dots, Q$ .

The algorithm is as follows:

Initialisation: set all  $f(i, q)$  and  $g(i, q)$  values to  $\infty$ .

For  $q = 0, \dots, Q - 1$  do:

(\*) Call Dijkstra's algorithm to compute  $f(i, q)$  for all  $i \in V$ .

For each  $(i, j) \in E_R$  such that  $q + q_{ij} \leq Q$  do:

If  $f(i, q) + c_{ij} - \pi_{ij} < f(j, q + q_{ij})$

set  $g(j, q + q_{ij}) := f(i, q) + c_{ij} - \pi_{ij}$ .

If  $f(j, q) + c_{ij} - \pi_{ij} < f(i, q + q_{ij})$

set  $g(i, q + q_{ij}) := f(j, q) + c_{ij} - \pi_{ij}$ .

Call Dijkstra's algorithm to compute  $f(i, Q)$  for all  $i \in V$ .

For  $q = 1, \dots, Q$  do:

If  $f(0, q) < 0$ , output the corresponding column.

The steps of this algorithm are self-explanatory, with the exception of the step marked with an asterisk. When  $q = 0$ , it suffices to call Dijkstra's algorithm on  $G$  with the depot as source (since the vehicle will always take a shortest path from the depot to the node at which servicing begins). When  $q \geq 1$ , however, things are more complicated. We have already computed  $g(i, q)$  for all  $i \in V_R$ , and we wish to use that information to compute  $f(i, q)$  for all  $i \in V$ . Of course, we have

$$f(i, q) := \min_{j \in V_R} \{g(j, q) + sp(j, i)\},$$

where  $sp(j, i)$  denotes the cost of the shortest path from  $j$  to  $i$  in  $G$  (and by convention  $sp(i, i) = 0$ ). Computing all-pairs shortest paths is, however, precisely what we want to avoid. Instead, we use Dijkstra's algorithm in the following way. We construct a copy of  $G$ , with edge-costs  $c_e$  as usual, and add a dummy vertex, say  $v^*$ . For each  $i \in V_R$ , we then add a directed arc from  $v^*$  to  $i$  with cost equal to  $g(i, q)$ . By definition,  $f(i, q)$  is then equal to the cost of the shortest path from  $v^*$  to  $i$  in this auxiliary graph. Thus, it suffices to call Dijkstra's algorithm once with  $v^*$  as source. (The presence of negative  $g(i, q)$  values does not cause any problem, since one can simply add a large positive constant to the cost of the directed arcs, and then subtract it from the cost of the shortest paths afterwards.)

We therefore have the following result:

**Theorem 1.** For the CARP, pricing with non-elementary routes permitted can be performed in  $\mathcal{O}(Q(|E| + |V| \log |V|))$  time.

**Proof.** We call Dijkstra's algorithm once on  $G$ , and  $Q$  times on a graph with  $|V| + 1$  vertices,  $|E|$  edges and  $|V|$  arcs. If we use a Fibonacci heap implementation of Dijkstra's algorithm (Fredman and Tarjan [20]), this takes  $\mathcal{O}(Q(|E| + |V| \log |V|))$  time. The remainder of the algorithm, scanning the required edges, takes only  $\mathcal{O}(Q|E_R|)$  time.  $\square$

We close this subsection with a remark. It usually happens that there are several required edges with negative reduced cost, i.e., edges for which  $\pi_e > c_e$ . When this occurs, it is possible for our pricing algorithm to generate a non-elementary route in which a small number of edges are serviced many times. This is analogous to the well-known phenomenon of  $k$ -cycles in the VRP literature; see for example Irnich and Villeneuve [21]. Just as in the case of the VRP, one can modify our pricing algorithm to prevent  $k$ -cycles, but this would be at the expense of increasing the running time.

#### 3.2. Heuristic pricing

Although the pricing algorithm described above exploits graph sparsity, it can still be rather slow for large CARP instances. This led us to devise fast *heuristics* for pricing, which can be called before invoking the exact pricing routines.

Intuitively, one would expect the columns that are basic at the optimal solution of the master LP to correspond to routes that include very little deadheading. Our first pricing heuristic tries to exploit this observation, by pricing over the (not necessarily elementary) routes that have the following 'three-phase' structure:

1. The vehicle departs from the depot and deadheads to the first edge to be serviced.
2. The vehicle services one or more required edges without any deadheading taking place.
3. The vehicle departs from the last edge serviced and deadheads back to the depot.

Phase 1 or 3, or both, can of course be 'empty', if the vehicle starts or ends by servicing a required edge that is incident on the depot.

