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A new separation algorithm for the Boolean quadric and cut polytopes

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ABSTRACT

A *separation algorithm* is a procedure for generating cutting planes. Up to now, only a few polynomial-time separation algorithms were known for the *Boolean quadric* and *cut* polytopes. These polytopes arise in connection with zero–one quadratic programming and the max-cut problem, respectively. We present a new algorithm, which separates over a class of valid inequalities that includes all odd bicycle wheel inequalities and $(2p+1, 2)$-circulant inequalities. It exploits, in a non-trivial way, three known results in the literature: one on the separation of $\{0, \frac{1}{2}\}$-cuts, one on the symmetries of the polytopes in question, and one on an affine mapping between the polytopes.

1. Introduction

A popular way to tackle hard combinatorial optimisation problems is to formulate them as Integer Linear Programs (ILPs), define an associated family of polytopes, and then derive linear inequalities that define faces (preferably facets) of those polytopes (see, e.g., [1,2]). These inequalities can then be used as cutting planes within a branch-and-cut framework (see, e.g., [2–4]).

In order actually to use a class of inequalities as cutting planes, one needs a *separation* algorithm. A separation algorithm, for a given family of polytopes and a given class of inequalities, is an algorithm that takes as input a point that does not lie in one of the polytopes, and outputs a violated inequality in the given class, if one exists [5]. A great deal of the research on separation algorithms has been carried out in the context of the *traveling salesman problem* (see, e.g., [6,7]). Even so, useful separation algorithms have been discovered for many other $\mathcal{NP}$-hard combinatorial optimisation problems; see [1–4] for surveys.

In this paper, we are concerned with the so-called *Boolean quadric* and *cut* polytopes. The Boolean quadric polytope, first defined by Padberg [8], arises in the context of *unconstrained zero–one quadratic programming*. The cut polytope, defined by Barahona and Mahjoub [9], arises in connection with the *max-cut* problem. Both problems have a wide array of important applications (see, e.g., [10,11]). The reason that we consider these polytopes together is that there is a well-known affine mapping from one to the other, known as the *covariance map* [8,11–13].

A vast array of valid and facet-defining inequalities have been discovered for the Boolean quadric and cut polytopes (see the survey in [11]). On the other hand, there exist relatively few separation algorithms (see the next section). In this paper, we present a new separation algorithm, and show that it separates over a class of inequalities that includes all of the so-called *odd bicycle wheel* inequalities [9] and $(2p+1, 2)$-*circulant* inequalities [14]. A separation algorithm with this property

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was already found by one of the authors [15], but it was impractical, being based on solving a series of large linear programs. Our new algorithm is much faster.

We remark that our algorithm exploits, in a non-trivial way, three known results in the literature:

1. A result, due to Caprara and Fischetti [16], on the complexity of separation for a class of cutting planes for general ILPs, called $\{0, \frac{1}{2}\}$-cuts.
2. A result, due to Barahona and Mahjoub [9], about the invariance of the cut polytope with respect to the so-called *switching* operation.
3. The result mentioned above, about the equivalence of the Boolean quadric and cut polytopes under the covariance map.

The structure of the paper is as follows. Section 2 is the literature review. In Section 3, we present some simple valid inequalities for the Boolean quadric polytope. These will be used later, in our separation algorithm, to generate more complex inequalities. The separation algorithm itself is presented in Section 4, along with the analysis of its running time. Section 5 presents the algorithm for the cut polytope, and then shows how it can be used to derive a second separation algorithm for the Boolean quadric polytope, which is slower, but separates over a wider class of inequalities. Finally, some concluding remarks are made in Section 6.

Throughout the paper, we let V_n and E_n denote the vertex and edge sets, respectively, of a complete undirected graph of order n. That is, V_n denotes $\{1, \ldots, n\}$ and E_n denotes $\{S \subset V_n : |S| = 2\}$.

2. Literature review

In this section, we review the relevant literature. We cover the polytopes in Section 2.1, the covariance map and switching in Section 2.2, valid inequalities in Section 2.3, separation routines in Section 2.4, and $\{0, \frac{1}{2}\}$-cuts in Section 2.5.

2.1. The polytopes

The *Boolean quadric polytope* of order n, denoted by BQP_n, is the convex hull of vectors $(x, y) \in \{0, 1\}^{V_n + E_n}$ satisfying $y_{ij} = x_i x_j$ for all $\{i, j\} \in E_n$ (Padberg [8]). A vector $(x, y) \in \mathbb{Z}^{V_n + E_n}$ is an extreme point of BQP_n if and only if it satisfies the following linear inequalities, due to Fortet [17]:

$$y_{ij} \geq 0 \quad (\{i, j\} \in E_n) \tag{1}$$

$$y_{ij} - x_i \leq 0 \quad (i \in V_n,\ j \in V_n \setminus \{i\}) \tag{2}$$

$$x_i + x_j - y_{ij} \leq 1 \quad (\{i, j\} \in E_n). \tag{3}$$

We will call these *trivial* inequalities.

Given any vertex set $S \subseteq V_n$, the edge-set

$$\delta(S) = \{\{i, j\} \in E_n : i \in S, j \in V_n \setminus S\}$$

is called a *cut*. The *cut polytope* of order n, denoted by CUT_n, is the convex hull of vectors $z \in \{0, 1\}^{E_n}$ that are incidence vectors of cuts (Barahona and Mahjoub [9]). As noted in [9], CUT_n is the convex hull of vectors $z \in \{0, 1\}^{E_n}$ satisfying the following linear inequalities:

$$z_{ij} - z_{ik} - z_{jk} \leq 0 \quad (\{i, j\} \in E_n,\ k \in V_n \setminus \{i, j\}) \tag{4}$$

$$z_{ij} + z_{ik} + z_{jk} \leq 2 \quad (\{i, j, k\} \subset V_n). \tag{5}$$

These are called *triangle* inequalities [9].

