SIAM J. OPTIM.
Vol. 24, No. 3, pp. 1294–1312

ITERATED CHVÁTAL–GOMORY CUTS AND THE GEOMETRY OF NUMBERS*

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Abstract. Chvátal–Gomory cutting planes (CG-cuts for short) are a fundamental tool in integer programming. Given any single CG-cut, one can derive an entire family of CG-cuts, by "iterating" its multiplier vector modulo one. This leads naturally to two questions: first, which iterates correspond to the strongest cuts, and, second, can we find such strong cuts efficiently? We answer the first question empirically, by showing that one specific approach for selecting the iterate tends to perform much better than several others. The approach essentially consists of solving a nonlinear optimization problem over a special lattice associated with the CG-cut. We then provide a partial answer to the second question, by presenting a polynomial-time algorithm that yields an iterate that is strong in a certain well-defined sense. The algorithm is based on results from the algorithmic geometry of numbers.

Key words. integer programming, cutting planes, covering radius, distribution of lattices

AMS subject classifications. Primary, 90C10; Secondary, 90C27, 52C17, 11H16, 11J71

DOI. 10.1137/130926389

1. Introduction.

Let $\boldsymbol{x} \in \mathbb{Z}^n$ be a vector of integer-constrained decision variables, and let $A\boldsymbol{x} \leq \boldsymbol{b}$ be a system of linear inequalities, where $A \in \mathbb{Z}^{m\times n}$ and $\boldsymbol{b} \in \mathbb{Z}^m$. A *Chvátal–Gomory cutting plane*, or *CG-cut* for short, is a linear inequality of the form

$$\left(\boldsymbol{\lambda}^T A\right)\boldsymbol{x} \leq \left\lfloor \boldsymbol{\lambda}^T \boldsymbol{b} \right\rfloor \tag{1.1}$$

for some multiplier vector $\boldsymbol{\lambda} \in \mathbb{R}^m_{\geq 0}$ with $\boldsymbol{\lambda}^T A \in \mathbb{Z}$. (Here, $\lfloor\cdot\rfloor$ denotes rounding down to the nearest integer. If $\boldsymbol{\lambda}^T \boldsymbol{b} \in \mathbb{Z}$, we call the CG-cut (1.1) *trivial*.)

CG-cuts are so-called because they were derived by Chvátal [13], based on earlier work of Gomory [19, 20]. They form a fundamental family of cutting planes for *integer linear programs* (ILPs); see, e.g., [35, 45].

A large number of papers have appeared that use CG-cuts either theoretically or algorithmically. We survey some of them in section 2. One well-known operation in the literature for creating new CG-cuts from old ones is to take a multiplier vector $\boldsymbol{\lambda}$ and an integer t, and create the new multiplier vector $t\boldsymbol{\lambda} \bmod 1 := t\boldsymbol{\lambda} - \lfloor t\boldsymbol{\lambda}\rfloor$. (When $\lfloor\cdot\rfloor$ is applied to a vector, each component of the vector is rounded down.) We call this operation "iterating modulo 1."

This leads naturally to two questions: first, which choices for the integer t correspond to strong cuts, and, second, can we find such strong cuts efficiently? In this paper, we answer the first question empirically, by showing that one specific approach for selecting t tends to perform much better than several others. The approach essentially amounts to solving a nonlinear optimization problem over a special lattice associated with the initial cut. To address the second question, we first show that for a

*Received by the editors June 26, 2013; accepted for publication (in revised form) April 14, 2014; published electronically August 14, 2014.
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"typical" cut the covering radius of the associated lattice is small. This result justifies using the covering radius for estimating the quality of the iterates. We then provide a partial answer to the second question, by showing the existence of a polynomial-time algorithm that computes an iterated CG-cut that is strong in a certain well-defined sense. The algorithm is based on results from the algorithmic geometry of numbers and computational Diophantine approximations.

The structure of this paper is as follows. The relevant literature is briefly reviewed in the next section. In section 3, we describe several rules, both known and new, for selecting the integer t, and study their empirical performance. In section 4, we study the properties of the iterates for the case in which $\boldsymbol{\lambda}$ is random. The polynomial-time algorithm mentioned above is presented in section 5. Finally, some concluding remarks are made in section 6.

2. Literature review. In this section, we review some relevant papers, introducing some useful notation and terminology along the way.

2.1. Gomory fractional cuts. The original method of Gomory [19] was designed for ILPs of the form

$$\max\left\{\boldsymbol{c}^T x:\ C\boldsymbol{x}=\boldsymbol{d},\ \boldsymbol{x}\in\mathbb{Z}^n_{\geq 0}\right\},$$

where $\boldsymbol{c}\in\mathbb{Z}^n$, $C\in\mathbb{Z}^{p\times n}$, and $\boldsymbol{d}\in\mathbb{Z}^p$. The first step is to solve the *linear program* (LP)

$$\max\left\{\boldsymbol{c}^T\boldsymbol{x}:\ C\boldsymbol{x}=\boldsymbol{d},\ \boldsymbol{x}\in\mathbb{R}^n_{\geq 0}\right\}$$

by the simplex method. Let $\boldsymbol{x}^*$ be the optimal solution to this LP, and suppose that $x^*_k\notin\mathbb{Z}$ for some $1\leq k\leq n$. Then x_k is basic, and there exists a row of the simplex tableau of the form

$$x_k+\sum_{i\in B}\alpha_i x_i=x^*_k, \tag{2.1}$$

where B is the set of nonbasic variables. Rounding down each coefficient to the nearest integer, we obtain the valid inequality

$$x_k+\sum_{i\in B}\lfloor\alpha_i\rfloor x_i\leq\lfloor x^*_k\rfloor.$$

Using (2.1), this inequality can be written as

$$\sum_{i\in B}\{\alpha_i\}x_i\geq\{x^*_k\}, \tag{2.2}$$

where $\{r\}=r-\lfloor r\rfloor$ is the fractional part of r. The inequality (2.2) has come to be known as the *Gomory fractional cut*. We will write GF-cut for short.

Gomory (see [20, section 4]) pointed out that, by taking integral combinations of the rows of the simplex tableau, one can create new equations, from which further GF-cuts can be derived. In this way, he derived a "group" of GF-cuts. He showed that, unless the original ILP possesses an unusual degree of symmetry, then the group is *cyclic*, which means that the entire group can be derived by taking integral multiples of one single equation in the tableau.

2.2. Separation of Chvátal–Gomory cuts. Returning to CG-cuts, we define the polyhedron

$$P = \{\boldsymbol{x} \in \mathbb{R}^n : \ A\boldsymbol{x} \leq \boldsymbol{b}\}, \tag{2.3}$$

and let P_I be the convex hull of $P \cap \mathbb{Z}^n$, i.e., the so-called *integral hull* of P. Chvátal [13] defined the *elementary closure* of the P, denoted by P', as the convex set that remains after all CG-cuts have been added. Clearly, $P_I \subseteq P' \subseteq P$. Schrijver [41] showed that P' is a polyhedron, or, equivalently, that a finite subset of the CG-cuts dominates all others.

Now we consider the separation problem for CG-cuts. If P is pointed and $\boldsymbol{x}^*$ is a fractional extreme point of P, then one can generate a violated CG-cut via the following four-step procedure: (i) add slack variables to convert the inequality system $A\boldsymbol{x} \leq \boldsymbol{b}$ into an equation system, (ii) express $\boldsymbol{x}^*$ as a basic feasible solution to that equation system, (iii) generate a GF-cut, and (iv) convert the GF-cut into a CG-cut by eliminating slack variables. (For details, see, e.g., sect. II.1.3 of [35].) For general $\boldsymbol{x}^*$, however, separation over P' is NP-hard (Eisenbrand [16]). Fischetti and Lodi [18] present an integer programming approach for separating over P' in practice. Fast separation heuristics have been presented, for example, in [9, 10, 32].

2.3. Cut strengthening. GF-cuts and CG-cuts may induce facets of P_I in certain cases (see again [9, 10]). In general, however, the GF-cuts generated by Gomory's method, or the CG-cuts generated by existing separation heuristics, can be rather weak. There is considerable literature on the derivation of general families of valid linear inequalities which dominate the GF-cuts and/or CG-cuts (e.g., [12, 14, 21, 33, 35, 36]). The drawback of the inequalities described in those papers is that their coefficients are typically numerically less stable than those of GF-cuts and CG-cuts. (Recall that CG-cuts have integer coefficients by definition, and that any GF-cut can be written as a CG-cut.)