It turns out that pricing over the routes with this structure can easily be performed in  $\mathcal{O}(Q|E_R|)$  time, assuming that we have already computed, for each  $i \in V_R$ , the cost  $c_i^*$  of the shortest path from the depot to  $i$ . The algorithm is as follows:

Set  $f(i, q) = \infty$  for all  $i \in V_R$  and for  $q = 1, \dots, Q$ . Set  $f(i, 0) = c_i^*$  for all  $i \in V_R$ .  
 For  $q = 0, \dots, Q - 1$  do:  
   For each  $\{i, j\} \in E_R$  such that  $q + q_{ij} \leq Q$  do:  
     If  $f(i, q) + c_{ij} - \pi_{ij} < f(j, q + q_{ij})$  set  $f(j, q + q_{ij}) := f(i, q) + c_{ij} - \pi_{ij}$ .  
     If  $f(j, q) + c_{ij} - \pi_{ij} < f(i, q + q_{ij})$  set  $f(i, q + q_{ij}) := f(j, q) + c_{ij} - \pi_{ij}$ .  
 For  $q = 1, \dots, Q$  do:  
 For each  $i \in V_R$  do:  
   If  $f(i, q) + c_i^* < 0$ , output the corresponding column.

This heuristic is very fast in practice and extremely easy to implement, but it suffers from an important drawback. If the subgraph of  $G$  induced by the required edges has many connected components, then many ‘useful’ routes may be lost. (The heuristic permits the vehicle to pass through at most one such component while servicing.)

Our second pricing heuristic attempts to get around the above-mentioned limitation, by permitting a limited amount of deadheading to occur in phase 2. Specifically, we ‘pretend’ that one unit of demand is consumed each time an edge is deadheaded. (Putting this another way, we insist that the total demand serviced by the vehicle, plus the number of edges deadheaded in phase 2, must not exceed  $Q$ .) One can price over the resulting set of routes in  $\mathcal{O}(Q|E|)$  time, simply by inserting the following steps in the above algorithm immediately after the start of the first ‘for  $q$ ’ loop:

For each  $\{i, j\} \in E$  do:  
   If  $f(i, q) + c_{ij} < f(j, q + 1)$  set  $f(j, q + 1) := f(i, q) + c_{ij}$ .  
   If  $f(j, q) + c_{ij} < f(i, q + 1)$  set  $f(i, q + 1) := f(j, q) + c_{ij}$ .

Although slightly slower and more complicated than the first heuristic, this second heuristic is still very fast in practice and much easier to implement than the exact pricing algorithm described in Section 3.

#### 4. Pricing with elementary routes

Although pricing with elementary routes is strongly  $\mathcal{NP}$ -hard, it turns out that one can often solve it exactly in a reasonable amount of time when  $G$  is sparse. The key is to formulate the pricing problem as a mixed-integer program (MIP) with only  $\mathcal{O}(|E|)$  variables and constraints, as we explain in the following subsections.

##### 4.1. A directed MIP formulation

We first present an MIP formulation that uses directed flow variables. For each  $\{i, j\} \in E_R$ , define two binary variables  $x_{ij}$  and  $x_{ji}$ . The variable  $x_{ij}$  takes the value 1 if and only if edge  $\{i, j\}$  is serviced from  $i$  to  $j$ , and similarly for  $x_{ji}$ . Then, for each  $\{i, j\} \in E$ , define two binary variables  $y_{ij}$  and  $y_{ji}$ , representing the number of times  $\{i, j\}$  is traversed (whether servicing or not) in either direction. (It is easy to show that there always exists at least one optimal solution to the pricing subproblem such that no edge is traversed more than once in any given direction.) Finally define for each  $\{i, j\} \in E$  two continuous variables  $f_{ij}$  and  $f_{ji}$ . The variable  $f_{ij}$  takes the value zero if  $y_{ij} = 0$ , but if  $y_{ij} = 1$  it represents the remaining load on the vehicle when the vehicle arrives at  $j$  from  $i$ . The variable  $f_{ji}$  is defined similarly.

