Strengthened Clique-Family Inequalities for the Stable Set Polytope

Adam N. Letchford* Paolo Ventura†

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Abstract

The *stable set polytope* is a fundamental object in combinatorial optimisation. Among the many valid inequalities that are known for it, the *clique-family* inequalities play an important role. Pêcher and Wagler showed that the clique-family inequalities can be strengthened under certain conditions. We show that they can be strengthened even further, using a surprisingly simple mixed-integer rounding argument.

Keywords: stable set problem; polyhedral combinatorics

1 Introduction

Let $G = (V, E)$ be a (loopless and simple) undirected graph. A set $S \subseteq V$ is called *stable* (or *independent*) if none of the vertices in S are adjacent. Given a weight vector $w \in \mathbb{Q}_+^V$, the *stable set problem* calls for a stable set of maximum total weight. This problem, and other problems that are equivalent to it, play a key role in integer programming, combinatorial optimisation and computational complexity theory (see, e.g., [1–3, 16, 17, 25]).

For each vertex $v \in V$, define a binary variable x_v, taking the value 1 if and only if v is to be included in a stable set. Then, a vector $x \in \{0,1\}^{|V|}$ is the incidence vector of a stable set if and only if $x_u + x_v \leq 1$ for all $\{u, v\} \in E$. The convex hull of the incidence vectors is called the *stable set polytope* and denoted by STAB(G) [16]. Many families of valid and facet-defining inequalities have been discovered for this polytope (e.g., [3–5, 16, 23, 24, 27, 30]). These inequalities have been used to good effect in exact algorithms for the stable set problem and related problems (e.g., [1, 21, 28, 29]).

*Department of Management Science, Lancaster University, Lancaster LA1 4YX, UK. Email: A.N.Letchford@lancaster.ac.uk

†Istituto di Analisi dei Sistemi ed Informatica (IASI) del CNR, Via dei Taurini 19, Roma 00185, Italy. Email: paolo.ventura@iasi.cnr.it

This paper is concerned with the *clique-family* inequalities, which were introduced by Oriolo [23]. These inequalities played a key role in the celebrated paper [10], which gives a complete linear description of STAB(G) for the case in which G is a so-called quasi-line graph. Nevertheless, there are cases in which clique-family inequalities do not define facets of STAB(G). Indeed, Pêcher and Wagler [26] showed how to strengthen the inequalities under certain conditions.

In this paper, we present some new strengthened clique family inequalities, which dominate those of Pêcher and Wagler. Surprisingly, our inequalities can be derived very easily, using a well-known mixed-integer rounding argument (see [15, 20, 22]).

The structure of the paper is very simple. Section 2 reviews the relevant literature, Section 3 presents the new inequalities, and Section 4 lists some interesting open questions.

Throughout the paper, given a vertex set $S \subseteq V$, we let $x(S)$ denote $\sum_{v \in S} x_e$.

2 Literature Review

For reasons of space, we cover only works of direct relevance in this section.

2.1 Some known inequalities for the stable set polytope

Padberg [24] showed the following:

- If $S \subseteq V$ is a maximal clique in G, then the *clique* inequality $x(S) \leq 1$ defines a facet of STAB(G).
- If $S \subseteq V$ induces a chordless odd cycle in G, then the *odd hole* inequality $x(S) \leq (|C| - 1)/2$ is valid for STAB(G).
- If $S \subseteq V$ induces the complement of an odd hole in G, then the *odd antihole* inequality $x(S) \leq 2$ is valid.
- Any odd hole or odd antihole inequality that does not define a facet of STAB(G) can be strengthened (lifted) to make it facet-defining.

Trotter [30] derived two additional families of inequalities, called *web* and *antiweb* inequalities. We will be interested in the latter. Let p and q be integers, with $q \geq 2$ and $p > 2q$, and suppose that p is not a multiple of q. A "(p, q)-antiweb" is a graph with vertex set $1, \ldots, p$ in which, for $i = 1, \ldots, p$, node i is adjacent to nodes $i + 1, i + 2, \ldots, i + q - 1$ and nodes $i - 1, i - 2, \ldots, i - q + 1$ (indices taken modulo p). Trotter showed that, if $S \subseteq V$ induces a (p, q)-antiweb in G, then the *antiweb inequality* $x(S) \leq \lfloor p/q \rfloor$ is valid for STAB(G). Note that antiweb inequalities reduce to odd hole inequalities when $q = 2$, and to odd antihole inequalities when $p = 2q + 1$.