2.2. The covariance map and switching

It was pointed out in [8,12,13] that a point (x^*, y^*) belongs to BQP_n if and only if the point z^* belongs to CUT_{n+1}, where:

$$z^*_{i,n+1} = x^*_i \quad (i \in V_n)$$
$$z^*_{ij} = x^*_i + x^*_j - 2y^*_{ij} \quad (\{i, j\} \in E_n).$$

This linear mapping is called the *covariance map*. A consequence of this map is that the inequality $\alpha^T z \leq \beta$ is valid for CUT_{n+1} if and only if the inequality

$$\sum_{i \in V_n} \left(\sum_{j \in V_{n+1} \setminus \{i\}} \alpha_{ij} \right) x_i - 2 \sum_{e \in E_n} \alpha_e y_e \leq \beta$$

is valid for BQP_n.

Another important mapping, called *switching*, was defined in [9]. Given any $S \subset V_n$, switching leaves z_e unchanged for all $e \in E_n \setminus \delta(S)$, but maps z_e onto $1 - z_e$ for all $e \in \delta(S)$. (Note that switching is affine but not linear.) It is shown in [9] that CUT_n is invariant under switching. It follows that, if the inequality $\lambda^T z \leq \gamma$ is valid for CUT_n, then the 'switched' inequality

$$\sum_{e \in E_n \setminus \delta(S)} \lambda_e z_e - \sum_{e \in \delta(S)} \lambda_e z_e \leq \gamma - \sum_{e \in \delta(S)} \lambda_e$$

is also valid, for any $S \subset V_n$.

Switching was adapted to BQP_n in [8]. For a given S, one must map:

- x_i onto $1 - x_i$ for all $i \in S$,
- y_{ij} onto $x_j - y_{ij}$ for all $i \in S$ and $j \in V_n \setminus S$,
- y_{ij} onto $1 - x_i - x_j + y_{ij}$ for all $\{i, j\} \subset S$.

Again, BQP_n is unchanged under this operation.

2.3. Valid inequalities

Padberg [8] showed that the trivial inequalities (1)–(3) define facets of BQP_n, along with the following *triangle* inequalities:

$$-x_k - y_{ij} + y_{ik} + y_{jk} \leq 0 \quad (\{i, j\} \in E_n,\ k \in V_n \setminus \{i, j\}) \tag{6}$$

$$x_i + x_j + x_k - y_{ij} - y_{ik} - y_{jk} \leq 1 \quad (\{i, j, k\} \subset V_n). \tag{7}$$

He also introduced some other facet-defining inequalities, called *clique*, *cut* and *generalised cut* inequalities. A huge class of valid inequalities, generalising all of Padberg's, was introduced by Boros and Hammer [18].

Barahona and Mahjoub [9] showed that the triangle inequalities (4), (5) define facets of CUT_n. They also proved the following two results:

Proposition 1 (*Barahona and Mahjoub [9]*). *For any $S \subseteq V_n$ with $|S| \geq 3$ and odd, the 'odd clique' inequality*

$$\sum_{e \subset S} z_e \leq \lfloor |S|^2/4 \rfloor \tag{8}$$

defines a facet of CUT_n.

Proposition 2 (*Barahona and Mahjoub [9]*). *Let $C \subset E_n$ be the edge set of a simple cycle of odd length, let s and t be distinct nodes not in the cycle, and let S be the set of 'spokes', i.e., edges connecting either s or t to a node in the cycle. The 'odd bicycle wheel' inequality*

$$z_{st} + \sum_{e \in C \cup S} z_e \leq 2|C| \tag{9}$$

defines a facet of CUT_n.

We will also need the following result:

Proposition 3 (*Poljak and Turzik [14]*). *Let $p \geq 2$ be an even integer, and let $v_1, \ldots, v_{2p+1}$ be distinct vertices in V_n. Then the* '$(2p + 1, 2)$*-circulant' inequality*

$$\sum_{i=1}^{2p+1} (z(v_i, v_{i+1}) + z(v_i, v_{i+2})) \leq 3p \tag{10}$$

defines a facet of CUT_n.

Note that odd clique inequalities with $|S| = 5$, odd bicycle wheel inequalities with $|C| = 3$ and $(5, 2)$-circulant inequalities are all equivalent.

For a detailed survey of other valid and facet-defining inequalities for BQP_n and CUT_n, see Deza and Laurent [11].

2.4. Known separation routines

To our knowledge, little has been published on separation for BQP_n. Of course, the trivial inequalities (1)–(3) and triangle inequalities (6), (7) can be separated in $\mathcal{O}(n^2)$ and $\mathcal{O}(n^3)$ time, respectively, by enumeration. In [19–21], separation *heuristics* are presented for various special cases of the Boros–Hammer inequalities. In [22], it is shown that a *weakened* version of the Boros–Hammer inequalities can be separated efficiently, via an eigenvector computation.