Since we are working with directed variables, it is helpful to define  $\delta^+(i)$  and  $\delta^-(i)$ , the set of directed arcs leaving and entering vertex  $i$ , respectively. We define  $\delta_R^+(i)$  and  $\delta_R^-(i)$  analogously. Letting

$0$  denote the depot vertex, the pricing subproblem then amounts to minimising

$$\sum_{\{i,j\} \in E} c_{ij}(y_{ij} + y_{ji}) - \sum_{\{i,j\} \in E_R} \pi_{ij}(x_{ij} + x_{ji})$$

subject to the following constraints:

$$x_{ij} + x_{ji} \leq 1 \quad (\{i, j\} \in E_R), \tag{2}$$

$$y_{ij} \geq x_{ij}, \quad y_{ji} \geq x_{ji} \quad (\{i, j\} \in E_R), \tag{3}$$

$$y(\delta^-(i)) = y(\delta^+(i)) \quad (i \in V'), \tag{4}$$

$$f(\delta^+(i)) = f(\delta^-(i)) - \sum_{\{i,j\} \in \delta_R^+(i)} q_{ij}x_{ij} \quad (i \in V'), \tag{5}$$

$$f(\delta^+(0)) - f(\delta^-(0)) + \sum_{\{0,j\} \in \delta_R^+(0)} q_{0j}x_{0j} \leq Q, \tag{6}$$

$$f_{ij} \leq Qy_{ij}, \quad f_{ji} \leq Qy_{ji} \quad (\{i, j\} \in E \setminus E_R), \tag{7}$$

$$f_{ij} \leq Qy_{ij} - q_{ij}x_{ij}, \quad f_{ji} \leq Qy_{ji} - q_{ji}x_{ji} \quad (\{i, j\} \in E_R), \tag{8}$$

$$f_{ij}, f_{ji} \geq 0 \quad (\{i, j\} \in E),$$

$$x_{ij}, x_{ji} \in \{0, 1\} \quad (\{i, j\} \in E_R),$$

$$y_{ij}, y_{ji} \in \{0, 1\} \quad (\{i, j\} \in E).$$

Constraints (2) ensure that required edges are serviced at most once and constraints (3) ensure that edges are traversed when they are serviced. Constraints (4) ensure that the vehicle leaves each vertex as many times as it enters. Constraints (5) ensure that servicing reduces vehicle load. Constraint (6) ensures that the total demand on the route is no more than  $Q$ . Constraints (7) and (8) restrict vehicle load *en route*. The remaining constraints are trivial non-negativity and binary integrality conditions.

This formulation, which is easily seen to be valid, has only  $\mathcal{O}(|E|)$  variables, constraints and non-zero constraint coefficients. Moreover, its LP relaxation is reasonably strong, as indicated by the following proposition:

**Proposition 1.** *The feasible region of the LP relaxation of the above MILP formulation satisfies the following inequalities:*

$$Qy(\delta^-(S)) \geq \sum_{\{i,j\} \in E_R(S) \cup \delta_R(S)} q_{ij}(x_{ij} + x_{ji}) \quad (\forall S \subset V'). \tag{9}$$

**Proof.** We use the projection technique of Gouveia [22]. Summing constraints (5) over all  $i \in S$ , and re-arranging gives

$$f(\delta^-(S)) = f(\delta^+(S)) + \sum_{\{i,j\} \subset S} q_{ij}x_{ij} + \sum_{i \in S, j \in V \setminus S} q_{ij}x_{ij}.$$

Since the  $f$  variables are non-negative we have

$$f(\delta^-(S)) \geq \sum_{\{i,j\} \subset S} q_{ij}x_{ij} + \sum_{i \in S, j \in V \setminus S} q_{ij}x_{ij}.$$

The result then follows from the bounds (7) and (8).  $\square$

Note that the inequalities (9) are exponential in number. That is, a linear number of constraints involving the  $f$  variables implies an exponential number of constraints in the subspace defined by the  $x$  and  $y$  variables.

We remark that constraints (5) and (6) also imply the ‘obvious’ valid inequality  $\sum_{\{i,j\} \in E_R} q_{ij}(x_{ij} + x_{ji}) \leq Q$ .

Although the formulation has these nice properties, there are two simple ways to improve it. First, one can add the constraint  $y(\delta^+(0)) = 1$ , since any feasible route in which  $y(\delta^+(0)) > 1$  can be decomposed into two or more simpler routes. Second, one can fix the variables  $x_{ij}$  and  $x_{ji}$  to zero whenever  $\pi_{ij} = 0$ . Note that, to use this second improvement, one must modify the formulation dynamically during the course of the column generation algorithm.