An alternative way to address the issue of cut weakness is to develop procedures which take one or more vectors $\boldsymbol{\alpha} \bmod 1$ (or, equivalently, one or more multiplier vectors $\boldsymbol{\lambda}$), and attempt to construct another vector with more desirable properties. (Here, $\boldsymbol{\alpha}$ is the vector with components α_i from (2.1).) Here are three examples of such procedures:

- Gomory (see [20, section 5]) pointed out that, if $\{x_i^*\} < 1/2$, then one can obtain a GF-cut that is at least as strong as the original, by multiplying (2.1) by the largest positive integer t such that $1/2 \leq t\{x_i^*\} < 1$.
- For the same case, Letchford and Lodi [33] suggested instead to multiply (2.1) by -1.
- Ceria, Cornuéjols, and Dawande [11] gave a heuristic, based on solving systems of linear congruences, to find a member of the group of GF-cuts with as many zero left-hand side coefficients as possible.

We follow the same approach in this paper, but use more sophisticated algorithmic tools.

We remark that sequences $t\boldsymbol{\lambda} \bmod 1$ have been investigated in a completely different context, that of the *method of good lattice points* in numerical integration. See, e.g., [28, 30, 43]. We remark also that this is not the first paper to apply tools from the geometry of numbers to integer programming; see the survey [17].

3. Rules for finding a good iterate. In this section, we examine various rules for finding a good iterated CG-cut, or, equivalently, for selecting the integer t.

Throughout this section, and in the following two, we make an important assumption. Let $\boldsymbol{x}^* \in P \setminus P'$ be a fractional point that we wish to separate, and let $\boldsymbol{\lambda}$ be an initial multiplier vector. The assumption is that $\boldsymbol{\lambda}^T(\boldsymbol{b} - A\boldsymbol{x}^*) = \boldsymbol{0}$, i.e., that all inequalities with a positive multiplier have zero slack at $\boldsymbol{x}^*$. This assumption holds, for example, when $\boldsymbol{x}^*$ is an extreme point of P and the CG-cut has been generated by the four-step procedure mentioned in subsection 2.2. It also holds when the CG-cut has been generated using the separation heuristics in [10, 32]. It has the important implication that, regardless of the integer t, every nontrivial iterated CG-cut will be violated by $\boldsymbol{x}^*$.

In the following three subsections, we present some useful notation, describe six specific rules for selecting an iterate, and present some preliminary computational results.

3.1. Some useful notation. This subsection follows from results in Schrijver [41] from which we can assume, without loss of generality, that $\boldsymbol{\lambda}$ is rational. Furthermore, the CG-cut $\left(\boldsymbol{\lambda}^T A\right)\boldsymbol{x} \leq \left\lfloor \boldsymbol{\lambda}^T \boldsymbol{b} \right\rfloor$ is implied by $A\boldsymbol{x} \leq \boldsymbol{b}$ and the CG-cut $\left((\boldsymbol{\lambda} \bmod 1)^T A\right)\boldsymbol{x} \leq \left\lfloor (\boldsymbol{\lambda} \bmod 1)^T \boldsymbol{b} \right\rfloor$. Thus we may also assume that $\boldsymbol{\lambda} \in [0,1)^m$.

Therefore, we can write

$$\boldsymbol{\lambda} = \left(\frac{p_1}{q}, \frac{p_2}{q}, \ldots, \frac{p_m}{q}\right)^T, \tag{3.1}$$

where q is a positive integer and $p_1, p_2, \ldots, p_m$ are nonnegative integers with the greatest common divisor $\gcd(p_1, p_2, \ldots, p_m, q) = 1$. Then, for any integer $1 \leq t < q$, the inequality

$$\left((t\boldsymbol{\lambda} \bmod 1)^T A\right)\boldsymbol{x} \leq \left\lfloor (t\boldsymbol{\lambda} \bmod 1)^T \boldsymbol{b} \right\rfloor \tag{3.2}$$

is a (possibly trivial) iterated CG-cut.

The family of iterated CG-cuts formed in this way is analogous to the group of GF-cuts described by Gomory, or, more precisely, to the subgroup of GF-cuts that can be derived by taking integer multiples of one single row of the tableau. Note that q can be exponentially large, and so can the family of iterated CG-cuts.

At this point, it is helpful to define the *slack vector* $\boldsymbol{s} = \boldsymbol{b} - A\boldsymbol{x}$ and the *rounding effect* $\nu = \left\{\boldsymbol{\lambda}^T \boldsymbol{b}\right\}$. Then, the iterated CG-cut (3.2) can be written in the alternative form

$$(t\boldsymbol{\lambda} \bmod 1)^T \boldsymbol{s} \geq \{t\nu\}. \tag{3.3}$$

Now, since we are assuming that $\boldsymbol{\lambda}^T \boldsymbol{s} = \boldsymbol{0}$ at $\boldsymbol{x}^*$, the left-hand side of (3.3) at $\boldsymbol{x}^*$ will be zero. This means that, provided that an iterated CG-cut is not trivial, it will be violated by $\boldsymbol{x}^*$.

3.2. Six specific rules. Now we consider how to select the integer t. A trivial strategy, which we call *Strategy* 0, is to select $t = 1$. As mentioned in subsection 2.3, Gomory [20] suggested setting $t = 1$ if $\nu < 1/2$, but to the largest integer such that $t\nu < 1$ otherwise; and Letchford and Lodi [33] suggested setting $t = 1$ if $\nu < 1/2$, but to -1 otherwise. We will call these approaches *Strategy* 1 and *Strategy* 2, respectively. Another approach, that we call *Strategy* 3, is to select an integer t such that the right-hand side of (3.3) is maximized.

The previous three strategies are concerned only with making the right-hand side of (3.3) (rounding effect) large. It is also desirable for the left-hand side to have small

norm. In this paper we propose optimizing these two quantities *simultaneously*. We consider two strategies, *multiplicative* and *additive*, to ensure that the norm of the multiplier vector is small, but the rounding effect is large.

The multiplicative strategy attempts to minimize the ratio

$$||t\boldsymbol{\lambda} \bmod 1||/\{t\nu\}$$

over all iterations with positive rounding effect $\{t\nu\}$. Here $||\cdot||$ denotes the Euclidean norm. That is, we are solving the following optimization problem:

$$\min\left\{||t\boldsymbol{\lambda} \bmod 1||/\{t\nu\} :\ t = 1, \ldots, q-1,\ \{t\nu\} > 0\right\}. \tag{3.4}$$

We will call this *Strategy* 4. Unfortunately, the complexity of this problem is unknown. We conjecture that it is NP-hard.

Let us now construct the augmented vector

$$\boldsymbol{\nu} = (\lambda_1, \ldots, \lambda_m, \nu)^T$$

and put for $\boldsymbol{x} = (x_1, \ldots, x_{d-1}, x_d)$

$$N(\boldsymbol{x}) = ||(x_1, \ldots, x_{d-1}, 1 - x_d)||.$$

The additive strategy attempts to find a vector $\boldsymbol{\xi} = t\boldsymbol{\nu} \bmod 1$ with minimum value $N(\boldsymbol{x})$ and positive last entry $\xi_d = \{t\nu\}$, which represents the rounding effect of the iterated cut. That is, we are solving the following optimization problem:

$$\min\left\{N(t\boldsymbol{\nu} \bmod 1) :\ t = 1, \ldots, q-1,\ \{t\nu\} > 0\right\}. \tag{3.5}$$

We call this *Strategy* 5. We conjecture that this problem, too, is NP-hard. In section 5, we show that both problems (3.4) and (3.5) can be solved approximately in polynomial time.

Note that the new Strategies 4 and 5 (as well as Strategies 0–3) do not depend on the objective function. Finding an effective strategy that employs the parameters of the objective function is a topic for future research.

3.3. Preliminary computational results. In order to gain some insight into the performance of the six strategies mentioned in the previous subsection, we performed some computational experiments on some small ILPs. We began by creating 45 random ILPs of the form

$$\max\left\{\boldsymbol{c}^T\boldsymbol{x} :\ A\boldsymbol{x} \le \boldsymbol{b},\ \boldsymbol{x} \in \mathbb{Z}^n_{\ge 0}\right\},$$

where $\boldsymbol{c} \in \mathbb{Z}^n_{\ge 0}$, $A \in \mathbb{Z}^{m\times n}_{\ge 0}$, and $\boldsymbol{b} \in \mathbb{Z}^m_{\ge 0}$. (Note that instances of this form are guaranteed to be feasible, since the origin is feasible.) For any pair (n, m) with $m \in \{5, 10, 15\}$ and $n \in \{10, 20, 30\}$, five such instances (m, n, k), $k \in \{1, \ldots, 5\}$, were constructed. The c_i were random integers distributed uniformly between one and five. The A_{ij} were random integers with a 50% chance of being distributed uniformly between one and five, but a 50% chance of being zero. This was to mimic the sparsity that is usually found in real-life ILPs. (If any column of A had fewer than two nonzeroes, the column was discarded and another one generated. This is to ensure boundedness.) The b_j were set to $\left\lceil \frac{1}{2}\sum_{i=1}^n A_{ij} \right\rceil$.

