Valid inequalities for mixed-integer programmes with fixed charges on sets of variables

Adam N. Letchford [a,*], Georgia Souli [b]

[a] *Department of Management Science, Lancaster University, Lancaster LA1 4YX, United Kingdom*
[b] *STOR-i Centre for Doctoral Training, Lancaster University, Lancaster LA1 4YR, United Kingdom*

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ABSTRACT

We consider mixed 0-1 linear programmes in which one is given a collection of (not necessarily disjoint) *sets* of variables and, for each set, a fixed charge is incurred if and only if at least one of the variables in the set takes a positive value. We derive strong valid linear inequalities for these problems, and show that they generalise and dominate a subclass of the well-known *flow cover inequalities* for the classical fixed-charge problem.

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1. Introduction

It is well known that a wide range of important optimisation problems can be modelled as *mixed-integer linear programmes* (MILPs). A key ingredient of modern exact MILP algorithms is the use of strong valid linear inequalities, also known as *cutting planes*, to strengthen the continuous relaxation of the problem (see, e.g., [4,5]).

One strand of the literature on cutting planes is concerned with MILPs that involve *fixed charges* (see, e.g., [1,9–11,14,16–18,21–24,26]). A fixed charge is an additional cost that is incurred whenever a certain variable takes a positive value. The textbook way to model fixed charges is as follows. Suppose that x_j is a continuous non-negative variable, and the fixed charge d_j is incurred whenever $x_j > 0$. Suppose also that we know an upper bound u_j on the value taken by x_j in an optimal solution. We define a new binary variable, say y_j, taking the value 1 if and only if the fixed charge is incurred. We then add $d_j y_j$ to the cost function, and add the linear constraint $x_j \le u_j y_j$.

Unfortunately, when the textbook approach is used, the continuous relaxation of the resulting MILP is often very weak. In this situation, cutting planes are essential. In particular, the so-called *flow cover* inequalities [21,23], and various extensions of them [9,10,18], have proven to be so effective that they are now generated by default not only in commercial MILP solvers (such as CPLEX, Gurobi and Xpress), but also in open-source solvers (such as CBC and SCIP).

In this paper, we consider a more general situation, in which fixed charges are associated with *sets* of variables. More precisely, we suppose that we have continuous variables $x_1, \ldots, x_n$, a collection of (not necessarily disjoint) sets $S_1, \ldots, S_m$ satisfying $\bigcup_{i=1}^m S_i = \{1, \ldots, n\}$, and a fixed charge d_i for $i = 1, \ldots, m$. The idea is that, for each i, the fixed charge d_i is incurred if $x_j > 0$ for *any* $j \in S_i$. (To see how this situation could arise in practice, consider a set of products that share a common machine. If any of the products are to be manufactured on a given day, then the machine must be set up at the start of that day.)

The paper has the following structure. Section 2 is a literature review. In Section 3, we consider the case in which the sets $S_1, \ldots, S_m$ are *nested*. We derive a family of valid inequalities for the associated polytope, and show that the inequalities both generalise and dominate a subclass of the flow cover inequalities. In Section 4, we extend some of our results to the general case, in which the sets can intersect in an arbitrary way. Throughout the paper, N and M denote $\{1, \ldots, n\}$ and $\{1, \ldots, m\}$, respectively.

2. Literature review

We now review the relevant literature. In Section 2.1, we briefly review valid inequalities for fixed-charge problems. In Section 2.2, for reasons which will become clear later, we review some papers on what we call *optimality cuts*.

* Corresponding author.
E-mail addresses: A.N.Letchford@lancaster.ac.uk (A.N. Letchford), G.Souli@lancaster.ac.uk (G. Souli).

2.1. Valid inequalities for the fixed charge polytope

Padberg et al. [21] introduced the *fixed-charge polytope*, defined as the convex hull of pairs $(x, y) \in \mathbb{R}_+^n \times \{0, 1\}^n$ satisfying

$$\sum_{j=1}^{n} x_j \leq d$$

$$x_j \leq u_j y_j \quad (j = 1, \ldots, n),$$

where d and the u_j are positive integers.