4.2. Undirected and mixed MIP formulations

The above directed MIP formulation suffers from a high degree of symmetry (many feasible solutions of the same cost). One can eliminate this symmetry, and reduce the number of variables, by using undirected  $x$  and  $y$  variables. However, to ensure that vertex degrees are even, one then needs to use an additional general integer variable for each (non-depot) vertex. This leads to the following formulation, which we call the *undirected* formulation:

$$\begin{aligned} \text{Min } & \sum_{e \in E} c_e y_e - \sum_{e \in E_R} \pi_e x_e \\ \text{s.t. } & y_e \geq x_e \quad (e \in E_R), \\ & y(\delta(i)) = 2z_i \quad (i \in V'), \\ & f(\delta^+(i)) = f(\delta^-(i)) - \frac{1}{2} \sum_{e \in \delta_R^-(i)} q_e x_e \quad (i \in V'), \\ & f(\delta^+(0)) - f(\delta^-(0)) + \frac{1}{2} \sum_{e \in \delta_R^-(0)} q_e x_e \leq Q, \\ & f_{ij}, f_{ji} \leq \frac{Q}{2} y_{ij} \quad ((i, j) \in E \setminus E_R), \\ & f_{ij}, f_{ji} \leq \frac{1}{2} (Q y_{ij} - q_{ij} x_{ij}) \quad ((i, j) \in E_R), \\ & f_{ij}, f_{ji} \geq 0 \quad ((i, j) \in E), \\ & x_e \in \{0, 1\} \quad (e \in E_R), \\ & y_e \in \{0, 1, 2\} \quad (e \in E), \\ & z_i \in \mathbb{Z}_+ \quad (i \in V'). \end{aligned}$$

We remark that, although this gives a valid formulation of the pricing problem, the  $f$  variables no longer have any natural physical interpretation.

By analogy with the directed formulation, one can strengthen the undirected formulation by adding the constraint  $y(\delta(0)) = 2$  and by fixing  $x_e$  to zero whenever  $\pi_e = 0$ .

It can be shown that the optimal solutions to the LP relaxations of the directed and undirected formulations have the same cost. The directed formulation does, however, have one advantage over the undirected one: if during the branch-and-bound process we impose the constraint  $y_{ij} = 1$ , constraints (4) immediately ensure that  $y(\delta^-(S)) \geq 1$  for all  $S$  with  $i \in S, j \notin S$ . In the undirected case, on the other hand, imposing  $y_e = 1$  has no analogous effect. The result is that the branch-and-bound tree tends to have more nodes when the undirected formulation is used, rather than fewer as one might expect.

The above considerations led us to devise a third formulation, which we call *mixed*, in which the  $y$  variables are directed but the  $x$  variables are undirected. The derivation of the mixed formulation is straightforward, and we omit details for brevity. Nevertheless, our computational experience (Section 5) indicates that the mixed formulation performs slightly better than the directed and undirected formulations.

4.3. Heuristic pricing

For large instances, solving the pricing subproblem exactly by mixed-integer programming can be very time-consuming. So, we

were led to consider fast heuristics for pricing with elementary routes. We experimented with various constructive heuristics, but did not find any method that consistently led to better overall performance. Instead, we found it more fruitful to adjust the parameters of the MIP solver itself, as follows. In a first step, we solve the MIP *approximately* by branch-and-bound, aborting whenever the gap between upper and lower bound drops below 0.01. (We do this in ILOG CPLEX by setting the parameters CPX\_PARAM\_EPAGAP and CPX\_PARAM\_MIPEMPHASIS to 0.01 and 1, respectively. Apart from that, all other settings were left at their default values.) If a column is found that has a reduced cost less than  $10^{-3}$ , we add it to the restricted master LP. Otherwise, we proceed to solve the MIP to proven optimality with a much higher degree of precision. (We do this in CPLEX by setting CPX\_PARAM\_EPAGAP to  $10^{-6}$  and CPX\_PARAM\_MIPEMPHASIS to 0. Again, no other parameters were changed.) If a column is found whose reduced cost is less than  $10^{-6}$ , we add it to the master.

5. Computational experiments

Four sets of CARP instances are commonly used in the literature. The set created by Golden et al. [23] contains 23 instances, defined on artificial graphs. The set of Benavent et al. [24] contains 34 instances, based on the road network in Valencia. The third set, due to Kiuchi et al. [25], contains six instances, defined on artificial graphs. In each of these three sets, the demands are randomly generated and all edges are required. The fourth set, due to Li [26] and Li and Eglese [27], contains 24 instances based on the road network in Cumbria. The complete data for these instances are available on the web (<http://www.uv.es/~belengue/carp.html>).