For each instance (m, n, k), the LP relaxation was solved to optimality and the optimal simplex tableau computed using exact rational arithmetic. (To avoid numerical problems, instances for which the determinant D of the basis matrix exceeded

TABLE 1
Computational results.

m	n	$A_0(m,n)$	$A_1(m,n)$	$A_2(m,n)$	$A_3(m,n)$	$A_4(m,n)$	$A_5(m,n)$
	10	19.29	35.46	30.98	34.99	45.94	45.02
5	20	18.89	29.14	23.56	33.67	36.00	40.84
	30	14.09	21.76	17.23	19.99	21.20	29.74
	10	4.52	5.95	4.66	6.68	13.22	10.46
10	20	3.14	5.90	4.97	7.13	11.37	8.90
	30	3.85	6.63	4.93	6.28	10.49	9.44
	10	5.89	9.02	7.53	10.02	15.92	16.39
15	20	1.86	2.94	2.51	3.68	14.16	12.15
	30	2.87	4.04	3.16	3.28	10.25	10.00
A_s :		8.27	13.43	11.06	13.97	19.83	20.33

$2 \cdot 10^6$ were discarded. The desire to keep D small also motivated the above restrictions on the coefficients.) Then, for each variable taking a fractional value in the LP solution, whether a structural variable or a slack variable, a GF-cut was generated and converted into a CG-cut. At the end, for the instance (m, n, k) and for each of the strategies $s \in \{0, \ldots, 5\}$, we stored the average $A_s(m, n, k)$, over all considered CG-cuts, of the percentage of the integrality gap closed by a CG-cut.

In Table 1, we compare all six strategies. For each value of (n, m) and for each of the strategies $s \in \{0, \ldots, 5\}$, we report the average $A_s(m, n) = (1/5) \sum_{k=1}^{5} A_s(m, n, k)$. In the last row of the table, $A_s = (1/9) \sum_{m,n} A_s(m, n)$ are the averages of $A_s(m, n, k)$ over all computed instances.

The computational results show that both Strategies 4 and 5 close significantly more of the integrality gap than the other four strategies. This indicates that the rounding effect and the norm of the multiplier vector should be simultaneously optimized for generating strong CG-cuts. To gain insight into the theoretical aspects of this problem, we study in the next section the behavior of the iterated cuts for a randomly chosen augmented vector $\boldsymbol{\nu}$.

4. Behavior of the iterates for a random vector. As illustrated by Figure 1, the values of the minima in (3.4) and (3.5) may vary significantly from one vector $\boldsymbol{\nu}$ to another, even for a fixed q. Intuitively, the chance of obtaining a good iterate is higher if the iterates are "spread" reasonably uniformly over the hypercube, as in cases C and D. This led us to examine the behavior of the iterates for "typical" vectors $\boldsymbol{\nu}$.

To formulate the obtained results, we need to introduce the following notation. Given a matrix $B \in \mathbb{R}^{d \times l}$ with linearly independent column vectors $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_l \in \mathbb{R}^d$, the set

$$L(\boldsymbol{b}_1, \ldots, \boldsymbol{b}_l) = \{u_1 \boldsymbol{b}_1 + \cdots + u_l \boldsymbol{b}_l : u_1, \ldots, u_l \in \mathbb{Z}\}$$

is called a *lattice* of *rank* (or *dimension*) l with *basis* $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_l$ and *determinant*

$$\det(L(\boldsymbol{b}_1, \ldots, \boldsymbol{b}_l)) = \sqrt{\det(B^T B)}\,.$$

For a comprehensive and extensive survey on lattices and Minkowski's geometry of numbers we refer the reader to the book of Gruber and Lekkerkerker [24].

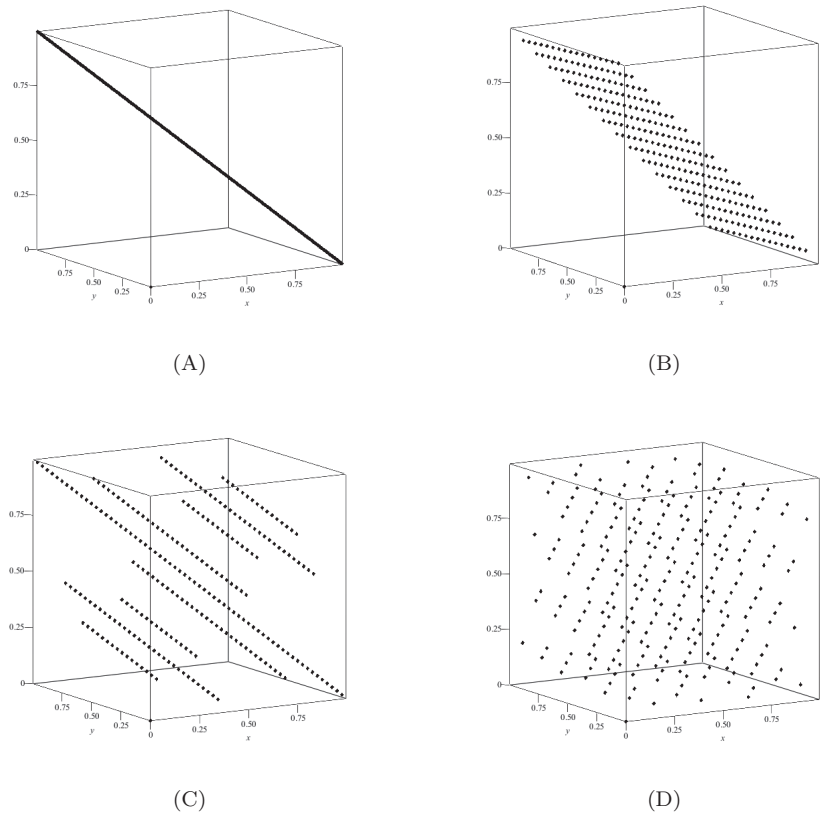

FIG. 1. *Examples of sequences* $\boldsymbol{\nu} \bmod 1$ *with* $q = 256$*:* (A) $\boldsymbol{\nu} = (1/256, 255/256, 255/256)$*;* (B) $\boldsymbol{\nu} = (1/256, 15/256, 255/256)$*;* (C) $\boldsymbol{\nu} = (1/256, 63/256, 127/256)$*; and* (D) $\boldsymbol{\nu} = (1/256, 15/256, 63/256)$.

Given lattice L, we will denote by L^* its *dual* lattice; that is,

$$L^* = \{\boldsymbol{y} \in \operatorname{span}_{\mathbb{R}}(L) : \boldsymbol{y}^T\boldsymbol{x} \in \mathbb{Z} \text{ for all } \boldsymbol{x} \in L\}\,.$$

Let $B^d(\boldsymbol{x}, r)$ denote a d-dimensional ball of radius r centered at $\boldsymbol{x}$. Given any l-dimensional lattice $\Lambda \subset \mathbb{R}^d$ we also denote by $\lambda_i = \lambda_i(\Lambda)$ its ith successive minimum

$$\lambda_i = \min\{r > 0 : \dim \operatorname{span}_{\mathbb{R}}(B^d(\mathbf{0}, r) \cap \Lambda) \geq i\}\,,\ 1 \leq i \leq l\,.$$

Recall that the *inhomogeneous minimum* of a set $S \subset \operatorname{span}_{\mathbb{R}}(L)$ with respect to a lattice L is defined as

$$\mu(S, L) = \inf\{\sigma > 0 : L + \sigma S = \operatorname{span}_{\mathbb{R}}(L)\}\,.$$

The *covering radius* $\tau(L)$ of a lattice L is the inhomogeneous minimum of the unit ball B in $\operatorname{span}_{\mathbb{R}}(L)$ with respect to L,

$$\tau(L) = \mu(B, L)\,.$$

Let also $\ll_d$ (resp., $\gg_d$) denote the Vinogradov symbol with the constant depending on d only. The notation $\gg\ll_d$ is interpreted as both $\ll_d$ and $\gg_d$ hold.

We will first study the "typical" behavior of the iterates, for a random vector $\boldsymbol{\nu}$ sampled from a certain natural distribution. In particular, we show that the covering radius of a lattice associated with $\boldsymbol{\nu}$ is relatively small on average. This is important because the quality of the approximation algorithm presented in section 5 will be defined in terms of the covering radius. (Of course, a multiplier vector obtained in a real cutting-plane algorithm will not be truly random. Nevertheless, the insights gained in this section will be useful for what follows.)