A set $C \subseteq N$ is called a *cover* if $\sum_{j \in C} u_j > d$. Given a cover C, we let λ denote the "excess capacity" $\sum_{j \in C} u_j - d$, we let u^+ denote $\max_{j \in C} u_j$, and we define the set $C^* = \{j \in C : u_j > \lambda\}$. Padberg et al. showed that, given a cover C and a (possibly empty) set $L \subseteq N \setminus C$, the following *flow cover inequality* (FCI) is valid:

$$\sum_{j \in C \cup L} x_j \leq d - \sum_{j \in C^*}(u_j - \lambda)(1 - y_j) + \sum_{j \in L} \alpha_j y_j,$$

where α_j is $\max\{u^+, u_j\} - \lambda$ for $j \in L$. As a special case, when $L = \emptyset$ we obtain:

$$\sum_{j \in C} x_j \leq d - \sum_{j \in C^*}(u_j - \lambda)(1 - y_j). \tag{1}$$

We will call inequalities of this type *simple* FCIs.

The FCIs were extended to *fixed-charge network flow* (FCNF) problems in [23]. They have been further generalised and strengthened in various ways [9,10,22,23]. A related family of inequalities, the *flow pack* inequalities, were studied by Atamtürk [1]. For a very different family of inequalities, derived from knapsack polytopes, see our recent paper [14].

2.2. Optimality cuts

By an *optimality* cut, we mean a linear inequality that is satisfied by all *optimal* MILP solutions, but may be violated by one or more sub-optimal solutions. Note that some authors expand this definition to include linear inequalities that are satisfied by *at least one* optimal solution. This concept appears, sometimes under different names, in many places (e.g., [2,3,6,7,11–13,19]).

Among these works, Hooker et al. [11] is itself concerned with FCNF problems. Consider a directed network in which, for each arc a, there is a continuous flow variable x_a and a binary fixed-charge variable y_a. Consider a node i. Let a be an arc entering node i, and let $\alpha, \ldots, \omega$ be the arcs leaving i. Suppose that $y_a = 1$ in an optimal solution. Then x_a must be positive, since, if that were not so, we could have made a cost saving by setting y_a to 0. The flow entering node i via arc a must then exit node i via one or more of the outgoing arcs. This in turn implies that at least one of the y variables associated with the outgoing arcs must take the value 1. Thus, the inequality $y_a \leq y_\alpha + \cdots + y_\omega$ is an optimality cut (called a *logic cut* in [11]).

3. The nested case

In this section, we consider the special case in which the sets $S_1, \ldots, S_m$ are *nested* (that is, for all pairs $\{i, i'\} \subset M$, we have $S_i \cap S_{i'} = \emptyset$, $S_i \subset S_{i'}$ or $S_{i'} \subset S_i$). The section is organised as follows. In Section 3.1, we give some notation and terminology. In Section 3.2, we prove a negative complexity result. In Section 3.3, we show that our problem is closely related to FCNF problems. In Section 3.4, we present our valid inequalities. Finally, in Section 3.5, we consider the effect on our inequalities when optimality cuts are present.
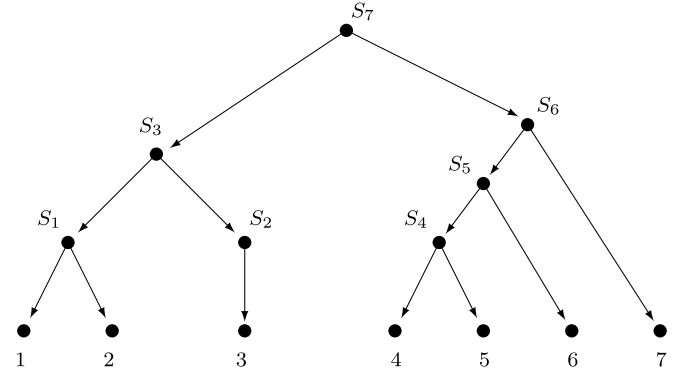

Fig. 1. Visualisation of nested sets using a directed tree.