Tables 1–4 present some detailed results for these instances. The first four columns give the instance name, the number of vertices and edges, and the minimum number of vehicles needed. The columns marked  $LB_{NE}$  and  $LB_E$  give the lower bounds obtained with our column generation algorithm, with non-elementary and elementary routes, respectively. The column marked  $LB_A$  gives, for comparison, the lower bound obtained by Ahr’s cutting plane algorithm [3], based on the exact separation of capacity and  $R$ -odd cut inequalities. (Ahr’s algorithm ran into time and memory problems for the 12 larger *eg1* instances. For those instances, the number reported within parentheses is the best lower bound obtained by his algorithm at the point

Table 1 Results for Golden et al. instances.

Inst.	V	E	$m$	$LB_{NE}$	$LB_E$	$LB_A$	Opt.
gdb1	12	22	5	282	285	<b>316</b>	<b>316</b>
gdb2	12	26	6	313	314	<b>339</b>	<b>339</b>
gdb3	12	22	5	248	250	<b>275</b>	<b>275</b>
gdb4	11	19	4	266	272	<b>287</b>	<b>287</b>
gdb5	13	26	6	358	359	<b>377</b>	<b>377</b>
gdb6	12	22	5	282	284	<b>298</b>	<b>298</b>
gdb7	12	22	5	288	293	<b>325</b>	<b>325</b>
gdb8	27	46	10	319	330	<b>344</b>	<b>348</b>
gdb9	27	51	10	291	294	<b>303</b>	<b>303</b>
gdb10	12	25	4	254	254	<b>275</b>	<b>275</b>
gdb11	22	45	5	364	364	<b>395</b>	<b>395</b>
gdb12	13	23	7	422	444	450	<b>458</b>
gdb13	10	28	6	525	525	<b>536</b>	<b>536</b>
gdb14	7	21	5	98	98	<b>100</b>	<b>100</b>
gdb15	7	21	4	56	56	<b>58</b>	<b>58</b>
gdb16	8	28	5	122	122	<b>127</b>	<b>127</b>
gdb17	8	28	5	85	85	<b>91</b>	<b>91</b>
gdb18	9	36	5	159	159	<b>164</b>	<b>164</b>
gdb19	8	11	3	47	<b>55</b>	<b>55</b>	<b>55</b>
gdb20	11	22	4	107	114	<b>121</b>	<b>121</b>
gdb21	11	33	6	151	151	<b>156</b>	<b>156</b>
gdb22	11	44	8	196	196	<b>200</b>	<b>200</b>
gdb23	11	55	10	<b>233</b>	<b>233</b>	<b>233</b>	<b>233</b>

**Table 2**  
Results for Kiuchi et al. instances.

Inst.	V	E	<i>m</i>	$LB_{NE}$	$LB_E$	$LB_A$	Opt.
kshs1	8	15	4	13 363	13 553	<b>14 661</b>	<b>14 661</b>
kshs2	10	15	4	8195	8723	<b>9863</b>	<b>9863</b>
kshs3	6	15	4	8401	8654	<b>9320</b>	<b>9320</b>
kshs4	8	15	4	11 442	<b>11 498</b>	11 098	<b>11 498</b>
kshs5	8	15	3	10 215	10 370	<b>10 957</b>	<b>10 957</b>
kshs6	9	15	3	9080	9232	<b>10 197</b>	<b>10 197</b>

**Table 3**  
Results for Benavent et al. instances.

Inst.	V	E	<i>m</i>	$LB_{NE}$	$LB_E$	$LB_A$	Opt.
val1a	24	39	2	220	220	<b>247</b>	<b>247</b>
val1b	24	39	3	223	225	<b>247</b>	<b>247</b>
val1c	24	39	8	294	313	309	<b>319</b>
val2a	24	34	2	270	277	<b>298</b>	<b>298</b>
val2b	24	34	3	300	304	328	<b>330</b>
val2c	24	34	8	515	<b>528</b>	526	<b>528</b>
val3a	24	35	2	92	93	<b>105</b>	<b>105</b>
val3b	24	35	3	99	101	<b>111</b>	<b>111</b>
val3c	24	35	7	153	155	159	<b>162</b>
val4a	41	69	3	478	478	<b>522</b>	<b>522</b>
val4b	41	69	4	490	492	<b>534</b>	<b>534</b>
val4c	41	69	5	511	515	<b>550</b>	<b>550</b>
val4d	41	69	9	615	621	642	<b>652</b>
val5a	34	65	3	524	524	<b>566</b>	<b>566</b>
val5b	34	65	4	545	548	586	<b>589</b>
val5c	34	65	5	574	578	610	<b>617</b>
val5d	34	65	9	682	689	714	<b>720</b>
val6a	31	50	3	300	305	<b>330</b>	<b>330</b>
val6b	31	50	4	309	315	336	<b>340</b>
val6c	31	50	10	397	405	414	<b>424</b>
val7a	40	66	3	352	358	<b>382</b>	<b>382</b>
val7b	40	66	4	354	361	<b>386</b>	<b>386</b>
val7c	40	66	9	401	407	430	<b>437</b>
val8a	30	63	3	489	489	<b>522</b>	<b>522</b>
val8b	30	63	4	501	502	<b>531</b>	<b>531</b>
val8c	30	63	9	633	638	645	<b>657</b>
val9a	50	92	3	407	407	<b>450</b>	<b>450</b>
val9b	50	92	4	412	412	<b>453</b>	<b>453</b>
val9c	50	92	5	419	419	<b>459</b>	<b>459</b>
val9d	50	92	10	481	484	505	<b>518</b>
val10a	50	97	3	590	590	<b>637</b>	<b>637</b>
val10b	50	97	4	597	597	<b>645</b>	<b>645</b>
val10c	50	97	5	608	609	<b>655</b>	<b>655</b>
val10d	50	97	10	691	695	731	<b>735</b>