In more detail, we study in this section the behavior of the points $t\boldsymbol{\nu}$ mod 1 for a random vector $\boldsymbol{\nu}$ uniformly chosen from the set of rational vectors of the form

$$\boldsymbol{\nu} = \left(\frac{p_1}{q}, \frac{p_2}{q}, \ldots, \frac{p_d}{q}\right)^T, \quad p_1, p_2, \ldots, p_d, q \in \mathbb{Z}_{>0}\,, \quad \max_{1\le i\le d} p_i < q, \quad \gcd(p_1, p_2, \ldots, p_d, q) = 1 \tag{4.1}$$

that have denominator $q \le T$ for some $T \ge 1$. Our aim is to understand how well the points $t\boldsymbol{\nu}$ mod 1 are distributed "on average."

The iterates $t\boldsymbol{\nu}$ mod 1 can be naturally embedded in the lattice

$$L_{\boldsymbol{\nu}} = \{\boldsymbol{z} + (t\boldsymbol{\nu} \bmod 1) : \boldsymbol{z} \in \mathbb{Z}^d\,,\ t = 1, \ldots, q-1\}\,. \tag{4.2}$$

Equivalently, $L_{\boldsymbol{\nu}} = \mathbb{Z}^d + \mathbb{Z}\boldsymbol{\nu}$. This observation allows us to use results from Minkowski's geometry of numbers and, via the transference principle (see, e.g., [7]), Schmidt's theorems [39] on the distribution of integer sublattices.

The first result of this paper aims to understand the "typical" behavior of the covering radius $\tau(L_{\boldsymbol{\nu}})$ for $\boldsymbol{\nu}$ of the form (4.1) with common denominator $q \le T$. Note that for any dimension d and any common denominator $q > 1$ there exist vectors ν such that the covering radius $\tau(L_{\boldsymbol{\nu}})$ is relatively large. For instance, it is easy to see that $\tau(L_{(1/q,\ldots,1/q)}) > 1/4$ for any integer $q > 1$. In what follows, we will show that for a "typical" vector ν the covering radius $\tau(L_{\boldsymbol{\nu}})$ has the order $q^{-1/d}$.

For technical reasons it is convenient to replace the rationals with bounded denominators by the primitive integer vectors in a bounded domain. Let $\widehat{\mathbb{N}}^{d+1}$ be the set of integer vectors in $\mathbb{R}^{d+1}$ with positive co-prime coefficients, and let

$$\mathcal{D}_{d+1} = \left\{(x_1, \ldots, x_d, x_{d+1}) \in \mathbb{R}^{d+1}_{\ge 0} : \max_{j=1,\ldots,d} x_j < x_{d+1} \le 1\right\}.$$

Then for $T \ge 1$, the elements $\boldsymbol{a} = (p_1, \ldots, p_d, q)$ of the set $\widehat{\mathbb{N}}^{d+1} \cap T\mathcal{D}_{d+1}$ will correspond to the rational vectors $\boldsymbol{\nu}$ of the form (4.1) and the common denominator $q \le T$. Since $L_{\boldsymbol{\nu}}$ is uniquely defined by the integer vector $\boldsymbol{a} = (p_1, \ldots, p_d, q)$, we will also denote the lattice $L_{\boldsymbol{\nu}}$ by $L_{\boldsymbol{a}}$.

For any $T \in \mathbb{R}_{\ge 1}$ and $R \in \mathbb{R}_{>0}$, we define the quantity

$$P_d(T, R) = \frac{1}{\#(\widehat{\mathbb{N}}^{d+1} \cap T\mathcal{D}_{d+1})} \#\left\{\boldsymbol{a} \in \widehat{\mathbb{N}}^{d+1} \cap T\mathcal{D}_{d+1} : \tau(L_{\boldsymbol{a}}) a_{d+1}^{1/d} > R\right\}.$$

Roughly speaking, $P_d(T, R)$ is the probability of uniformly picking up a rational vector $\boldsymbol{\nu}$ of the form (4.1) with denominator $q \le T$, such that the iterations $t\boldsymbol{\nu}$ mod 1 are relatively badly distributed in $[0, 1)^d$ or, more precisely, such that the covering radius of the lattice $L_{\boldsymbol{\nu}}$ is bigger than $Rq^{-1/d}$.

THEOREM 4.1. *Let $d \ge 2$. Then*

$$P_d(T, R) \ll_d R^{-d}\,, \tag{4.3}$$

uniformly over all $T \geq 1$ *and all* $R > 0$. *Furthermore,*

$$P_d(T, R) = 0 \text{ whenever } R > \frac{\sqrt{d}}{2} T^{1/d}. \tag{4.4}$$

A celebrated result of Kannan [29] implies that the *Frobenius number* associated with an integer vector $\boldsymbol{a} \in \widehat{\mathbb{N}}^{d+1}$ can be estimated in terms of the covering radius of the dual lattice $L_{\boldsymbol{a}}^*$. (For more details we refer the reader to the book of Ramirez Alfonsin [38].) The following proof of Theorem 4.1 is based on a recent far-reaching refinement due to Strömbergsson [44] of the approach used in [2] and [3] for estimating the expected value of Frobenius numbers, combined with the Banaszczyk transference theorem [7]. The approach is built on results from the Minkowski's geometry of numbers (see, e.g., [23, 24]) and results on the distribution of integer lattices obtained by Schmidt in [39].

Proof of Theorem 4.1. Observe that $\mathbb{Z}^d$ is a sublattice of $L_{\boldsymbol{a}}$, and hence

$$\tau(L_{\boldsymbol{a}}) \leq \tau(\mathbb{Z}^d) = \frac{\sqrt{d}}{2}. \tag{4.5}$$

Note also that for all $\boldsymbol{a} \in \widehat{\mathbb{N}}^{d+1} \cap T\mathcal{D}_{d+1}$ we have $a_{d+1} \leq T$. Hence the inequality (4.5) implies (4.4).

Let us now prove that the inequality (4.3) holds. For a subset $Y \subset \mathbb{R}^{d+1}$ we denote by $\pi_{d+1}(Y)$ the orthogonal projection of Y onto the coordinate hyperplane $x_{d+1} = 0$; we view $\pi_{d+1}(Y)$ as a subset of $\mathbb{R}^d$. Given $\boldsymbol{a} \in \widehat{\mathbb{N}}^{d+1}$, we define the lattice

$$\Lambda_{\boldsymbol{a}} = \{\boldsymbol{x} \in \mathbb{Z}^{d+1} : \boldsymbol{x}^T \boldsymbol{a} = 0\}$$

and set $M_{\boldsymbol{a}} = \pi_{d+1}(\Lambda_{\boldsymbol{a}})$. Then $M_{\boldsymbol{a}}$ is a sublattice of $\mathbb{Z}^d$ of determinant $\det(M_{\boldsymbol{a}}) = a_{d+1}$ (see, e.g., [1, section 2]). It is well known that $M_{\boldsymbol{a}} = L_{\boldsymbol{a}}^*$ (see, e.g., [1]).

By Banaszczyk transference theorem [7], we have

$$\lambda_1(M_{\boldsymbol{a}}) \leq \frac{d}{2\tau(L_{\boldsymbol{a}})}.$$

Since $M_{\boldsymbol{a}}$ embedded in $\mathbb{R}^{d+1}$ is the orthogonal projection of $\Lambda_{\boldsymbol{a}}$ on the coordinate hyperplane $x_{d+1} = 0$ and $a_{d+1} = \max_i a_i$, we have $\lambda_1(\Lambda_{\boldsymbol{a}}) \leq \sqrt{d+1}\lambda_1(M_{\boldsymbol{a}})$ and, consequently,

$$\lambda_1(\Lambda_{\boldsymbol{a}}) \leq \frac{d\sqrt{d+1}}{2\tau(L_{\boldsymbol{a}})}. \tag{4.6}$$

In the rest of this subsection we modify the proof of Theorem 3 in [44] for our case. Roughly speaking, the main difference is that, due to the transference principle reflected in the inequality (4.6), we need to work with the first successive minimum $\lambda_1(\Lambda_{\boldsymbol{a}})$, whilst in the case of the Frobenius number the last successive minimum $\lambda_d(\Lambda_{\boldsymbol{a}})$ plays the major role.

Note first that $\#(\widehat{\mathbb{N}}^{d+1} \cap T\mathcal{D}_{d+1}) \gg\ll_d T^{d+1}$ uniformly over all $T \geq 1$ and that $\det(\Lambda_{\boldsymbol{a}}) = ||\boldsymbol{a}|| \geq a_{d+1}$. Therefore,

$$\begin{aligned} P_d(T, R) &\ll_d T^{-(d+1)} \\ &\times \#\left\{\Lambda \in \mathcal{L}_d : \det(\Lambda) \leq \sqrt{d+1}T, \lambda_1(\Lambda) < \frac{d\sqrt{d+1}\det(\Lambda)^{1/d}}{2R}\right\}, \end{aligned} \tag{4.7}$$

where $\mathcal{L}_d$ is the set of all d-dimensional sublattices of $\mathbb{Z}^{d+1}$.