Oriolo [23] introduced the *clique-family* inequalities. Let p and q be integers as before, and suppose again that p is not a multiple of q. Let $r = p \bmod q$, and let $\mathcal{C} = \{C_1, \dots, C_p\}$ be an arbitrary collection of maximal cliques. Let V^q denote the set of nodes that are contained in *at least* q of the cliques in the collection, and V^{q-1} denote the set of nodes that are contained in *exactly* $q-1$ cliques. Then the following *clique-family* inequality is valid:

$$(q-r)x\big(V^q\big) + (q-r-1)x\big(V^{q-1}\big) \le (q-r)\lfloor p/q \rfloor. \tag{1}$$

Note that antiweb inequalities can be expressed as clique-family inequalities with V^{q-1} empty.

Now suppose that $r \le q-2$. For $k = r+1, \dots, q-1$, let V^k be the set of nodes contained in exactly k cliques. Pêcher and Wagler [26] showed that the following inequality (hereafter called a *Pêcher-Wagler* inequality) is valid:

$$\sum_{0 \le j < q-r} (q-r-j)\; x\big(V^{q-j}\big) \le (q-r)\lfloor p/q \rfloor. \tag{2}$$

If $V^k \neq \emptyset$ for some index k with $r+1 \le k < q-1$, then the inequality (2) is stronger than the clique-family inequality (1).

2.2 Complete descriptions

Chvátal [7] used a result of Edmonds [9] to show that, when G is a *line graph*, STAB(G) is completely described by clique inequalities, non-negativity inequalities, and one other family of inequalities. Oriolo [23] showed that these latter inequalities are clique-family inequalities with $q = 2$ and $r = 1$.

Line graphs are a special case of *quasi-line* graphs. Oriolo [23] conjectured that, when G is quasi-line, STAB(G) is described by the clique, non-negativity and clique-family inequalities. This conjecture was proved in [10].

Quasi-line graphs in turn are a special case of *claw-free* graphs. It is shown in [12, 13] that, when G is claw-free and has stability number larger than 4, STAB(G) is described by the clique, non-negativity and clique-family inequalities, together with the "geared" inequalities from [11].

2.3 Chvátal-Gomory cuts and mixed-integer rounding

Let $P \subset \mathbb{R}^n_+$ be a polyhedron, and let $\alpha^T x \le \beta$ be a valid inequality for P, with $\alpha \in \mathbb{Q}^n$ and $\beta \in \mathbb{Q} \setminus \mathbb{Z}$. Chvátal [6] pointed out that the following inequality is satisfied by all integer points in P:

$$\sum_{i=1}^{n} \lfloor \alpha_i \rfloor x \le \lfloor \beta \rfloor.$$

Such inequalities are now called *Chvátal-Gomory* (CG) cuts, since they appeared implicitly in the earlier work of Gomory [14].

Pêcher & Wagler [27] showed that clique-family inequalities with $r \in \{1, q-1\}$ can be derived as CG cuts from the clique inequalities. On the other hand, Oriolo [23] found a quasi-line graph for which STAB(G) has a facet-defining clique-family inequality that is not a CG cut.

CG cuts can be strengthened as follows [15]. Given a real number t, let $\phi(t)$ denote $t - \lfloor t \rfloor$, the so-called *fractional part* of t. Also define the set

$$T = \Big\{ i \in \{1, \ldots, n\} : \phi(\alpha_i) > \phi(\beta) \Big\}.$$

The strengthened CG cut takes the form:

$$\sum_{i=1}^{n} \lfloor \alpha_i \rfloor x_i + \sum_{i \in T} \frac{\phi(\alpha_i) - \phi(\beta)}{1 - \phi(\beta)} x_i \le \lfloor \beta \rfloor.$$

We will follow Nemhauser and Wolsey [22] in calling these inequalities *mixed-integer rounding* (MIR) inequalities.