where the run was aborted.) The last column displays the optimal solution value (in the case of the *gdb*, *val* and *kshs* instances), or the best known upper bound at the time of writing (in the case of the *egl* instances). The optimal solution values are taken from Belenguer and Benavent [17], Longo et al. [5], Baldacci and Maniezzo [10] and Ghiani et al. [15]. The upper bounds are taken from Brandão and Eglese [28]. Bounds that have been proven optimal are in bold font.

We point out that not all of the *gdb* and *kshs* instances are sparse. Indeed, the instances *gdb16* and *kshs3* are defined on complete graphs.

Two main conclusions can be drawn from these tables. First, forbidding non-elementary routes leads to significantly improved bounds, especially in the case of the *egl* instances. Second, the bounds obtained with column generation alone are weaker than those obtained with cutting planes alone. However,  $LB_E$  is stronger than  $LB_A$  in a few cases (specifically, in two *val* instances, one *kshs* instance and eight *egl* instances).

To explore further the quality of the various bounds, we constructed Table 5. This table shows, for the three sets of instances that have been solved to optimality and for several lower bounds, the average ratio between the bound and the optimum. The mean-

**Table 4**  
Results for Li and Eglese instances.

Inst.	V	E	$E_R$	<i>m</i>	$LB_{NE}$	$LB_E$	$LB_A$	UB
egl-e1-A	77	98	51	5	2983	3425	3516	<b>3548</b>
egl-e1-B	77	98	51	7	3791	4291	4436	<b>4498</b>
egl-e1-C	77	98	51	10	4931	5472	5481	5595
egl-e2-A	77	98	72	7	4221	4832	4963	<b>5018</b>
egl-e2-B	77	98	72	10	5463	6105	6271	6317
egl-e2-C	77	98	72	14	7679	8187	8155	8335
egl-e3-A	77	98	87	8	5076	5706	5866	<b>5898</b>
egl-e3-B	77	98	87	12	6882	7541	7649	7777
egl-e3-C	77	98	87	17	9434	10 086	10 119	10 305
egl-e4-A	77	98	98	9	5634	6233	6378	6456
egl-e4-B	77	98	98	14	8048	8678	8838	9000
egl-e4-C	77	98	98	19	10 770	11 416	11 376	11 601
egl-s1-A	140	190	75	7	4170	4985	(4975)	<b>5018</b>
egl-s1-B	140	190	75	10	5542	6284	(6180)	6388
egl-s1-C	140	190	75	14	7716	8423	(8286)	8518
egl-s2-A	140	190	147	14	8867	9667	(9718)	9956
egl-s2-B	140	190	147	20	12 146	12 801	(12 835)	13 165
egl-s2-C	140	190	147	27	15 618	16 262	(16 216)	16 524
egl-s3-A	140	190	159	15	9190	9925	(9991)	10 260
egl-s3-B	140	190	159	22	12 752	13 388	(13 520)	13 807
egl-s3-C	140	190	159	29	16 390	17 014	(16 958)	17 234
egl-s4-A	140	190	190	19	11 314	11 905	(12 007)	12 341
egl-s4-B	140	190	190	27	15 266	15 891	(15 897)	16 462
egl-s4-C	140	190	190	35	19 651	20 197	(20 176)	20 591

**Table 5**  
Average ratios computed for the test sets and five lower bounds.

Set	$LB_{NE}$	$LB_E$	$LB_{BB}$	$LB_A$	$LB_C$
<i>gdb</i>	0.9368	0.9524	0.9987	0.9987	0.9990
<i>val</i>	0.9230	0.9327	0.9940	0.9941	0.9964
<i>kshs</i>	0.9103	0.9315	0.9942	0.9942	–