As for CUT_n, it is well known that the triangle inequalities (4), (5) can be separated in $\mathcal{O}(n^3)$ time, by enumeration. Gerards [23] presented a separation algorithm for the odd bicycle wheel inequalities (9), that can be implemented to run in $\mathcal{O}(n^5)$ time. In [24–26], separation heuristics are presented for the so-called *hypermetric* inequalities, and switchings of them. In [26], a separation heuristic is also presented for the odd clique inequalities (8), and their switchings. In [27], it is shown that some additional valid inequalities, that are never facet-defining, can be separated efficiently, via an eigenvector computation. (The results in [22,27] are equivalent under the covariance map.)

The key result of relevance to this paper, presented by one of the authors in [15], is that one can separate over a class of inequalities that includes all of the odd bicycle wheel inequalities (9) and $(2p+1, 2)$-circulant inequalities (10), using lift-and-project techniques. However, this necessitates the solution of $\binom{n}{2}$ linear programs, each with $\mathcal{O}(n^3)$ variables and $\mathcal{O}(n^2)$ constraints, and so is impractical. The main purpose of the present paper is to present an alternative separation algorithm that does the same, but is fast enough to be of practical use.

Finally, we mention that there are also several separation algorithms designed for max-cut instances on sparse graphs (see, e.g., [9,12,23,28–30]). For the sake of brevity, we do not give details.

2.5. $\{0, \frac{1}{2}\}$-cuts

Finally, we recall some results about $\{0, \frac{1}{2}\}$-cuts.

Definition 1 (*Caprara and Fischetti [16]*)**.** Let $Ax \le b$ be a system of linear inequalities, where $A \in \mathbb{Z}^{p \times q}$, $x \in \mathbb{Z}^q$ and $b \in \mathbb{Z}^p$, and let

$$P_I = \mathrm{conv}\left\{x \in \mathbb{Z}^q : Ax \le b\right\}$$

be the associated integral polyhedron. A '$\{0, \frac{1}{2}\}$-Chvátal–Gomory cut' (or '$\{0, \frac{1}{2}\}$-cut' for short) is a valid inequality for P_I of the form $(\lambda^T A)x \le \lfloor \lambda^T b \rfloor$, where $\lambda \in \{0, \frac{1}{2}\}^p$ is such that $\lambda^T A \in \mathbb{Z}^q$, but $\lambda^T b \notin \mathbb{Z}$.

Caprara and Fischetti showed that the separation problem for $\{0, \frac{1}{2}\}$-cuts is $\mathcal{NP}$-hard in general, but solvable in polynomial time under certain conditions. One of these conditions, of relevance here, is that the matrix A has at most two odd coefficients per row.

For the purpose of what follows, we recall their separation algorithm for this special case. One constructs a weighted labelled graph, which we call the *auxiliary* graph, in which there is a node for each variable and an edge for each constraint. There is also a dummy node, say node $q+1$. If a constraint has odd left-hand side coefficients for variables i and j, then the corresponding edge is $\{i, j\}$. If it has an odd left-hand side coefficient for only one variable, say i, then the corresponding edge is $\{i, q+1\}$. Each edge is given a weight equal to the slack of the corresponding constraint. If the right-hand side of a constraint is odd, then the corresponding edge is labelled odd; otherwise it is labelled even. Then, every odd cycle of weight less than 1 in the resulting auxiliary graph corresponds to a violated $\{0, \frac{1}{2}\}$-cut, and vice-versa. So, to find a violated $\{0, \frac{1}{2}\}$-cut, it suffices to find a minimum weight odd cycle in the auxiliary graph. This can be done, e.g., using the approach described in [5,9], which involves running Dijkstra's single-source shortest path algorithm [31] $q+1$ times in a graph that has twice as many nodes and edges as the auxiliary graph. We will call this last graph the *expanded* graph.

For the case in which A does not satisfy the condition mentioned, Caprara and Fischetti suggested *weakening* the linear system in order to obtain a system that does meet the condition.

3. Three linear systems for BQP_n

The purpose of this section is to show that one can construct, in polynomial time, a system of valid linear inequalities for BQP_n, such that:

- the system meets the condition mentioned in Section 2.5 (i.e., each inequality has no more than two odd left-hand side coefficients), and
- the family of $\{0, \frac{1}{2}\}$-cuts that can be derived from the system contains an exponential number of facet-defining members.

In the following three subsections, we consider three candidates for this linear system, in increasing order of complexity.

3.1. A simple weakened linear system

As a starting point, we consider the following simple linear system:

$$-y_{ij} \le 0 \quad (\{i, j\} \in E_n) \tag{11}$$

$$-x_i + y_{ij} \le 0 \quad (i \in V_n,\ j \in V_n \setminus \{i\}) \tag{12}$$

$$x_i \le 1 \quad (i \in V_n) \tag{13}$$

$$x_i + x_j - 2y_{ij} \le 1 \quad (\{i, j\} \in E_n). \tag{14}$$

We call this 'system I'. Note that the inequalities (14) are a weakened version of the trivial inequalities (3). Note also that, although the inequalities (13) are dominated by the trivial inequalities (2) and (3), they are not dominated by the other inequalities in system I.

The following theorem characterises the valid inequalities for BQP_n that can be derived as $\{0, \frac{1}{2}\}$-cuts from system I:

Theorem 1. *Let P^1 be the polytope defined by the inequalities in system* I*, together with all inequalities that can be derived as $\{0, \frac{1}{2}\}$-cuts from system* I*. A complete and non-redundant linear description of P^1 is given by the trivial inequalities (1)–(3) and the triangle inequalities (6)–(7).*

Proof. If we multiply the inequalities $x_i + x_j - 2y_{ij} \leq 1, x_i \leq 1$ and $x_j \leq 1$ by $\frac{1}{2}$ and sum them together, we obtain the inequality $x_i + x_j - y_{ij} \leq \frac{3}{2}$. This shows that the trivial inequality (3) is a $\{0, \frac{1}{2}\}$-cut.