3 Strengthened Clique-Family Inequalities

This section presents our strengthened clique-family inequalities. The theoretical results are in Subsection 3.1. In Subsection 3.2, we give some examples that may be of interest.

3.1 Theoretical results

As before, let $C_1, \ldots, C_p$ be an arbitrary collection of maximal cliques, let q be an integer that does not divide p, such that $2 \le q < p/2$, and let r denote $p \bmod q$. For each $v \in V$, let μ_v be the "multiplicity" of v, by which we mean the number of cliques in the given collection that contain v. That is,

$$\mu_v = \Big| \big\{ j \in \{1, \ldots, p\} : v \in C_j \big\} \Big|.$$

Also, for each $v \in V$, let r_v denote $\mu_v \bmod q$. Our main result is as follows.

Theorem 1 *The following "strengthened clique-family" (SCF) inequality*

$$(q - r) \sum_{v \in V} \lfloor \mu_v / q \rfloor x_v + \sum_{v \in T} \big(r_v - r \big) x_v \le (q - r) \lfloor p/q \rfloor, \tag{3}$$

with $T = \big\{ v \in V : r_v > r \big\}$, *is valid for STAB(G).*

PROOF. Summing together the clique inequalities associated with $C_1, \ldots, C_p$, and dividing the result by q, we obtain:

$$\sum_{v \in V} \big(\mu_v / q \big) x_v \le p/q. \tag{4}$$

Then, for all $v \in V$, we have $\phi\big(\mu_v/q\big) = r_v/q$. One can check that the MIR inequality associated with (4) takes the form:

$$\sum_{v \in V} \lfloor \mu_v/q \rfloor x_v + \sum_{v \in T} \left(\frac{r_v - r}{q - r} \right) x_v \leq \lfloor p/q \rfloor.$$

Multiplying this inequality by $q - r$, we obtain inequality (3). □

We also have the following result.

Proposition 1 *The SCF inequality (3) dominates the inequality (2).*

PROOF. Since the two inequalities have the same right-hand side, it suffices to compare the left-hand side coefficients. One can check that, when $\mu_v \leq 2q - r$, the coefficient of x_v is the same in (2) and (3). When $\mu_v > 2q - r$, however, the coefficient in (2) is only $q - r$, whereas the coefficient in (3) is larger than $q - r$. □

We remind the reader that the inequality (2) dominates the clique-family inequality (1). Thus, SCF inequalities dominate clique-family inequalities as well.

3.2 Examples

We now give some examples of SCF inequalities that may be of interest.

Example 1: Let $G = (V, E)$ be a $(37, 8)$-antiweb, $G' = (V', E')$ a $(37, 7)$-antiweb, and let G^+ be the graph obtained by joining each vertex j of G with the vertices $j', (j+1)', \ldots, (j+13)'$ of G' (with indices taken modulo 37). One can check that G^+ is quasi-line. Giles and Trotter [31] proved that the inequality

$$3 \sum_{v \in V} x_v + 2 \sum_{v \in V'} x_v \leq 12 \tag{5}$$

is facet-defining for $\text{STAB}\big(G^+\big)$. Oriolo [23] noted that (5) is a clique-family inequality, obtained with $\mathcal{C}$ being the set of the $p = 37$ (maximal) cliques of the form $\{j, j+1, ..., j+7\} \cup \big\{(j+7)', (j+8)', ..., (j+13)'\big\}$, and with q and r being 8 and 5, respectively. Moreover, he showed that (5) cannot be derived as a CG cut from clique inequalities.

Now, let G^{++} be obtained from G^+ by adding a clique Q whose vertices are adjacent to all vertices of G^+. We extend each clique in $\mathcal{C}$ to a maximal clique in G^{++}, by inserting all of the nodes in Q. We have $\mu_v = 37$ for all $v \in Q$. Thus, the following SCF inequality

$$3 \sum_{v \in V} x_v + 2 \sum_{v \in V'} x_v + 12 \sum_{v \in Q} x_v \leq 12. \tag{6}$$

is valid for $\mathrm{STAB}(G^{++})$. $\square$

It turns out that the inequality (6) has some interesting properties.