**Table 6**  
Time to solve the master LP under six pricing strategies.

Set	NE-C	NE-S	NE-SH	E-D	E-U	E-M
<i>gdb</i>	2.26	0.45	0.17	28.61	54.23	20.10
<i>val</i>	64.33	31.99	10.81	4364.19	6696.91	3531.59
<i>ksh</i>	0.85	0.22	0.09	3.75	3.86	3.75
<i>egl</i>	410.20	500.14	385.40	36 890.00	57 932.94	30 542.09

ing of the headings  $LB_{NE}$ ,  $LB_E$  and  $LB_A$  is as before;  $LB_{BB}$  corresponds to the bound obtained by Belenguer and Benavent’s cutting plane algorithm [17], when heuristics are used for separation of capacity and *R*-odd cut inequalities; and  $LB_C$  to the bound obtained with the cut-and-price algorithm of Gómez-Cabrero et al. [18], which used non-elementary routes in the pricing routine, and heuristic separation of capacity and *R*-odd cut inequalities. Based on these figures, we conjecture that a cut-and-price algorithm using exact pricing with elementary routes, or exact separation of capacity and *R*-odd cut inequalities, or both, would give extremely strong lower bounds.

Finally, Table 6 reports, for each set of instances and for six different pricing strategies, the average time (in seconds) taken to solve the master LP to optimality. As before, the columns headed *NE* correspond to the case in which non-elementary routes are permitted, whereas the columns headed *E* correspond to the case in which only elementary routes are permitted. For the column *NE-C*, we used the traditional pricing routine, based on a transformation to a *complete* graph. For the column *NE-S*, we used our *sparse* pricing routine, and for the column *NE-SH* we also used our pricing *heuristics*. The columns *E-D*, *E-U* and *E-M* correspond to the *directed*, *undirected* and *mixed* MIP formulations, respectively.

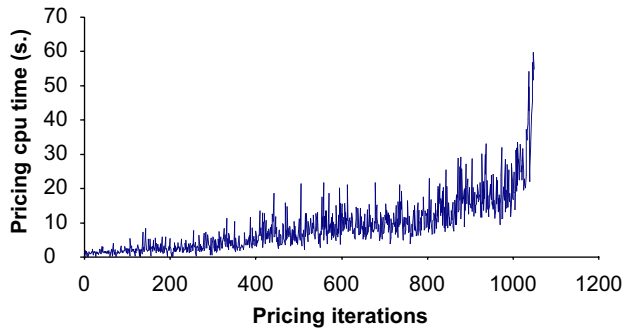


Fig. 1. Pricing cpu time when only the elementary routes are generated.

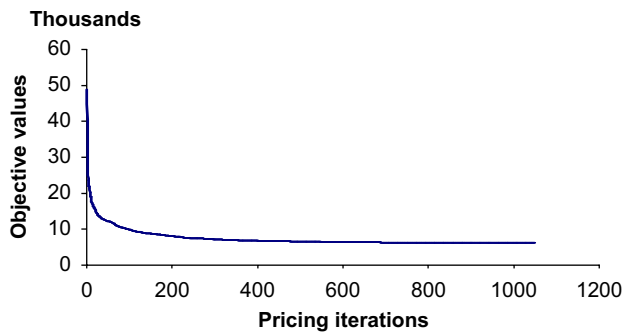


Fig. 2. Tail-off effect curve when only the elementary routes are generated.

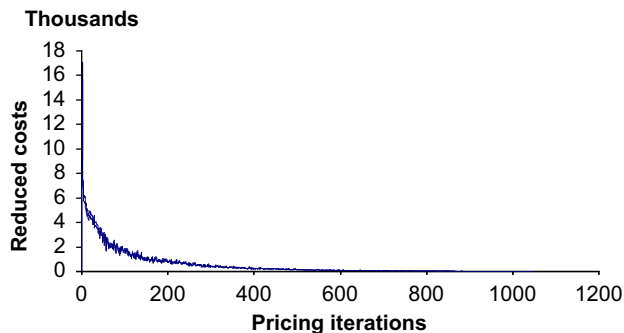


Fig. 3. Reduced costs when only the elementary routes are generated.