Let

$$\rho_j(\Lambda) = \lambda_{j+1}(\Lambda)/\lambda_j(\Lambda)\,, \quad j = 1, \dots, d-1\,.$$

For any $\boldsymbol{r} = (r_1, \dots, r_{d-1}) \in \mathbb{R}^{d-1}_{\geq 1}$ we set

$$\mathcal{L}_d(\boldsymbol{r}) = \{\Lambda \in \mathcal{L}_d : \rho_j(\Lambda) \geq r_j\,,\ 1 \leq j \leq d-1\}\,.$$

Also let X_d be the set of all lattices $L \subset \mathbb{R}^d$ of determinant one, and let μ_d be Siegel's measure (see [42]) on X_d, normalized to be a probability measure. The main ingredient of the proof is the following result.

THEOREM 4.2 (Schmidt [39]). *For any $\boldsymbol{r} \in \mathbb{R}^{d-1}_{\geq 1}$ and $T > 0$ we have*

$$\begin{aligned}\#\{\Lambda \in \mathcal{L}_d(\boldsymbol{r}) : \det(\Lambda) \leq T\} &= \frac{\pi^{\frac{d+1}{2}}}{2\Gamma\left(1+\frac{d+1}{2}\right)} \left(\prod_{j=2}^{d} \zeta(j)\right) \\ &\times \mu_d\left(\{L \in X_d : \rho_j(L) \geq r_j\,,\ 1 \leq j \leq d-1\}\right) T^{d+1} \\ &+ O_d\left(\left(\prod_{j=1}^{d-1} r_j^{-(j-\frac{1}{d})(d-j)}\right) T^{d+1-\frac{1}{d}}\right)\,.\end{aligned} \tag{4.8}$$

Furthermore,

$$\mu_d(\{L \in X_d : \rho_j(L) \geq r_j\,,\ 1 \leq j \leq d-1\}) \gg\ll_d \prod_{j=1}^{d-1} r_j^{-j(d-j)}\,. \tag{4.9}$$

From the above theorem we get the upper bound

$$\begin{aligned}&\#\{\Lambda \in \mathcal{L}_d(\boldsymbol{r}) : \det(\Lambda) \leq T\} \ll_d T^{d+1} \\ &\times \prod_{j=1}^{d-1} r_j^{-j(d-j)} \left(1 + T^{-\frac{1}{d}} \prod_{j=1}^{d-1} r_j^{\frac{1}{d}(d-j)}\right)\,.\end{aligned} \tag{4.10}$$

By Minkowski's second theorem, for any d-dimensional lattice Λ we have

$$\lambda_1(\Lambda)^d = \frac{\prod_{j=1}^{d} \lambda_j(\Lambda)}{\prod_{j=1}^{d-1} \rho_j(\Lambda)^{d-j}} \gg\ll_d \frac{\det(\Lambda)}{\prod_{j=1}^{d-1} \rho_j(\Lambda)^{d-j}}\,. \tag{4.11}$$

Thus there exists a constant $c_1 = c_1(d) > 0$ such that for any d-dimensional lattice Λ and any $R > 0$, we have

$$\lambda_1(\Lambda) < \frac{d\sqrt{d+1}\det(\Lambda)^{1/d}}{2R} \Rightarrow \prod_{j=1}^{d-1} \rho_j(\Lambda)^{d-j} > c_1 R^d\,. \tag{4.12}$$

Assume without loss of generality that $R > e c_1^{-\frac{1}{d}}$ (the inequality (4.3) is trivial when $R \ll 1$ as $P_d(T, R) \leq 1$), put

$$B = \lfloor \log(c_1 R^d) - d \rfloor \in \mathbb{Z}_{\geq 0}\,, \tag{4.13}$$

and denote

$$\mathcal{R}(d,R)=\Big\{(e^{b_1/(d-1)},e^{b_2/(d-2)},\dots,e^{b_{d-2}/2},e^{b_{d-1}}) : \boldsymbol{b}\in\mathbb{Z}_{\ge 0}^{d-1},\sum_{j=1}^{d-1}b_j=B\Big\}. \tag{4.14}$$

If Λ is an d-dimensional lattice with $\prod_{j=1}^{d-1}\rho_j(\Lambda)^{d-j}>c_1R^d$, then for $b_j=\lfloor (d-j)\log\rho_j(\Lambda)\rfloor$ we have

$$\begin{aligned}\sum_{j=1}^{d-1}b_j&>\sum_{j=1}^{d-1}((d-j)\log\rho_j(\Lambda)-1)>\log(c_1R^d)-(d-1)\\&>\log(c_1R^d)-d\ge B\,.\end{aligned} \tag{4.15}$$

Thus we can decrease some of the numbers b_j's so as to make $\sum_{j=1}^{d-1}b_j=B$, while keeping $\boldsymbol{b}=(b_1,\dots,b_{d-1})\in\mathbb{Z}_{\ge 0}^{d-1}$. The new vector $\boldsymbol{b}$ still satisfies $b_j\le(d-j)\log\rho_j(\Lambda)$ for each j; that is, $\rho_j(\Lambda)\ge e^{b_j/(d-j)}$. Therefore, for any d-dimensional lattice Λ with $\prod_{j=1}^{d-1}\rho_j(\Lambda)^{d-j}>c_1R^d$, there exists some $\boldsymbol{r}\in\mathcal{R}(d,R)$ such that $r_j\le\rho_j(\Lambda)$ for all j.

By (4.12), the set in the right-hand side of (4.7) is contained in the union of $\mathcal{L}_d(\boldsymbol{r})$ over all $\boldsymbol{r}\in\mathcal{R}(d,R)$. Hence we have for all $T\ge 1$ and all $R\ge ec_1^{\frac{1}{d}}$,

$$\begin{aligned}P_d(T,R)&\ll_d T^{-(d+1)}\\&\times\sum_{\boldsymbol{r}\in\mathcal{R}(d,R)}\#\Big\{\Lambda\in\mathcal{L}_d(\boldsymbol{r}):\det(\Lambda)\le\sqrt{d+1}T\Big\}\,.\end{aligned} \tag{4.16}$$

By (4.10),

$$\begin{aligned}P_d(T,R)\ll_d&\sum_{\substack{\boldsymbol{b}\in\mathbb{Z}_{\ge 0}^{d-1}\\ b_1+\cdots+b_{d-1}=B}}\exp\left\{-\sum_{j=1}^{d-1}jb_j\right\}\\&+T^{-\frac{1}{d}}\sum_{\substack{\boldsymbol{b}\in\mathbb{Z}_{\ge 0}^{d-1}\\ b_1+\cdots+b_{d-1}=B}}\exp\left\{-\sum_{j=1}^{d-1}\left(j-\frac{1}{d}\right)b_j\right\}.\end{aligned} \tag{4.17}$$

If $d=2$, we get

$$P_2(T,R)\ll R^{-2}+T^{-\frac{1}{2}}R^{-1}.$$

If $R\le\frac{\sqrt{2}}{2}T^{1/2}$, then this implies $P_2(T,R)\ll R^{-2}$. On the other hand, if $R>\frac{\sqrt{2}}{2}T^{1/2}$ then $P_2(T,R)=0$ by (4.4).

Let us now assume $d\ge 3$. Observe that for any $\boldsymbol{b}\in\mathbb{Z}_{\ge 0}^{d-1}$ with $b_1+\cdots+b_{d-1}=B$ and $b_2+\cdots+b_{d-1}=s$, we have

$$\sum_{j=1}^{d-1}jb_j\ge B+s$$

and

$$\sum_{j=1}^{d-1}\left(j-\frac{1}{d}\right)b_j \geq \left(1-\frac{1}{d}\right)B+s\,.$$

Next, for $s\in\{0,1,\dots,B\}$ there are exactly $\binom{s+d-3}{d-3}$ vectors $\boldsymbol{b}\in\mathbb{Z}^{d-1}_{\geq 0}$ with $b_1+\cdots+b_{d-1}=B$ and $b_2+\cdots+b_{d-1}=s$. Therefore,

$$\begin{aligned}P_d(T,R) &\ll_d \sum_{s=0}^{B}\binom{s+d-3}{d-3}e^{-B-s}\\ +T^{-\frac{1}{d}}\sum_{s=0}^{B}\binom{s+d-3}{d-3}e^{-(1-\frac{1}{d})B-s} &\ll_d e^{-B}+T^{-\frac{1}{d}}e^{-(1-\frac{1}{d})B}\\ &\ll_d R^{-d}(1+T^{-\frac{1}{d}}R)\,.\end{aligned}$$

If $R\leq\frac{\sqrt{d}}{2}T^{1/d}$, then this implies $P_d(T,R)\ll R^{-d}$. On the other hand, if $R>\frac{\sqrt{d}}{2}T^{1/d}$, then $P_d(T,R)=0$ by (4.4). The proof is now complete. ∎

5. The approximation algorithm. We will assume for this section that $d\geq 2$. Theorem 4.1 shows that the quantity $1/q^{1/d}$ is a good predictor for the covering radius of the lattice $L_{\boldsymbol{\nu}}$. Let $\mathcal{S}=[0,+\infty)^{d-1}\times(-\infty,1)$. The following result states the existence of a polynomial-time algorithm which computes a point of the set $L_{\boldsymbol{\nu}}\cap\mathcal{S}$ in a certain ball of radius bounded in terms of $\tau(L_{\boldsymbol{\nu}})$. The obtained bound will be used to estimate the quality of polynomial-time approximations for the multiplicative and additive strategies (i.e., Strategies 4 and 5) introduced in section 3.