3.1. Notation and terminology

From now on, we let P denote the convex hull of the pairs $(x, y) \in \mathbb{R}_+^n \times \{0, 1\}^m$ satisfying

$$\sum_{j \in S_i} x_j \leq u_i y_i \ (i = 1, \ldots, m). \tag{2}$$

Note that, in the nested case, we can assume without loss of generality that $S_m = N$. Indeed, if there did not exist some set S_i that contained all the other sets, then P would be the Cartesian product of simpler polytopes of the same kind.

Given $i, i' \in M$, we will call i a *descendant* of i' if $S_i \subset S_{i'}$. If, in addition, there is no $k \in M$ such that $S_i \subset S_k \subset S_{i'}$, we call i a *child* of i'. We let $\delta(i)$ and $\chi(i)$ denote the set of descendants and children, respectively, of i. We also let $\rho(i)$ denote $S_i \setminus \bigcup_{k \in \chi(i)} S_k$. (Note that at most one of $\chi(i)$ and $\rho(i)$ can be empty.) We also represent the sets $S_1, \ldots, S_m$ as a rooted directed tree, as follows. There is one node for each set S_i, and one node for each $j \in N$. There is an arc from S_i to $S_{i'}$ if i' is a child of i, and there is an arc from S_i to j if $j \in \rho(i)$.

The above notation and terminology is illustrated in the following example.

Example 1. Let $n = 7$ and $m = 7$, and suppose that $S_1 = \{1, 2\}$, $S_2 = \{3\}$, $S_3 = \{1, 2, 3\}$, $S_4 = \{4, 5\}$, $S_5 = \{4, 5, 6\}$, $S_6 = \{4, 5, 6, 7\}$ and $S_7 = \{1, \ldots, 7\}$. The corresponding tree is shown in Fig. 1. One can check that, for example, 4 is a descendent of 6 and 1 is a child of 3. One can also check that $\delta(2) = \emptyset$, $\delta(6) = \{4, 5\}$, $\chi(3) = \{1, 2\}$, $\chi(4) = \emptyset$, $\rho(3) = \emptyset$ and $\rho(5) = \{6\}$. $\square$

3.2. Complexity

Our first result is that, even in the nested case, optimisation over P is difficult.

Proposition 1. *Even in the nested case, optimising a linear function over P is $\mathcal{NP}$-hard.*

Proof. We reduce SUBSET SUM, shown to be $\mathcal{NP}$-complete in [8], to the problem in question. An instance of SUBSET SUM is given by positive integers $p_1, \ldots, p_k$ and a "target" t. The task is to determine if there exists a subset of $\{p_1, \ldots, p_k\}$ whose sum is t. To convert this to our problem, set n to k and m to $k + 1$. For $i \in N$, set S_i to $\{i\}$, set u_i to p_i, give x_i a profit of 1 and give y_i a cost of $p_i(1 - \epsilon)$, where ϵ is a small positive quantity. Finally, set S_m to N, u_m to t, and give y_m a cost of zero. Since all u_i values are integers, there exists an optimal solution in which x is integral. Then, for any given $i \in N$, it is not worth setting y_i to 1 unless x_i

is set to p_i, in which case we gain a profit of ϵp_i. So the answer to SUBSET SUM is "yes" if and only if there is a point $(x, y) \in P$ with profit ϵt. (A suitable value for ϵ is $1/n$.) □

In light of this result, we cannot expect to obtain a complete description of P in terms of linear inequalities. So, we must be content with a partial linear description.