Similarly, if we multiply the inequalities $-x_i+y_{ik} \leq 0,\ -x_k+y_{ik} \leq 0, -x_j+y_{jk} \leq 0, -x_k+y_{jk} \leq 0$ and $x_i+x_j-2y_{ij} \leq 1$ by $\frac{1}{2}$ and sum them together, we obtain the inequality $-x_k - y_{ij} + y_{ik} + y_{jk} \leq \frac{1}{2}$. This shows that the triangle inequality (6) is a $\{0, \frac{1}{2}\}$-cut.

Moreover, if we multiply the inequalities $x_i + x_j - 2y_{ij} \leq 1,\ x_i + x_k - 2y_{ik} \leq 1$ and $x_j + x_k - 2y_{jk} \leq 1$ by $\frac{1}{2}$ and sum them together, we obtain the inequality $x_i + x_j + x_k - y_{ij} - y_{ik} - y_{jk} \leq \frac{3}{2}$. This shows that the triangle inequality (7) is a $\{0, \frac{1}{2}\}$-cut.

The above three results show that all points in P^1 satisfy the trivial and triangle inequalities. Now, it was shown by Boros et al. [32] that the only non-redundant inequalities that can be derived as CG-cuts from the system (1)–(3) are the triangle inequalities. Therefore, every point satisfying the trivial and triangle inequalities must lie in P^1. □

Theorem 1 is rather disappointing, since one can easily solve the separation problem for the triangle inequalities by mere enumeration, without invoking the machinery of $\{0, \frac{1}{2}\}$-cuts. To obtain more interesting $\{0, \frac{1}{2}\}$-cuts, we must enlarge our linear system.

3.2. A more sophisticated linear system

We now present two more classes of valid inequalities, each having only two odd left-hand side coefficients, which will turn out to be very useful. First, by summing together one triangle inequality of the form (6) and one trivial inequality of the form (2), we obtain:

$$-2x_k - y_{ij} + y_{ik} + 2y_{jk} \leq 0 \quad (\{i, j, k\} \subset V_n). \tag{15}$$

Second, by summing together two triangle inequalities of the form (6) and two trivial inequalities of the form (1), we obtain:

$$-2x_i + y_{ij} + y_{ik} + 2y_{i\ell} - 2y_{j\ell} - 2y_{k\ell} \leq 0 \quad (\{i, j, k, \ell\} \subset V_n). \tag{16}$$

We can now form a new linear system, called 'system II', by adding the inequalities (15) and (16) to those already present in system I.

It follows from Theorem 1 that the trivial inequalities (3) and the triangle inequalities (6) and (7) can be derived as $\{0, \frac{1}{2}\}$-cuts from system II. The following theorem shows that an exponentially-large family of non-trivial inequalities can be derived in the same way.

Theorem 2. *Let $C \subset E_n$ be the edge set of a simple cycle of odd length, let h be a node not in the cycle, and let T be the set of 'spokes', i.e., edges connecting each node in the cycle to h. The 'odd wheel' inequality*

$$-\left\lfloor \frac{|C|}{2} \right\rfloor x_h - \sum_{e \in C} y_e + \sum_{e \in T} y_e \leq 0 \tag{17}$$

is a $\{0, \frac{1}{2}\}$-cut with respect to system II.

Proof. Let c denote $|C|$. Without loss of generality, assume that the cycle passes through nodes $1, \ldots, c$, in order. If we sum together the following $\lfloor \frac{c}{2} \rfloor$ inequalities of type (16):

$$-2x_h + y_{1h} + 2y_{2h} + y_{3h} - 2y_{12} - 2y_{23} \leq 0$$
$$-2x_h + y_{3h} + 2y_{4h} + y_{5h} - 2y_{34} - 2y_{45} \leq 0$$
$$\cdots$$
$$-2x_h + y_{c-2,h} + 2y_{c-1,h} + y_{ch} - 2y_{c-2,c-1} - 2y_{c-1,c} \leq 0$$

together with the following inequalities from system I:

$$-x_1 + y_{1h} \leq 0$$
$$-x_c + y_{ch} \leq 0$$
$$x_1 + x_c - 2y_{1c} \leq 1$$

we obtain:

$$-(c-1)x_h + 2\sum_{i=1}^{c} y_{ih} - 2\sum_{i=1}^{c} y_{i,i+1} \le 1.$$

Dividing this inequality by 2 and rounding down the right-hand side, we obtain the odd wheel inequality (17). □

We will show in Section 5 that every odd wheel inequality is equivalent (via the covariance map) to a switching of an odd bicycle wheel inequality. Therefore, every odd wheel inequality defines a facet of BQP_n.

We now introduce a second family of inequalities, also exponentially-large, that can be derived as $\{0, \frac{1}{2}\}$-cuts from system II.

Theorem 3. *Let $p \ge 4$ be an even integer, and let $v_1, \ldots, v_{2p}$ be distinct vertices in V_n. The inequality*

$$-x(v_1) - x(v_{2p}) - \sum_{i=3}^{2p-2} x(v_i) - y(v_1, v_{2p}) - \sum_{i=1}^{p-1} y(v_{2i}, v_{2i+1}) + \sum_{i=1}^{p} y(v_{2i-1}, v_{2i}) + \sum_{i=1}^{2p-2} y(v_i, v_{i+2}) \le 0 \qquad (18)$$

is a $\{0, \frac{1}{2}\}$-cut with respect to system II.