Proposition 2 *The SCF inequality (6) is facet-defining for STAB$\big(G^{++}\big)$. It is not a Pêcher-Wagler inequality (and therefore is not a clique-family inequality either). Moreover, it cannot be derived as a CG cut from clique inequalities.*

PROOF. The fact that (6) is facet-defining for $\mathrm{STAB}\big(G^{++}\big)$ can be shown easily by taking the inequality (5) and applying the sequential lifting procedure of Padberg [24] to the nodes in Q.

Next, we prove by contradiction that (6) is not a Pêcher-Wagler inequality. Suppose that (6) can be written in the form (2). Comparing the coefficients for the nodes in Q with the right-hand side of (6), we must have $Q = V^q$ and $q - r = (q-r)\lfloor p/q \rfloor = 12$. This implies that $\lfloor p/q \rfloor = 1$, which in turn implies that $r = p - q$ and $p = 2q - 12$. Now, the coefficient of the vertices of V is $q - r - j = 3$, which yields $j = 9$ for those vertices. Moreover, since each vertex of V has to be covered by $q - j$ cliques of $\mathcal{C}$, $|V| = 37$, and each clique of $\mathcal{C}$ covers at most 8 vertices of V, we have $p \geq \frac{37(q-9)}{8}$. Because $p = 2q - 12$, this implies that $q \leq 11$, contradicting the fact that $q - r = 12$.

Finally, we construct a vector $x^* \in [0,1]^{V \cup V' \cup Q}$ by setting x^*_i to $1/8$ for all $i \in V$, and all other x^* values to 0. One can check that x^* satisfies all clique inequalities. Moreover, the left-hand side of (6), evaluated with respect to x^*, is $111/8$. So x^* violates (6) by $15/8$. Now, it is shown in [18] that, when the convex hull of feasible solutions to an integer program is full-dimensional, the amount by which any CG cut is violated by a solution to the LP relaxation is less than 1. So, given that $\mathrm{STAB}(G)$ is full-dimensional, (6) cannot be a CG cut. $\square$

The previous example was obtained by taking a known facet-defining clique-family inequality and adding a single node. The following example is obtained more directly.

Example 2: Let $G = (V, E)$ be a $(32, 5)$-antiweb, $G' = (V', E')$ a $(32, 9)$-antiweb, and let G^+ be the graph obtained by joining each vertex j of G with the vertices $j', (j+1)', \ldots, (j+12)'$ of G' (with indices taken modulo 32). Observe that G^+ is not claw-free. Now, consider the SCF inequality obtained with $\mathcal{C}$ consisting of the $p = 32$ (maximal) cliques of the form $\{j, j+1, \ldots, j+4\} \cup \big\{(j+4)', (j+5)', \ldots, (j+12)'\big\}$, and with q and r being 5 and 2, respectively. We have $\mu_i = 5$ for $i \in V$ and $\mu_{i'} = 9$ for $i' \in V'$. The resulting SCF inequality is therefore:

$$3 \sum_{i \in V} x_i + 5 \sum_{i' \in V'} x_{i'} \leq 18. \tag{7}$$

One can check (either by hand or with the help of a computer) that this SCF inequality defines a facet of $\mathrm{STAB}(G^+)$. One can also show (using the same method as for Example 1) that it is neither a Pêcher-Wagler inequality nor a CG cut. $\square$

The next example shows that an SCF inequality can define a facet even if some cliques appear in $\mathcal{C}$ more than once.