3.3. Relation to FCNF

Given a set of nodes V and a set of arcs $\mathcal{A}$, the FCNF polytope is the convex hull of the pairs $(x, y) \in \mathbb{R}_+^{|\mathcal{A}|} \times \{0, 1\}^{|\mathcal{A}|}$, such that

$$\sum_{j \in V} x_{ji} - \sum_{j \in V} x_{ij} = b_i \qquad (i \in V)$$

$$x_{ij} \le u_{ij} y_{ij} \qquad ((i, j) \in \mathcal{A}).$$

Next, we will show that, in the nested case, there is a strong connection between the polytope P and FCNF problems. Consider Fig. 1 once more. For $i \in M$, let f_i denote the flow entering the node that represents S_i, and set the capacity of the corresponding arc to u_i. We also interpret the binary variable y_i as an indicator of whether that arc is being used or not. Finally, for $j \in N$, we let node j be a "sink" node, and we interpret x_j as representing the flow entering node j. By construction, a triple $(x, y, f) \in \mathbb{R}_+^n \times \{0, 1\}^n \times \mathbb{R}_+^n$ represents a feasible flow if and only if (x, y) satisfies (2).

This observation is formalised in the following proposition.

Proposition 2. Suppose that $S_1, \ldots, S_m$ are nested. Let $M' = \{i \in M : \chi(i) = \emptyset\}$, and let P^f be the convex hull of the triples $(x, y, f) \in \mathbb{R}_+^n \times \{0, 1\}^m \times \mathbb{R}_+^m$ satisfying

$$f_i = \sum_{j \in S_i} x_j \qquad (i \in M') \tag{3}$$

$$f_i = \sum_{k \in \chi(i)} f_k + \sum_{j \in \rho(i)} x_j \quad (i \in M \setminus M') \tag{4}$$

$$f_i \le u_i y_i \qquad (i \in M). \tag{5}$$

Then (a) P^f is an FCNF polytope of an acyclic graph, (b) P is the projection of P^f onto (x, y)-space, and (c) P is affinely congruent to P^f.

Proof. By construction, Eqs. (3) and (4) enforce conservation of flow in an acyclic graph, and the constraints (5) enforce arc capacities, as well as ensuring that no flow can pass through an arc unless the arc is open. This proves claim (a). Now, by conservation of flow, we must have $f_i = \sum_{j \in S_i} x_j$ for all $i \in M$. Thus, if we eliminate the f variables, using Eqs. (3) and (4), the constraints (5) reduce to (2). This proves claim (b). To see that claim (c) holds, it suffices to note that there is an invertible affine transformation which maps each extreme point of P to an extreme point of P_f. This transformation consists of leaving x and y unchanged, and setting f_i to $\sum_{j \in S_i} x_j$ for all $i \in M$. □

This link with FCNF polytopes has an interesting consequence:

Corollary 1. Consider the following set function, which maps sets $T \subseteq N$ to $\mathbb{Z}_+$:

$$\phi(T) = \max \left\{ \sum_{j \in T} x_j : \sum_{j \in S_i} x_j \le u_i \ (i \in M), \ x_j \ge 0 \ (j \in N) \right\}. \tag{6}$$

If the sets $S_1, \ldots, S_m$ are nested, then ϕ is submodular.

Proof. Wolsey ([26], Th. 4) proved the following. Let $G = (V, A)$ be a directed graph, let $u \in \mathbb{R}_+^A$ be a vector of arc capacities,

and let $p \in \mathbb{R}_+^A$ be a vector of arc profits. Given any $S \subset V$, the maximum-profit flow through the arcs leaving S is a submodular function of the set of arcs that are open. To apply this result to our problem, it suffices to set S to the set of nodes representing $S_1, \ldots, S_m$, and set p_j to 1 for each arc that connects a node in $S_1, \ldots, S_m$ to a node in N. □

3.4. The new inequalities

From now on, we assume w.l.o.g. that, for each $i \in M$ such that $\rho(i) = \emptyset$, the condition $u_i \le \sum_{k \in \chi(i)} u_k$ holds. (If it did not hold, then one could decrease u_i without losing any feasible solutions.)