Proof. Without loss of generality, assume that $v_i = i$ for all i. Sum together the following three inequalities from system I:

$$-x_1 + y_{1,2} \le 0, \qquad -x_{2p} + y_{2p-1,2p} \le 0, \qquad x_1 + x_{2p} - 2y_{1,2p} \le 1$$

together with the following inequalities of type (15):

$$-2x_{2i-1} + y_{2i-1,2i} - y_{2i,2i+1} + 2y_{2i-1,2i+1} \le 0 \quad (i = 1, \ldots, p-1)$$
$$-2x_{2i+2} + y_{2i+1,2i+2} - y_{2i,2i+1} + 2y_{2i,2i+2} \le 0 \quad (i = 1, \ldots, p-1).$$

Dividing the resulting inequality by 2 and rounding down the right-hand side yields (18). □

We will show in Section 5 that every inequality (18) is equivalent (via the covariance map) to a switching of a $(2p+1, 2)$-circulant inequality. Therefore, every inequality (18) defines a facet of BQP_n as well.

3.3. A linear system that is closed under switching

Although system II does yield two exponentially-large families of facet-defining $\{0, \frac{1}{2}\}$-cuts, it has a rather undesirable feature: the family of $\{0, \frac{1}{2}\}$-cuts is not closed with respect to the switching operation. To see this, consider the inequality that we obtain if we take odd wheel inequality (17) and switch on node h. It takes the form:

$$\left\lfloor \frac{c}{2} \right\rfloor x_h + \sum_{i \in V(C)} x_i - \sum_{e \in C \cup T} y_e \le \left\lfloor \frac{c}{2} \right\rfloor, \qquad (19)$$

where $V(C)$ is the set of nodes in the cycle C.

Lemma 1. *System* II *does not yield the inequality (19) as a $\{0, \frac{1}{2}\}$-cut.*

Proof. By definition, a $\{0, \frac{1}{2}\}$-cut cannot be violated by more than 1/2, if the point to be separated satisfies the inequalities in the given linear system. So consider the point $(x^*, y^*) \in [0, 1]^{V_n + E_n}$ with $x_i^* = \frac{1}{2}$ for all $i \in V_n$ and $y_e^* = 0$ for all $e \in E_n$. It satisfies the inequalities in system II, but violates (19) by $(c+1)/4$, which exceeds 1/2 (since $c \ge 3$). □

This problem can be resolved by creating an even larger linear system, in which there are not only additional inequalities, but also the following additional 'artificial' variables:

$$x'_i = 1 - x_i \quad (\forall i \in V_n)$$
$$y'_{ij} = x_i x'_j = x_i - y_{ij} \quad (\forall i \in V_n, j \in V_n \setminus \{i\})$$
$$y''_{ij} = x'_i x'_j = 1 - x_i - x_j + y_{ij} \quad (\forall \{i, j\} \in E_n).$$

Note that the y' variables are 'directed', in the sense that y'_{ij} is not equal to y'_{ji}. Note also that the constraints $y'_{ij} = x_i x'_j$ and $y''_{ij} = x'_i x'_j$ are quadratic, so we cannot insert them into our linear system. Also, the constraints $y'_{ij} = x_i - y_{ij}$ and $y''_{ij} = 1 - x_i - x_j + y_{ij}$ have more than two odd left-hand side coefficients, so they cannot be inserted either. Nevertheless, we can derive useful additional linear inequalities that can be so inserted. In particular, we can insert:

- inequalities of the form $-y'_{ij} \le 0$ and $-y''_{ij} \le 0$, which are analogous to the inequalities (11);
- inequalities of the form $y'_{ij} \le x_i$, $y'_{ij} \le x'_j$, $y''_{ij} \le x'_i$ and $y''_{ij} \le x'_j$, which are analogous to (12);

- inequalities of the form $x'_i \le 1$, which are analogous to (13);
- inequalities of the form $x_i + x'_j \le 1 + 2y'_{ij}$ and $x'_i + x'_j \le 1 + 2y''_{ij}$, which are analogous to (14).

In exactly the same way, one can insert 'switched' versions of the inequalities (15) and (16). We call the resulting linear system 'system III'.

Now, consider any inequality that can be derived as a $\{0, \frac{1}{2}\}$-cut from system II. In order to obtain a particular switching of this inequality, we just apply the same switching to all the original inequalities that are used to generate the cut. For example, to derive the switched odd wheel inequality (19), we take the inequalities that were used in the proof of Theorem 2, and switch on node h. That is, we use the inequalities:

$$-2x'_h + y'_{1h} + 2y'_{2h} + y'_{3h} - 2y_{12} - 2y_{23} \le 0$$
$$-2x'_h + y'_{3h} + 2y'_{4h} + y'_{5h} - 2y_{34} - 2y_{45} \le 0$$
$$\dots$$
$$-2x'_h + y'_{c-2,h} + 2y'_{c-1,h} + y'_{ch} - 2y_{c-2,c-1} - 2y_{c-1,c} \le 0$$

together with the inequalities

$$-x_1 + y'_{1h} \le 0$$
$$-x_c + y'_{ch} \le 0$$
$$x_1 + x_c - 2y_{1c} \le 1.$$

This yields the following $\{0, \frac{1}{2}\}$-cut:

$$-\left\lfloor \frac{c}{2} \right\rfloor x'_h + \sum_{i=1}^{c} y'_{ih} - \sum_{i=1}^{c} y_{i,i+1} \le 0.$$

Expressing this $\{0, \frac{1}{2}\}$-cut in terms of the original variables, we obtain (19).