The cpu times corresponding to the case in which non-elementary routes are permitted show clearly that the sparse approach performs better than the complete approach for all sets of instances, except the *eg1* instances. A closer look at the results for the *eg1* instances revealed that the complete approach works better than the sparse approach when there is a significant number of non-required edges. Therefore, the sparse approach appears to be more suited for instances where most or all of the edges are required.

As for the case where only elementary routes are generated, the undirected MIP formulation is the slowest of the three formulations. Although it has fewer variables and constraints than the other MIP formulations, and the LP relaxation is typically solved more quickly, the number of branch-and-bound nodes tends to be higher. The directed MIP formulation performs well for small instances, but rather poorly for larger ones, probably due to the relatively large number of variables and constraints. Finally, the mixed MIP formulation is solved considerably more quickly than the others.

Even using the mixed MIP formulation, running times are excessive for the *val* and *eg1* instances. In an attempt to understand this, we produced Figs. 1–3. The first figure shows how the time

taken to solve the pricing subproblem typically evolves: the subproblems faced in the later iterations tend to be the hardest ones. This may be because the later routes tend to accommodate more customers, which makes the ‘knapsack’ aspect of the pricing subproblem harder. The second and third figures show that the column generation process also exhibits a strong ‘tailing-off’ phenomenon, i.e., small changes in the objective function value and in the reduced costs in the later iterations. This behaviour is typical of column generation algorithms.

## 6. Concluding remarks

In this paper, we have argued that exact algorithms for the CARP should be capable of exploiting the sparsity of typical real-life CARP instances. We have also argued that algorithms based on the set covering formulation are ideal for doing this. To back up this claim, we have shown how to exploit sparsity in a core component of such algorithms, namely the pricing subproblem.

There are several potential topics for future research. First, one could experiment with *k-cycle elimination* in the exact routine for non-elementary routes, as mentioned in Section 3.1. Second, in the case of elementary routes, one might consider solving the pricing MIPs themselves by branch-and-cut, instead of just feeding them to a standard MIP solver. Third, one could attempt to improve the convergence of the column generation process by using *stabilisation* techniques (see, e.g., du Merle et al. [29]). Fourth, one could develop a full *branch-cut-and-price* algorithm for sparse CARP instances, using one or more of our pricing algorithms as subroutines. Some thought would be needed to devise branching rules that effectively exploited graph sparsity.

Finally, we remark that our algorithm for pricing with non-elementary routes can be adapted with little modification to other VRPs or ARPs defined on sparse graphs. For example, one can easily obtain an  $\mathcal{O}(QT|E|)$  pricing algorithm for the *capacitated arc routing problem with time windows* (CARPTW), where  $T$  is the number of time periods. However, the approach we have given for pricing with elementary routes is not so easily adapted. For example, we do not know how to formulate the pricing problem for the CARPTW as a mixed-integer program with only  $\mathcal{O}(|E|)$  variables.

## Appendix

**Proposition 1.** *In the case of elementary routes, there exists at least one optimal solution to the pricing subproblem such that no edge is traversed more than once in any given direction.*

**Proof (Sketch).** Given an optimal solution to the pricing subproblem, let  $G'$  be the corresponding Eulerian multigraph, with one copy of each edge for each time that it is traversed by the vehicle. If  $G'$  is a graph (i.e., if it does not contain any parallel edges), then there is nothing to prove. So suppose that  $G'$  contains parallel edges. Note that, for a fixed  $(i, j) \in E$ , there cannot exist more than two parallel edges in  $G'$  between  $i$  and  $j$ . (If there were, we could delete two of them to obtain an Eulerian multigraph of lower cost, contradicting the fact that  $G'$  represents an optimal solution to the pricing subproblem.) So we can assume that, if there are any parallel edges, then they come in pairs. Let  $e^1$  and  $e^2$  form a pair of parallel edges in  $G'$ , and let  $G^-$  denote the (multi)graph obtained from  $G'$  by deleting  $e^1$  and  $e^2$ . We consider two cases:

Case 1:  $G^-$  is disconnected. In this case, the vehicle must travel from  $i$  to  $j$  via  $e^1$ , and travel from  $j$  to  $i$  via  $e^2$ , or vice versa.

Case 2:  $G^-$  is connected. In this case, we can construct an Eulerian traversal of  $G^-$ . We can then convert it into an Eulerian traversal of  $G'$  as follows. When the vehicle arrives at node  $i$ , traverse the edge

$e^1$  to arrive at node  $j$ , and then immediately traverse the edge  $e^2$  to return to node  $i$ .

In either case, we can assume that  $e^1$  and  $e^2$  are traversed in opposite directions. The proof can be completed via induction, since we can apply the same argument to  $G^-$ .  $\square$

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