Now, let $T \subseteq N$ be any set such that $\phi(T) = u_i$ for some $i \in M$. The following inequality is trivially valid and supporting for P:

$$\sum_{j \in T} x_j \le u_i. \tag{7}$$

We will use sequential lifting [20,25] to strengthen (7). More precisely, let $I = \{k \in M : S_k \cap T \ne \emptyset\}$, let c denote $|I|$, and let $s(1), \ldots, s(c)$ be an arbitrary ordering of the elements of I. We will compute a valid inequality of the form

$$\sum_{j \in T} x_j \le u_i - \sum_{k=1}^{c} \beta_k (1 - y_{s(k)}), \tag{8}$$

where the (non-negative) coefficients β_k are computed according to the given ordering.

The coefficients β_k can be calculated as follows. Let $\phi(T, k)$ denote the maximum value that $\sum_{j \in T} x_j$ can take when $y_{s(k)} = 0$. If $\phi(T, k) = u_i$, then $\beta_k = 0$. Otherwise, let $\Delta(k) = \{i \in \{1, \ldots, k-1\} : s(i) \in \delta(s(k))\}$ and set

$$\beta_k = u_i - \phi(T, k) - \sum_{\ell \in \Delta(k)} \beta_\ell. \tag{9}$$

Note that, for fixed T and k, one can compute $\phi(T, k)$ and β_k in $O(m + n)$ time. Thus, for fixed T, one can compute all lifting coefficients in $O(m(m + n))$ time. The following example shows that it is possible for this procedure to yield many inequalities that define facets of P. It also shows that it is possible for different lifting sequences to lead to different facets.

Example 2. Suppose that $n = 3$, $m = 5$, $S_i = \{i\}$ for $i = 1, 2, 3$, $S_4 = \{1, 2\}$, $S_5 = \{1, 2, 3\}$, $u_i = 4$ for $i = 1, 2, 3$, $u_4 = 7$ and $u_5 = 10$. Taking $T = \{1, 2\}$, the inequality (7) is $x_1 + x_2 \le 7$. We have $I = \{1, 2, 4, 5\}$. Lifting in the order 1, 2, 4, 5, we obtain $\Delta(1) = \Delta(2) = \emptyset$, $\Delta(3) = \{1, 2\}$ and $\Delta(4) = \{1, 2, 3\}$. This yields $\beta_1 = \beta_2 = 7 - 4 = 3$, $\beta_3 = 7 - 0 - 3 - 3 = 1$ and $\beta_4 = 7 - 0 - 3 - 3 - 1 = 0$. The resulting valid inequality is

$$x_1 + x_2 \le 3y_1 + 3y_2 + y_4. \tag{10}$$

One can check that different lifting orders yield seven additional inequalities:

$$x_1 + x_2 \le 3y_1 + 3y_2 + y_5$$
$$x_1 + x_2 \le 3y_1 + 4y_4$$
$$x_1 + x_2 \le 3y_1 + 4y_5$$
$$x_1 + x_2 \le 3y_2 + 4y_4$$
$$x_1 + x_2 \le 3y_2 + 4y_5$$
$$x_1 + x_2 \le 7y_4$$
$$x_1 + x_2 \le 7y_5.$$

One can also check (either by hand or with the help of a software package such as PANDA [15]) that all eight lifted inequalities

define facets of P. Finally, one can check that a further sixteen facets of P can be obtained by taking $T = \{1, 2, 3\}$ and using different lifting sequences. □

We also have the following lemma:

Lemma 1. *The inequalities* (8) *generalise and dominate the simple flow cover inequalities* (1).

Proof. Let n, d, u and C be given, as in Section 2.1. To put this into our framework, set m to $n + 1$, and set S_i to $\{i\}$ for $i = 1, \ldots, n$. Also set S_m and N to $\{1, \ldots, n + 1\}$, u_m to d, and T to C. We have $I = C \cup \{m\}$. Let r denote $d - \sum_{j \in C^*}(u_j - \lambda)$, and note that $r > 0$. One can check that, if we lift y_m last, we obtain:

$$\sum_{i=1}^{n} x_j \le d - \sum_{j \in C^*}(u_j - \lambda)(1 - y_j) - r(1 - y_m).$$

This dominates (1). □

An interesting question is whether there can exist facet-defining inequalities of the form (8) that cannot be obtained by lifting sequentially.