4. The algorithm

In this section, we present our separation algorithm for BQP_n. A naive version, which runs in $\mathcal{O}(n^6)$ time, is given in Section 4.1. An improved version, which runs in $\mathcal{O}(n^4)$ time, is given in Section 4.2.

4.1. Naive approach

Since every inequality in system III has at most two odd left-hand side coefficients, the result of Caprara and Fischetti, mentioned in Section 2.5, implies that there exists a polynomial-time separation algorithm for the associated family of $\{0, \frac{1}{2}\}$-cuts. Observe, however, that system III contains $\mathcal{O}(n^2)$ variables and $\mathcal{O}(n^4)$ inequalities. Then, if we apply the Caprara–Fischetti scheme as described in Section 2.5, the expanded graph will contain $\mathcal{O}(n^2)$ nodes and $\mathcal{O}(n^4)$ edges. We have to run Dijkstra's algorithm $\mathcal{O}(n^2)$ times in that graph. Each Dijkstra call will take $\mathcal{O}(n^4)$ time, leading to a total running time of $\mathcal{O}(n^6)$. Clearly, a separation algorithm with such a high running time is unlikely to be of practical use.

4.2. Reducing the running time

In this subsection, we present two key results, each of which enables us to reduce the running time of the separation algorithm by a factor of $\mathcal{O}(n)$. The first is the following:

Proposition 4. *Given a specific point $(x^*, y^*) \in [0, 1]^{V_n + E_n}$ to be separated, one can extract, in $\mathcal{O}(n^4)$ time, a subset $\mathcal{S}$ of the inequalities in system* III *with the following properties:*

- *There are only $\mathcal{O}(n^3)$ inequalities in $\mathcal{S}$.*
- *If (x^*, y^*) violates a $\{0, \frac{1}{2}\}$-cut from system* III*, then a $\{0, \frac{1}{2}\}$-cut that is violated by at least as much can be derived using only inequalities from $\mathcal{S}$.*

Proof. The inequalities that come from system I, namely (11)–(14), are only $\mathcal{O}(n^2)$ in number, so we put all of those into $\mathcal{S}$ immediately, along with their switched versions. The inequalities (15) are only $\mathcal{O}(n^3)$ in number, so we also put all of those into $\mathcal{S}$, along with their switched versions.

The inequalities (16), on the other hand, are $\mathcal{O}(n^4)$ in number. Notice however that only the variables y_{ij} and y_{ik} have odd coefficients. Therefore, for a fixed ordered triple (i, j, k), the inequalities (16) that are obtained by varying ℓ all correspond to parallel even edges in the weighted labelled graph described in Section 2.5. Now, a minimum weight odd cycle will never contain more than one of those edges, and, if it does include one, it will include one of minimum weight. Therefore, for each triple, it suffices to put into $\mathcal{S}$ just one of the inequalities, namely, the one with smallest slack. For a given triple, this can be done in $\mathcal{O}(n)$ time, by choosing the index $\ell \in V_n \setminus \{i, j, k\}$ that maximises $y^*_{i\ell} - y^*_{j\ell} - y^*_{k\ell}$. Repeating this for each of the $\mathcal{O}(n^3)$ triples, and also for the switched versions of the inequalities, we construct the desired set $\mathcal{S}$ in $\mathcal{O}(n^4)$ time. □

Once Proposition 4 has been applied, the expanded graph has only $\mathcal{O}(n^3)$ edges, rather than $\mathcal{O}(n^4)$ as before. If we use the Fibonacci heap variant of Dijkstra's algorithm (see [33]), then each of the $\mathcal{O}(n^2)$ shortest-path computations will take only $\mathcal{O}(n^3)$ time. Therefore, the total running time of the separation algorithm has been reduced from $\mathcal{O}(n^6)$ to $\mathcal{O}(n^5)$.

To obtain a further reduction in running time, we will reduce the number of shortest-path computations that are needed. To do that, we will need the following lemma:

Lemma 2. *If a* $\{0, \frac{1}{2}\}$*-cut is derived from system* III, *then at least one of the constraints* (13) *or* (14), *or at least one of their switched versions, must be used in the derivation. That is, the multiplier* λ_j *must equal* $1/2$ *for at least one of those constraints.*

Proof. In Definition 1, $\lambda^T b$ must be fractional. Since the constraints (11), (12), (15) and (16) all have a right-hand side of 0, the only way to make $\lambda^T b$ fractional is for λ_j to equal 1/2 for at least one of the constraints (13) or (14). □

We now present the main result of this subsection:

Theorem 4. *The separation problem for the* $\{0, \frac{1}{2}\}$*-cuts derived from system* III *can be solved in* $\mathcal{O}(n^4)$ *time.*

Proof. Note that a constraint of the form (13) has an odd left-hand side coefficient for the variable x_i. Therefore, the associated odd edge in the auxiliary graph is incident on the node that represents x_i. Similarly, an odd edge that corresponds to a constraint of the form (14) is incident on the nodes that represent the variables x_i and x_j. The situation with the 'switched' versions of the constraints is similar, except that the edges may be incident on nodes that represent the switched variables x'_i and/or x'_j.