For $r\in\mathbb{R}$ set

$$\boldsymbol{c}(r)=(r,\dots,r,1-r)\in\mathbb{R}^d\,.$$

THEOREM 5.1. *There is a polynomial-time algorithm which, given a rational vector $\boldsymbol{\nu}$ of the form* (4.1) *and any rational $\epsilon\in(0,1)$, finds a point $\boldsymbol{\xi}\in L_{\boldsymbol{\nu}}\cap\mathcal{S}$, such that*

$$\boldsymbol{\xi}\in B(\boldsymbol{c}(r),2^{d/2}\tau(L_{\boldsymbol{\nu}}))\ \textit{with}\ 0<r\leq 2^{d/2}\tau(L_{\boldsymbol{\nu}})+\epsilon\,. \tag{5.1}$$

The proof is constructive. We present the polynomial-time algorithm in section 5.1.

5.1. Proof of Theorem 5.1. We need to find in polynomial time a point of the set $L_\nu\cap\mathcal{S}$ in a ball $B^d(\boldsymbol{c}(r),r)$. The main challenge of the proof is to choose the radius $r\ll_d\tau(L_{\boldsymbol{\nu}})$ as small as possible. Note that computing the covering radius of a lattice is conjectured in [34] to be NP-hard (see also [27, 25, 15]). The Banaszczyk transference theorem [7] gives the estimate

$$\tau(L_{\boldsymbol{\nu}})\leq\frac{d}{2\lambda_1(L^*_{\boldsymbol{\nu}})}\,,$$

which allows us to approximate $\tau(L_{\boldsymbol{\nu}})$ in polynomial time within the factor $d2^{d/2-1}$ using the celebrated LLL algorithm [31]. The approximation can then be used for computing a relatively small radius r.

In this paper we use a slightly different approach. We will choose a suitable radius r by combining binary search in a certain interval with Babai's *nearest plane algorithm*. The nearest plane algorithm finds in polynomial time an approximation to a solution of the *closest vector problem*. The quality of the approximation is given by the following result.

THEOREM 5.2 (Babai [5]). *Let L be a lattice of rank d in $\mathbb{Q}^d$. Given any basis of L and any $\boldsymbol{c} \in \mathbb{Q}^d$ as input, the nearest plane algorithm computes a vector $\boldsymbol{x} \in L$ such that*

$$||\boldsymbol{x} - \boldsymbol{c}|| \leq 2^{d/2} \min_{\boldsymbol{y} \in L} ||\boldsymbol{y} - \boldsymbol{c}|| . \tag{5.2}$$

Babai's nearest plane algorithm is based on using the LLL algorithm and, in fact, makes use also of the transference principle, via Gram–Schmidt orthogonalization. Note also that the approximation factor $2^{d/2}$ in (5.2) can be replaced by $2^{O(d(\log\log d)^2/\log d)}$ by applying the algorithm of Schnorr [40].

We shall now give a high level description of a polynomial-time algorithm that satisfies the conditions stated in Theorem 5.1. Given rational $\boldsymbol{\nu}$ of the form (4.1), we first compute a basis $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_d$ of the $L_{\boldsymbol{\nu}}$. To perform this step, we use a link between iterations of $\boldsymbol{\nu}$ modulo one and the computational Diophantine approximations. Next, we use a version of binary search to find in the interval $[0, 1]$ two rationals r^- and r^+ with $r^- < r^+$, satisfying the following properties. First, the numbers r^-, r^+ are relatively close to each other, so that $r^+ - r^- < \epsilon$. Second, Babai's nearest plane algorithm applied to $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_d$ and $\boldsymbol{c} = \boldsymbol{c}(r^+)$ finds a lattice point $\boldsymbol{\xi} \in L_{\boldsymbol{\nu}}$ such that $\boldsymbol{\xi} \in \mathcal{S}$ and the same algorithm applied to $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_d$ and $\boldsymbol{c} = \boldsymbol{c}(r^-)$ fails to find a lattice point in $\mathcal{S}$. This will imply that $\boldsymbol{\xi}$ satisfies conditions of Theorem 5.1.

The algorithm is given below.

Algorithm

Input: $\boldsymbol{\nu}$ of the form (4.1) and rational $\epsilon \in (0, 1)$.
Output: $\boldsymbol{\xi} \in L_{\boldsymbol{\nu}}$ satisfying conditions of Theorem 5.1.
Step 0: Set $r^- := 0$, $r^+ := 1$, and $\boldsymbol{\xi} := \boldsymbol{c}(1)$.
Step 1: Compute a basis $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_d$ of the lattice $L_{\boldsymbol{\nu}}$.
Step 2: **While** $r^+ - r^- > \epsilon$ **do**
2.1 Set $m := (r^- + r^+)/2$ and $\boldsymbol{c} := \boldsymbol{c}(m)$.
2.2 Apply Babai's algorithm for finding a nearby lattice point to the basis $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_d$ and the point $\boldsymbol{c}$. The algorithm returns a lattice point $\boldsymbol{\chi} \in L_{\boldsymbol{\nu}}$.
2.3 **If** $\boldsymbol{\chi} \in \mathcal{S}$, **then** set $r^+ := m$ **else** set $r^- := m$ **end if**.
end while.
Step 3: Output vector $\boldsymbol{\xi}$.

Let us now analyze the algorithm. Clearly, Step 0 can be done in polynomial time. In Step 1 we can compute a basis of $L_{\boldsymbol{\nu}}$ as follows. Consider the matrix $G(\boldsymbol{\nu}) \in \mathbb{Q}^{d \times (d+1)}$ defined as

$$G(\boldsymbol{\nu}) = \begin{pmatrix} 1 & 0 & \ldots & 0 & p_1/q \\ 0 & 1 & \ldots & 0 & p_2/q \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & p_d/q \end{pmatrix}$$

and denote by $\boldsymbol{g}_i$ its ith column vector. Observe that $L_{\boldsymbol{\nu}} = \{y_1\boldsymbol{g}_1 + \cdots + y_{d+1}\boldsymbol{g}_{d+1} : y_1, \ldots, y_{d+1} \in \mathbb{Z}\}$. Thus we can find a basis of $L_{\boldsymbol{\nu}}$ in polynomial time by Corollary 5.4.8 of [22] (see also [8]).

The *while loop* at Step 2 is performing a binary search in the interval $[0,1]$ with approximation error bounded by ϵ and thus will be executed $O(l(\epsilon))$ times, where $l(\epsilon)$ is the length of the binary expansion of the rational number ϵ. The algorithm of Babai (see [5]), applied at Step 2.2, runs in polynomial time. Step 2.3 can be done in polynomial time as well.

Thus it is now enough to show that the vector $\boldsymbol{\xi}$ output at Step 3 satisfies conditions of Theorem 5.1. By Theorem 5.2, we clearly have $\boldsymbol{\xi} \in B(\boldsymbol{c}(r^+), 2^{d/2}\tau(L_{\boldsymbol{\nu}}))$. Next, since Babai's algorithm applied to $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_d$ and the point $\boldsymbol{c}(r^-)$ returns a lattice point outside of $\mathcal{S}$, we also conclude by Theorem 5.2 that $r^- \le 2^{d/2}\tau(L)$. The latter inequality, together with $r^+ - r^- \le \epsilon$, then implies

$$r^+ \le 2^{d/2}\tau(L) + \epsilon\,.$$

Therefore, the point $\boldsymbol{\xi}$ satisfies the conditions of Theorem 5.1.

Remark. It is easy to see that, in fact, we are solving in the above proof a problem of simultaneous Diophantine approximation of rationals $p_1/q, \ldots, p_d/q$. Indeed, all points of the lattice $L_{\boldsymbol{\nu}}$ have the form $(y_1 - y_{d+1}p_1/q, \ldots, y_d - y_{d+1}p_d/q)$ with integer numbers y_i. It may also be worthwhile using another standard approach to computing Diophantine approximations with bounded denominators for a given rational vector. In this case, we construct a basis of a special lattice $\Omega \in \mathbb{Q}^{d+1}$ with $\pi_{d+1}(\Omega) = L_{\boldsymbol{\nu}}$. For details, see the proof of Theorem 5.3.19 in [22] or, for a more recent approach, Chapter 6 in [37].