3.5. Optimality cuts

We end this section with some remarks on optimality cuts. Note that if y_i takes the value 0 in an optimal solution, then y_k should also take the value 0 for all $k \in \chi(i)$. Therefore, one can strengthen the LP relaxation by adding the optimality cut $y_k \le y_i$ for all $i, k \in M$ such that $k \in \chi(i)$. Given the connection between our problem and FCNF problems mentioned in Section 3.2, these can be viewed as a special kind of logic cuts.

Note that, once these optimality cuts have been added, the convex hull of the feasible pairs (x, y) is no longer an FCNF polytope. Nevertheless, our procedure for generating valid inequalities still applies. It turns out, however, that most lifting sequences no longer yield facets. Indeed, a necessary condition for obtaining a facet is that, for each $i \in I$ with $\delta(i) \ne \emptyset$, the descendants of i are lifted before i itself is lifted. The effect of this is that, for a given T, one of the valid inequalities dominates all of the others. This is illustrated in the following example.

Example 2 (*cont.*). We add the optimality cuts $y_1 \le y_4$, $y_2 \le y_4$, $y_3 \le y_5$ and $y_4 \le y_5$. Taking $T = \{1, 2\}$ and lifting in the order 1,2,4,5, as before, we obtain inequality (10). Together with the optimality cuts, this inequality dominates the other seven inequalities mentioned in Example 2. One can verify that it also defines a facet of the modified polytope. □

In other words, the optimality cuts eliminate some of the facets of the polytope P. The effect is to make the polytope simpler. Note that Lemma 1 still holds for the modified polytope.

4. The general case

In this last section, we consider the general case, in which the sets S_i do not need to be nested. It turns out that this case is much more complicated, in several respects.

A first complication is that one can no longer assume that $S_m = N$. For example, if $n = 3$, $m = 2$, $S_1 = \{1, 2\}$ and $S_2 = \{2, 3\}$, then the polytope P is not the Cartesian product of simpler polytopes.

A second complication is that optimising over P becomes even more difficult.

Proposition 3. *In the general case, optimising a linear function over P is $\mathcal{NP}$-hard in the strong sense.*

Proof. We reduce the maximum independent set problem, proven to be strongly $\mathcal{NP}$-hard in [8], to the problem in question. Let $G = (V, E)$ be a graph with n nodes and p edges. Set m to $n+p$. For $i = 1, \ldots, n$, set S_i to $\{i\}$, give x_i a profit of 3 and give y_i a cost of 2. For $r = 1, \ldots, p$, let S_{r+i} contain the end-nodes of the rth edge. Set all u values to 1. There exists an independent set of size k in G if and only if there is a solution with profit k. □

A third complication is that one can sometimes decrease one or more u values, by solving a series of LPs. This is shown in the following example.

Example 3. Let $n = 3$, $m = 7$, $S_1 = \{1\}$, $S_2 = \{2\}$, $S_3 = \{3\}$, $S_4 = \{1, 2\}$, $S_5 = \{1, 3\}$, $S_6 = \{2, 3\}$ and $S_7 = \{1, 2, 3\}$. Also, let $u_1 = u_2 = u_3 = 2$, $u_4 = u_5 = u_6 = 3$ and $u_7 = 5$. Recall the function ϕ from Section 3.3. Solving the LP (6) with $T = S_7$, we find that $\phi(S_7) = 4.5$. Hence u_7 can be reduced from 5 to 4.5. □

This example also shows that, even when the original vector u is integral, it is possible for $\phi(T)$ to be fractional for a given $T \subseteq N$.

A fourth complication is that the polytope P is no longer an FCNF polytope in general. In fact, the function ϕ is no longer submodular.