Together with Lemma 2, this implies that any odd cycle in the auxiliary graph must pass through at least one node that represents an x variable or an x' variable. This means in turn that one needs to run Dijkstra's algorithm only from nodes that represent x or x' variables in the expanded graph. Since there are only n x variables and n x' variables, the total number of Dijkstra calls needed is only $\mathcal{O}(n)$. Since each Dijkstra call takes only $\mathcal{O}(n^3)$ time when Proposition 4 has been applied, the total running time is only $\mathcal{O}(n^4)$. □

Although the running time of $\mathcal{O}(n^4)$ is still rather high, note that our algorithm can generate several violated $\{0, \frac{1}{2}\}$-cuts in a single call, rather than just one. Indeed, there are $\mathcal{O}(n)$ odd cycle computations in the auxiliary graph, and every odd cycle of weight less than 1 corresponds to a violated $\{0, \frac{1}{2}\}$-cut.

5. To the cut polytope (and back again)

In this section, we will use the covariance map (see Section 2.2) to show that there exists a polynomial-time separation algorithm for CUT_n which separates over a family of inequalities that includes all odd bicycle wheel and $(p, 2)$-circulant inequalities, along with their switchings. After that, we will show how a second application of the covariance map enables us to separate over a broader class of inequalities for the Boolean quadric polytope.

5.1. Adaptation to the cut polytope

Recall that the cut polytope CUT_n is the convex hull of the points $z \in \{0, 1\}^{E_n}$ that satisfy the triangle inequalities (4), (5). Observe that the right-hand side of every triangle inequality is even. Therefore, it is impossible to derive $\{0, \frac{1}{2}\}$-cuts for CUT_n from the triangle inequalities. To get around this difficulty, we will use the covariance map.

So, let $z^* \in [0, 1]^{E_{n+1}}$ be a fractional point that lies outside of CUT_{n+1}. Applying the covariance map, this can be transformed into a fractional point $(x^*, y^*) \in [0, 1]^{V_n + E_n}$ that lies outside BQP_n. Suppose we apply the separation algorithm described in the previous section to (x^*, y^*). If a violated inequality is found, then the covariance map can be used to construct a valid inequality for CUT_{n+1} that is violated by z^*.

The following two propositions imply that, in this way, one can separate over a class of inequalities for CUT_{n+1} that includes all switched odd wheel and $(2p + 1, 2)$-circulant inequalities that 'involve' node $n + 1$.

Proposition 5. *If an odd bicycle wheel inequality* (9) *for* CUT_{n+1} *satisfies* $s = n + 1$ *or* $t = n + 1$, *then the corresponding valid inequality for* BQP_n *(obtained via the covariance map) can be derived as a* $\{0, \frac{1}{2}\}$*-cut from system* III.

Proof. By symmetry, it suffices to consider the case $t = n + 1$. Consider an odd bicycle wheel inequality (9) for CUT_{n+1}, in which $t = n + 1$. If we switch on nodes s and t, we obtain:

$$z_{st} + \sum_{e \in C} z_e - \sum_{e \in S} z_e \leq 0.$$

If we apply the covariance map to this latter inequality, and set $s = h$, we obtain the odd wheel inequality (17), which Theorem 2 states can be derived as a $\{0, \frac{1}{2}\}$-cut from system II. Now, the set of inequalities that can be derived as $\{0, \frac{1}{2}\}$-cuts from system III includes all switchings of all inequalities that can be derived as $\{0, \frac{1}{2}\}$-cuts from system II. The corresponding set of inequalities for CUT_{n+1} therefore includes all switchings of the given odd bicycle wheel inequality. □

Proposition 6. *If a* $(2p+1, 2)$*-circulant inequality* (10) *for* CUT_{n+1} *satisfies* $n+1 \in \{v_1, \ldots, v_{2p+1}\}$*, then the corresponding valid inequality for* BQP_n *(obtained via the covariance map) can be derived as a* $\{0, \frac{1}{2}\}$*-cut from system* III.

Proof. If $p = 2$, then the circulant inequality is also an odd bicycle wheel inequality, and the result follows from Proposition 5. So, suppose that $p \geq 4$. By symmetry, it suffices to consider the case $n+1 = v_{2p+1}$. If we take the circulant inequality (10) and switch on nodes $\bigcup_{i=1}^{p/2}\{4i-3, 4i\}$, we obtain:

$$z(v_1, v_{2p}) + z(v_2, v_{2p+1}) + z(v_{2p-1}, v_{2p+1}) - z(v_1, v_{2p+1}) - z(v_{2p}, v_{2p+1}) + \sum_{i=1}^{p-1} z(v_{2i}, v_{2i+1}) - \sum_{i=1}^{p} z(v_{2i-1}, v_{2i}) - \sum_{i=1}^{2p-2} z(v_i, v_{i+2}) \leq 0.$$

If we apply the covariance map to this latter inequality, we obtain the inequality (18), which Theorem 3 states can be derived as a $\{0, \frac{1}{2}\}$-cut from system II. The rest of the proof is similar to that of Proposition 5. □

We are now ready to prove the key result of this section:

Theorem 5. *There exists an algorithm, running in* $\mathcal{O}(n^5)$ *time, that solves the separation problem for a class of valid inequalities for* CUT_{n+1} *that includes all odd bicycle wheel and* $(2p+1, 2)$*-circulant inequalities.*

Proof. It follows from Theorem 4 and Propositions 5 and 6 that one can separate in $\mathcal{O}(n^4)$ time over a class of inequalities that includes all odd bicycle wheel and $(2p+1, 2)$-circulant inequalities that 'involve' node $n+1$. If we take a node $i \neq n+1$, and re-number the nodes so that i becomes $n+1$ and vice-versa, then we can separate in $\mathcal{O}(n^4)$ time over a class of inequalities that includes all odd bicycle wheel and $(2p+1, 2)$-circulant inequalities that 'involve' node i. Doing this for all i yields the desired $\mathcal{O}(n^5)$ algorithm. □

We remark that our algorithm has the same running time as the one by Gerards [23], yet separates over a much wider family of inequalities. Moreover, it is much faster than the one by Letchford [15] (see Section 2.4).