Example 3: Let G be an odd hole on 7 nodes. Add an eight node, along with the edges $\{1,8\}$, $\{4,8\}$ and $\{5,8\}$. Let $\mathcal{C}$ consist of the cliques $\{1,2\}$, $\{1,7\}$, $\{1,8\}$, $\{3,4\}$, $\{5,6\}$, together with the following cliques counted twice each: $\{2,3\}$, $\{6,7\}$ and $\{4,5,8\}$. Set p to 11, q to 3 and r to 2. We have $\mu_i = 3$ for $i = 1,\ldots,8$, and therefore the SCF inequality is simply $\sum_{i=1}^{8} x_i \leq 3$. This inequality defines a facet of $\mathrm{STAB}(G)$, as one can see by taking the odd hole inequality and lifting on node 8. (It also happens to be both a clique-family inequality and a CG cut.) $\square$

One can check that the graphs in Examples 1 to 3 contain claws. A natural question is whether SCF inequalities can define non-trivial facets for graphs that are claw-free. This is indeed the case.

Example 4: Consider the *fish* graph F whose complement is depicted in Figure 1. Giles & Trotter [31] introduced F and noted that it is claw-free. One can check that the following two inequalities define facets of $\mathrm{STAB}(F)$:

$$x_1 + x_2 + 2x_3 + 2x_4 + x_5 + x_6 + x_7 + x_8 + 2x_9 + 2x_{10} \leq 3,$$

$$2x_1 + 2x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + 2x_9 + 2x_{10} \leq 3.$$

These inequalities are not clique-family inequalities (since it is impossible for $q - r$ to be 2 and $(q - r)\lfloor p/q \rfloor$ to be 3 simultaneously). One can also show (using the same method as for Example 1) that they are not Pêcher-Wagler inequalities either. On the other hand, they can be derived as SCF inequalities. Indeed, for the first one, it suffices to let $\mathcal{C}$ contain the 7 cliques $\{3,4,6,8\}$, $\{2,3,7,8\}$, $\{1,2,7,9\}$, $\{3,4,6,10\}$, $\{3,4,9,10\}$, $\{1,5,9,10\}$ and $\{4,5,9,10\}$, and then let $p = 7$, $q = 2$ and $r = 1$. (One can check that they are also CG cuts.) $\square$

4 Some Open Questions

There are three open questions that we find interesting. The first is whether it is possible to devise an effective (exact or heuristic) separation algorithm for the SCF inequalities. We remark that one cannot simply use an existing separation algorithm for MIR inequalities, such as the ones in [8, 20],

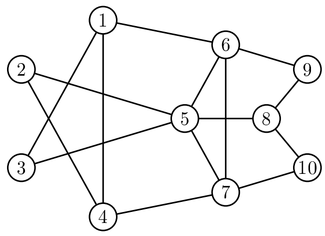

Figure 1: The graph $\bar{F}$, complement of the *fish* graph introduced in [31].

due to the fact that the number of maximal cliques in a graph is typically exponential in the number of nodes.

The second open question is whether there exist SCF inequalities that define facets for claw-free graphs, yet are neither clique-family inequalities nor CG cuts. We have checked the "geared" inequalities from [11] and they are not SCF inequalities. Together with results from [12, 13], this implies that, if an SCF inequality of the above-mentioned type exists, it can only be found in a claw-free graph with stability number 3. Unfortunately, we could not produce facet-defining inequalities of this type from the *fish* examples of claw-free graphs with stability number 3 provided in [19].

Finally, recall that a graph G is *circular interval* if its vertices can be mapped on a circle C so that the cliques of G correspond with intervals of C. (Note that antiwebs are circular interval, and circular interval graphs are quasi-line.) Let us say that a graph $G = (V, E)$ is a *knit* if its vertices can be partitioned in two subsets, V_1 and V_2, such that: i) the induced subgraphs $G[V_1]$ and $G[V_2]$ are circular interval graphs with circles C_1 and C_2, respectively; ii) the neighborhood in G_2 of each vertex of V_1 corresponds to an interval of C_2; iii) the neighborhood in G_1 of each vertex of V_2 corresponds to an interval of C_1. We pose the following conjecture:

Conjecture 1 *Let G be a knit. Then $STAB(G)$ is completely described by clique, SCF and non-negativity inequalities.*

We remark that the graph mentioned in Example 2 is a knit.

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