Example 3 (*cont.*). We have $\phi(S_4) = \phi(S_5) = 3$, $\phi(S_4 \cup S_5) = \phi(S_7) = 4.5$, and $\phi(S_4 \cap S_5) = \phi(S_1) = 2$. Since $3 + 3 < 4.5 + 2$, ϕ is not submodular. □

On the positive side, the function ϕ remains subadditive.

Proposition 4. *In the general case, ϕ is subadditive.*

Proof. Let S, T be subsets of N. Let x^* be the vector that maximises $\sum_{j \in S \cup T} x_j$ in (6). Let $x^1, x^2 \in \mathcal{X}$ be defined as follows

$$x_j^1 = \begin{cases} x_j^*, & \text{if } j \in S \\ 0, & \text{if } j \notin S \end{cases} \qquad x_j^2 = \begin{cases} 0, & \text{if } j \in S \\ x_j^*, & \text{if } j \notin S. \end{cases}$$

We have

$$\phi(S \cup T) = \sum_{j \in S \cup T} x_j^* = \sum_{j \in S} x_j^1 + \sum_{j \in T \setminus S} x_j^2 \le \sum_{j \in S} x_j^1 + \sum_{j \in T} x_j^2$$
$$\le \phi(S) + \phi(T). \quad \square$$

A fifth complication is that the closed formula (9) that we presented in Section 3.3 can no longer be used to compute the lifting coefficients. Moreover, even when the correct coefficients are used, the resulting inequality is no longer guaranteed to be facet-defining.

Example 3 (*cont.*). Suppose we take $T = \{1, 2, 3\}$. The inequality (7) is $x_1 + x_2 + x_3 \le 4.5$. We have $I = M$. If we lift 4,5 and 6 first, the formula (9) yields $\beta_1 = \beta_2 = \beta_3 = 4.5 - 2 = 2.5$, which yields the invalid inequality $x_1 + x_2 + x_3 \le -3 + 2.5(y_4 + y_5 + y_6)$. If we lift exactly (by solving a sequence of small mixed 0-1 LPs, as in [20,25]), then we obtain $\beta_1 = 2.5$, $\beta_2 = 2$ and $\beta_3 = 0$. The resulting valid inequality $x_1 + x_2 + x_3 \le 2.5y_4 + 2y_5$ does not define a facet of P. □

In fact, we do not know whether lifting can be performed in polynomial time in the general case, and we do not have a necessary and sufficient condition for a lifted inequality (8) to define a facet. On the positive side, it turns out that the lifted inequalities still define facets of P in many cases.

Example 3 (*cont.*). Suppose we change u_7 from 5 to 4. Taking $T = \{1, 2, 3\}$ again, the inequality (7) is $x_1 + x_2 + x_3 \le 4$. Lifting in the order 1, 2, 3, 7, 4, 5, 6, we get

$$x_1 + x_2 + x_3 \le y_1 + y_2 + y_3 + y_7. \tag{11}$$

Lifting in the order 1, 2, 5, 6, 3, 4, 7, we get

$$x_1 + x_2 + x_3 \le y_1 + y_2 + y_5 + y_6. \qquad (12)$$

One can check that these two inequalities define facets of P. One can also check that, using different lifting sequences, one can obtain an additional 42 facets. $\square$

Finally, we mention that, as in the nested case, when optimality cuts are present, most lifting sequences no longer yield facets. Interestingly, however, we can now obtain more than one facet for a given T.

Example 3 (*cont.*)**.** Suppose that, as before, $u_7 = 4$ and $T = \{1, 2, 3\}$. We add the optimality cuts $y_1 \le y_4$, $y_1 \le y_5$ and so on. One can check that the inequalities (11) and (12) remain facet-defining, and so do the following two inequalities:

$$x_1 + x_2 + x_3 \le y_1 + y_3 + y_4 + y_6$$
$$x_1 + x_2 + x_3 \le y_2 + y_3 + y_4 + y_5. \quad \square$$

An interesting open question is whether, in the general case, one can obtain facet-defining inequalities with fractional coefficients, even if the original u values are integers.

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