5.2. Moving back to the Boolean quadric polytope

In the previous subsection, we managed to separate over a wider class of valid inequalities for CUT_{n+1} by calling our original separation algorithm $n+1$ times instead of only once. Now, recall that BQP_n and CUT_{n+1} are affinely congruent under the covariance map. It follows that, by calling our original separation algorithm $n+1$ times, it should also be possible to separate over a wider class of valid inequalities for BQP_n. This leads naturally to the following separation scheme for BQP_n:

- Let $(x^*, y^*) \in [0, 1]^{V_n+E_n}$ be a fractional point that lies outside BQP_n.
- Apply the covariance map, to obtain a fractional point $z^* \in [0, 1]^{E_{n+1}}$ that lies outside of CUT_{n+1}.
- Run the separation algorithm for CUT_{n+1} presented in the previous subsection.
- If a valid inequality for CUT_{n+1} is found that is violated by z^*, apply the covariance map to convert it into a valid inequality for BQP_n that is violated by (x^*, y^*).

Note that this separation scheme runs in $\mathcal{O}(n^5)$ time. The following example shows that it can yield cutting planes that cannot be derived using the original $\mathcal{O}(n^4)$ algorithm.

Example. Let $n = 5$ and consider the fractional point (x^*, y^*) obtained by setting x_i^* to 1/2 for $i = 1, \ldots, 5$ and y_{ij}^* to 1/6 for $1 \leq i < j \leq 5$. One can check (either by hand or on a computer, with the help of the separation algorithm presented in Section 4), that (x^*, y^*) does not violate any inequality in system III, or any $\{0, \frac{1}{2}\}$-cut that can be derived from system III. Now, applying the covariance map, we obtain the fractional point z^* with $z_{ij}^* = 2/3$ for $1 \leq i < j \leq 5$ and $z_{i,6}^* = 1/2$ for $1 \leq i \leq 5$. Now, in one of the major iterations of the separation algorithm presented in the previous subsection, we will re-number the nodes so that 5 becomes 6 and vice-versa, and apply the covariance map again. This yields the new fractional point $(\tilde{x}, \tilde{y})$, with $\tilde{x}_i = 2/3$ for $1 \leq i \leq 4$, $\tilde{x}_5 = 1/2$ and $\tilde{y}_{ij} = 1/3$ for $1 \leq i < j \leq 5$. Running our original separation algorithm on this new point, we obtain the switched odd wheel inequality

$$2\sum_{i=1}^{4} x_i - \sum_{1 \leq i < j \leq 4} y_{ij} \leq 3,$$

which is violated by 1/3. This inequality corresponds, under the covariance map, to the odd clique inequality

$$\sum_{1 \leq i < j \leq 5} z_{ij} \leq 6,$$

which the point z^* violates by 2/3. Applying the covariance map once more, we obtain the inequality

$$2\sum_{i=1}^{5} x_i - \sum_{1\le i<j\le 5} y_{ij} \le 3,$$

which the original point (x^*, y^*) violates by 1/3.

6. Conclusion

Although the Boolean quadric and cut polytopes have been studied in great depth, there are only a few classes of facet-defining inequalities for which exact polynomial-time separation algorithms are known. We have derived a separation algorithm for the Boolean quadric polytope that runs in $\mathcal{O}(n^4)$ time and separates over a class of inequalities that includes all odd wheel inequalities. Then, using the covariance map and switching, we have derived a separation algorithm for the cut polytope that runs in $\mathcal{O}(n^5)$ time and separates over a class of inequalities that includes all odd bicycle wheel and $(2p+1, 2)$-circulant inequalities. This latter algorithm is much faster and simpler than the one presented in [15], which involved the solution of $\mathcal{O}(n^2)$ linear programs, each with $\mathcal{O}(n^3)$ variables and $\mathcal{O}(n^2)$ constraints.

We have performed some preliminary computational experiments on small unconstrained 0–1 quadratic programming and max-cut instances, including some of our own and some from the `BiqMac` library [34]. For the unconstrained 0–1 quadratic programming instances, we found that the $\mathcal{O}(n^4)$ separator typically closes around 65% of the integrality gap between the optimum and the bound obtained using the trivial inequalities (1)–(3) and the triangle inequalities (6), (7). For the max-cut instances, we found that the $\mathcal{O}(n^5)$ separator typically closes around 75% of the integrality gap between the optimum and the bound obtained using the triangle inequalities (4), (5).

Unfortunately, as one might expect, the time taken by the $\mathcal{O}(n^5)$ separator was excessive. If one wished to use it in a branch-and-cut algorithm for the max-cut problem, it would be a good idea to give priority to other, faster separation routines (such as the ones described in [19–21,26]), and only call the $\mathcal{O}(n^5)$ separator when those faster routines fail.

This leads naturally to two possible topics for future research: whether the algorithms can be put to good use in a branch-and-cut algorithm, and whether they can made faster, either theoretically or empirically. Another topic that would be worth studying is whether our algorithm could be somehow adapted to max-cut instances that are defined on *sparse* graphs, rather than complete graphs.

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