5.2. Approximation for the multiplicative strategy. For the rest of the paper we set $d = m+1$. Given the multiplier vector $\boldsymbol{\lambda}$ of the form (3.1), we construct the augmented vector

$$\boldsymbol{\nu} = (\lambda_1, \ldots, \lambda_m, \nu)^T \in (0,1)^d,$$

and attempt to find a vector $\boldsymbol{\xi} = t\boldsymbol{\nu} \bmod 1$, $\xi_d > 0$, with minimum ratio

$$r(\boldsymbol{\xi}) = ||\pi_d(\boldsymbol{\xi})||/\xi_d\,.$$

Recall that for $Y \subset \mathbb{R}^d$ by $\pi_d(Y)$ we understand the orthogonal projection of Y onto the coordinate hyperplane $x_d = 0$; we view $\pi_d(Y)$ as a subset of $\mathbb{R}^{d-1}$.

As it was remarked in section 4, for any given common denominator $q > 1$ there exist rational vectors $\boldsymbol{\nu}$ of the form (4.1) with $\tau(L_{\boldsymbol{\nu}}) \gg 1$. However, due to Theorem 4.1, for a typical $\boldsymbol{\nu}$ the covering radius of the lattice $L_{\boldsymbol{\nu}}$ is of order $q^{-1/d}$. In the following we show the existence of a vector $\boldsymbol{\xi} = t\boldsymbol{\nu} \bmod 1$, with ratio $r(\boldsymbol{\xi})$ bounded in terms of the covering radius. We also show the existence of a polynomial-time algorithm which computes an approximation of that vector $\boldsymbol{\xi}$.

For $0 < R < 1/2$ set

$$a(d,R) = \frac{(1-R)((d-1)R^2 - 2R + 1)^{1/2} - (d-1)^{1/2}R^2}{dR^2 - 2R + 1}$$

and

$$r(d,R) = \begin{cases} (a(d,R)^{-2} - 1)^{1/2} & \text{for } 0 < R < 1/2\,, \\ +\infty & \text{otherwise.} \end{cases}$$

We will first prove a simple geometric lemma.

LEMMA 5.1. *Let* $0 < R < 1/2$. *Then*

$$\max\{r(\boldsymbol{x}) : \boldsymbol{x} \in B^d(\boldsymbol{c}(R), R)\} = r(d,R)\,. \tag{5.3}$$

Proof. For any fixed $1 - R \le y \le 1$, the maximum

$$\max\{r(\boldsymbol{x}) : \boldsymbol{x} = (x_1, \dots, x_{d-1}, y) \in B^d(\boldsymbol{c}(R), R)\}$$

is attained at a point of the form $(x, \dots, x, y)$. Thus we can consider only two variables, x and y, and (5.3) reduces to solving a two-dimensional trigonometric problem. Straightforward computation gives

$$\max\left\{\frac{\sqrt{d-1}\,x}{y} : (x, \dots, x, y) \in B^d(\boldsymbol{c}(R), R)\right\} = r(d, R)\,. \qquad (5.4)$$

By (5.4), we also have $r(d, R) \ll_d R$ when $0 < R < 1/2$.

PROPOSITION 5.1. *There exists a point* $\boldsymbol{\xi} = t\boldsymbol{\nu} \bmod 1$, $1 \le t \le q-1$, *with*

$$r(\boldsymbol{\xi}) \le \min\{r(d, \tau(L_{\boldsymbol{\nu}})), 2\sqrt{d-1}\}\,. \qquad (5.5)$$

Proof. Observe first that there is a positive integer t_0 such that $1/2 \le \{t_0\nu\} < 1$. Thus for $\boldsymbol{\xi} = t_0\boldsymbol{\nu} \bmod 1$, we have

$$r(\boldsymbol{\xi}) < 2\sqrt{d-1}\,.$$

This justifies the second bound in (5.5).

Recall that the iterations $t\boldsymbol{\nu} \bmod 1$ can be naturally embedded in the lattice $L_{\boldsymbol{\nu}}$. Thus, it is enough to show that there exists a nonzero point $\boldsymbol{\xi} \in L_{\boldsymbol{\nu}} \cap [0,1)^d$ that satisfies the first inequality in (5.5). If $\tau(L_{\boldsymbol{\nu}}) \ge 1/2$, the latter inequality holds by the definition of $r(d, R)$. Suppose that $\tau(L_{\boldsymbol{\nu}}) < 1/2$. By the definition of the covering radius there exists a point $\boldsymbol{\xi} \in L_{\boldsymbol{\nu}} \cap B^d(\boldsymbol{c}(\tau(L_{\boldsymbol{\nu}})), \tau(L_{\boldsymbol{\nu}}))$. Since $\tau(L_{\boldsymbol{\nu}}) < 1/2$, the point $\boldsymbol{\xi}$ is in $[0,1)^d$. The first inequality in (5.5) now holds by Lemma 5.1. ∎

On the algorithmic side, Theorem 5.1 implies the following result.

COROLLARY 5.1. *There is a polynomial-time algorithm which, given an augmented vector* $\boldsymbol{\nu} = (\lambda_1, \dots, \lambda_m, \nu)$ *of the form* (4.1) *and any rational* $\epsilon \in (0,1)$, *finds a point* $\boldsymbol{\xi} = t\boldsymbol{\nu} \bmod 1$, $1 \le t \le q-1$, *with*

$$r(\boldsymbol{\xi}) < \min\{r(d, 2^{d/2}\tau(L_{\boldsymbol{\nu}}) + \epsilon), 2\sqrt{d-1}\}\,. \qquad (5.6)$$

Proof. The first bound in (5.6) immediately follows from Theorem 5.1 and Lemma 5.1, where we take $R = 2^{d/2}\tau(L_{\boldsymbol{\nu}}) + \epsilon$.

Next, if $\nu \ge 1/2$, we have $r(\boldsymbol{\nu}) < 2\sqrt{d-1}$, so the second bound in (5.6) holds for $\boldsymbol{\xi} = \boldsymbol{\nu}$. If $0 < \nu < 1/2$, then we can take $\boldsymbol{\xi} = t_0\boldsymbol{\nu} \bmod 1$ with $t_0 = \lfloor 1/\nu \rfloor$ when $\lfloor 1/\nu \rfloor \nu \ne 1$ and $t_0 = \lfloor 1/\nu \rfloor - 1$ otherwise. ∎

5.3. Approximation for the additive strategy. Now we move on to the additive strategy. As in the previous section, for a nontrivial CG-cut (1.1) with $\boldsymbol{\lambda}$ of the form (3.1) we construct the augmented vector $\boldsymbol{\nu} = (\lambda_1, \dots, \lambda_m, \nu)^T$. One can easily obtain the following bound for problem (3.5).

PROPOSITION 5.2. *There exists a point* $\boldsymbol{\xi} = t\boldsymbol{\nu} \bmod 1$, $1 \le t \le q-1$, *with*

$$N(\boldsymbol{\xi}) \le (1 + \sqrt{d})\tau(L_{\boldsymbol{\nu}})\,. \qquad (5.7)$$

Furthermore,

$$\{t\nu\} > 0 \text{ whenever } \tau(L_{\boldsymbol{\nu}}) < 1/2\,. \qquad (5.8)$$

Proof. Observe first that the set $B^d(\boldsymbol{c}(0),(1+\sqrt{d})\tau(L_{\boldsymbol{\nu}}))\cap\mathcal{S}$ contains the ball $B^d(\boldsymbol{c}(\tau(L_{\boldsymbol{\nu}})),\tau(L_{\boldsymbol{\nu}}))$. By the definition of the covering radius there exists a point $\boldsymbol{\chi}\in L_{\boldsymbol{\nu}}\cap B^d(\boldsymbol{c}(\tau(L_{\boldsymbol{\nu}})),\tau(L_{\boldsymbol{\nu}}))$, so that $N(\boldsymbol{\chi})\le(1+\sqrt{d})\tau(L_{\boldsymbol{\nu}})$. If $\boldsymbol{\chi}\in\mathbb{Z}^d$, then we may assume without loss of generality that $\boldsymbol{\chi}=\boldsymbol{c}(1)$. Thus in this case we can take $\boldsymbol{\xi}=\boldsymbol{\nu}$. Otherwise, since $\boldsymbol{\chi}\in\mathcal{S}\setminus\mathbb{Z}^d$, we have $0<N(\boldsymbol{\chi}\bmod 1)\le N(\boldsymbol{\chi})$. Thus, the point $\boldsymbol{\xi}=\boldsymbol{\chi}\bmod 1$ satisfies condition (5.7).

Suppose now that $\tau(L_{\boldsymbol{\nu}})<1/2$. Then for all sufficiently small $\epsilon>0$ the ball $B^d(\boldsymbol{c}((\tau(L_{\boldsymbol{\nu}})+\epsilon)),\tau(L_{\boldsymbol{\nu}}))$ contains a point of the set $L_{\boldsymbol{\nu}}\cap(0,1)^d$. Since $L_{\boldsymbol{\nu}}$ is a discrete set, we conclude that there exists a point $\boldsymbol{\xi}\in L_{\boldsymbol{\nu}}\cap B^d(\boldsymbol{c}(\tau(L_{\boldsymbol{\nu}})),\tau(L_{\boldsymbol{\nu}}))\cap(0,1)^d$. This point clearly satisfies (5.2). ∎

On the other hand, Theorem 5.1 implies the following corollary.

COROLLARY 5.2. *There is a polynomial-time algorithm which, given an augmented vector $\boldsymbol{\nu}=(\lambda_1,\dots,\lambda_m,\nu)$ of the form* (4.1) *and any rational $\delta\in(0,1)$, finds a point $\boldsymbol{\xi}=t\boldsymbol{\nu}\bmod 1$, $1\le t\le q-1$, with*

$$N(\boldsymbol{\xi})<(1+\sqrt{d})2^{d/2}\tau(L_{\boldsymbol{\nu}})+\delta\,. \tag{5.9}$$

Furthermore,

$$\{t\nu\}>0 \text{ whenever } \tau(L_{\boldsymbol{\nu}})<2^{-d/2-1}(1-\delta/\lceil\sqrt{d}\rceil)\,. \tag{5.10}$$

Proof. By Theorem 5.1, given $\boldsymbol{\nu}=(\lambda_1,\dots,\lambda_m,\nu)$ and $\epsilon=\delta/\lceil\sqrt{d}\rceil\in(0,1)$ we can compute in polynomial time a point $\boldsymbol{\xi}\in L_{\boldsymbol{\nu}}\cap\mathcal{S}$ such that $\boldsymbol{\xi}\in B(\boldsymbol{c}(r),2^{d/2}\tau(L_{\boldsymbol{\nu}}))$ with $0<r\le 2^{d/2}\tau(L_{\boldsymbol{\nu}})+\epsilon$. Thus $N(\boldsymbol{\xi})\le N(\boldsymbol{c}(r))+2^{d/2}\tau(L)$ and, consequently,

$$N(\boldsymbol{\xi})\le(2^{d/2}\tau(L)+\epsilon)\sqrt{d}+2^{d/2}\tau(L)\le(1+\sqrt{d})2^{d/2}\tau(L)+\delta.$$

Therefore, the point $\boldsymbol{\xi}$ satisfies the inequality (5.9).

Suppose now that $\tau(L_{\boldsymbol{\nu}})<2^{-d/2-1}(1-\delta/\lceil\sqrt{d}\rceil)$. Clearly, $\{t\nu\}=\{\xi_d\}$, so it is enough to show that $\xi_d\in(0,1)$. Since $\boldsymbol{\xi}\in\mathcal{S}$, the number ξ_d is positive. On the other hand, we have

$$\xi_d\le r+2^{d/2}\tau(L)\le 2^{d/2+1}\tau(L)+\delta/\lceil\sqrt{d}\rceil<1\,. \qquad ∎$$

5.4. Approximation error. As is shown in sections 5.2 and 5.3, the computed approximations of the optimal values of $r(\boldsymbol{\xi})$ and $N(\boldsymbol{\xi})$ are bounded in terms of the covering radius and thus are small for a typical augmented vector. We conjecture that the iterated CG-cuts found by the algorithms obtained in Corollaries 5.1 and 5.2 solve problems (3.4) and (3.5), respectively, with the multiplicative approximation error $2^{O(d)}$. In this section we prove the second conjecture for the special case $\tau(L_{\boldsymbol{\nu}})\ll q^{-1/d}$, where $\ll$ is the Vinogradov symbol.

Let $\boldsymbol{\nu}=(\lambda_1,\dots,\lambda_m,\nu)$ be a vector of the form (4.1), and let $\delta\in(0,1)\cap\mathbb{Q}$. We will denote by $m_{add}=m_{add}(\boldsymbol{\nu})$ the value of the minimum in (3.5); that is,

$$m_{add}(\boldsymbol{\nu})=\min\{N(t\boldsymbol{\nu}\bmod 1):\ t=1,\dots,q-1,\ \{t\nu\}>0\}\,.$$

We will also denote by $\boldsymbol{\xi}_{add}=\boldsymbol{\xi}_{add}(\boldsymbol{\nu},\delta)$ the output vector of the algorithm obtained in Corollary 5.2.

PROPOSITION 5.3. *Let $\boldsymbol{\nu}$ be a vector of the form* (4.1) *with common denominator q. Then*

$$\frac{N(\boldsymbol{\xi}_{add}(\boldsymbol{\nu},1/q))}{m_{add}(\boldsymbol{\nu})}<2^{3d/2-1}(1+\sqrt{d})\tau(L_{\boldsymbol{\nu}})^d q+1\,. \tag{5.11}$$

Proof. Recall that $L_{\boldsymbol{\nu}} = \mathbb{Z}^d + \mathbb{Z}\boldsymbol{\nu}$. Therefore, for the first successive minimum $\lambda_1 = \lambda_1(L_{\boldsymbol{\nu}})$ we obtain the inequalities $1/q \le \lambda_1 \le m_{add}(\boldsymbol{\nu})$. Together with (5.9) this observation implies the inequality

$$\frac{N(\boldsymbol{\xi}_{add}(\boldsymbol{\nu}, 1/q))}{m_{add}(\boldsymbol{\nu})} < \frac{(1+\sqrt{d})2^{d/2}\tau(L_{\boldsymbol{\nu}})}{\lambda_1} + 1\,. \tag{5.12}$$

By Minkowski's Second theorem for spheres, $1/q = \det(L_{\boldsymbol{\nu}}) \le \lambda_1\lambda_2\cdots\lambda_d$ and hence

$$\lambda_1 \ge \frac{1}{q\lambda_d^{d-1}}\,. \tag{5.13}$$

Next, by Jarnik's inequalities (cf. [24, pp. 99 and 106]) we have $\lambda_d \le 2\tau(L_{\boldsymbol{\nu}})$. Consequently, by (5.13)

$$\lambda_1 \ge \frac{1}{2^{d-1}q\tau(L_{\boldsymbol{\nu}})^{d-1}}\,. \tag{5.14}$$

Combining (5.12) and (5.14), we obtain the inequality (5.11). ∎

Proposition 5.3 implies the inequality $N(\boldsymbol{\xi}_{add}(\boldsymbol{\nu}, 1/q)) < 2^{O(d)}m_{add}(\boldsymbol{\nu})$, provided $\tau(L_{\boldsymbol{\nu}}) \ll q^{-1/d}$.

A natural step towards establishing both conjectures would be to show that the approximation error is independent of the common denominator q. In this light, Proposition 5.3, together with Theorem 4.3, imply that for a *typical* input vector $\boldsymbol{\nu}$ problem (3.5) can be approximated with the multiplicative approximation error that only depends on d.

6. Concluding remarks. Although Chvátal–Gomory cuts have been around for over 50 years and have been studied in depth, many important questions about them remain unanswered. We have studied the behavior of the iterated CG-cuts for a randomly chosen augmented vector and have shown the existence of a polynomial-time algorithm that computes approximations for problems (3.4) and (3.5). For computed approximations the values of $r(\boldsymbol{\xi})$ and $N(\boldsymbol{\xi})$ are bounded in terms of the covering radius and thus are small for a typical augmented vector. On the other hand, we do not know the precise approximation ratio that this algorithm yields. Nor do we know the precise approximability (or inapproximability) status of problems (3.4) and (3.5). Moreover, our algorithm seems at present of mainly theoretical interest, though this may change in the near future, given the intensive recent work on algorithms for integer lattices (see the survey [26]).

We also remark that the strategy presented in this paper is designed to optimize individual CG-cuts only. On the other hand, since the work of Balas et al. [6], most integer programmers prefer to work with collections of cutting planes rather than individual ones. (Specifically, given a fractional simplex tableau, one can generate one GF-cut for each fractional variable, and add all such GF-cuts to the LP relaxation.) It is not clear that optimizing each CG-cut in a collection will improve the effectiveness of the entire collection. Indeed, in our computational experiments, we often observed that different CG-cuts led to the same strengthened iterated CG-cut, so that a large collection of weak CG-cuts was converted into a small collection of strong ones. This suggests that a suitable topic for future research might be the simultaneous optimization of a collection of CG-cuts. A method for strengthening a collection of Gomory *mixed-integer* cuts, rather than GF-cuts, was presented in [4].

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