On the Complexity of Surrogate and Group Relaxation for Integer Linear Programs

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Abstract

Surrogate and *group* relaxation have been used to compute bounds for various integer linear programming problems. We prove that (a) when only inequalities are surrogated, the surrogate dual is $\mathcal{NP}$-hard, but solvable in pseudo-polynomial time under certain conditions; (b) when equations are surrogated, the surrogate dual exhibits unusual complexity behaviour; (c) the group relaxation is $\mathcal{NP}$-hard for the integer and 0-1 knapsack problems, and strongly $\mathcal{NP}$-hard for the set packing problem.

Keywords: integer programming; surrogate relaxation; group relaxation

1 Introduction

Many important $\mathcal{NP}$-hard problems have a natural formulation as an *integer linear program* or ILP (see, e.g., Conforti *et al.* [6]). To obtain a bound on the optimal value, one can solve the *continuous relaxation* of the ILP, which is obtained by permitting variables to take fractional values. The resulting bound can however be very weak in some cases. Popular ways to obtain stronger bounds include *cutting planes* (e.g., [7, 28]), *Lagrangian relaxation* (e.g., [12, 19]) and *Dantzig-Wolfe decomposition* (e.g., [1, 9]).

Two further methods for obtaining strong bounds, which are less well known, are *surrogate* and *group* relaxation (see [13, 17] and [15, 16], respectively). The existing literature on these techniques leaves unanswered several natural questions concerned with computational complexity. In an attempt to address this gap, we show the following:

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- For the case in which only inequalities may be surrogated, the surrogate dual is $\mathcal{NP}$-hard, but solvable in pseudo-polynomial time under certain conditions.
- For the case in which only equations may be surrogated, the surrogate dual exhibits unusual complexity behaviour: computing the bound is $\mathcal{NP}$-hard in the strong sense, but optimal multipliers can be found in polynomial time.
- The group relaxation is $\mathcal{NP}$-hard for the integer knapsack and 0-1 knapsack problems, and strongly $\mathcal{NP}$-hard for the set packing problem.

The paper has a simple structure. The literature is reviewed in Section 2. The subsequent three sections present the three theoretical results mentioned above. Some concluding remarks are made in Section 6.

Throughout the paper, we let n denote the number of variables, and let N denote $\{1, \ldots, n\}$. We call surrogate and group relaxation "SR" and "GR", respectively. Given a vector $v \in \mathbb{Q}^p_+$, we let $||v||_1$ denote $\sum_{i=1}^p v_i$. Given a rational scalar s, vector v or matrix M, we let "size(s)", "size(v)" and "size(M)" denote the number of bits needed to represent s, v or M, respectively. We assume that the reader is familiar with the basics of computational complexity theory, including ordinary and strong $\mathcal{NP}$-hardness, and pseudo-polynomial time (see [11]). Finally, we remind the reader that a function $f : S \mapsto \mathbb{R}$ with convex domain C is called *quasi-convex* if

$$f\big(\lambda x + (1-\lambda)y\big) \leq \max\big\{f(x), f(y)\big\} \qquad \big(\forall x, y, \in C,\ \lambda \in (0,1)\big).$$

2 Literature Review

In this section, we recall the key papers on SR and GR.

2.1 Surrogate relaxation

Consider an ILP of the form

$$\max\big\{c^T x :\ Ax \leq b,\ x \in \mathcal{X}\big\}, \tag{1}$$

where $c \in \mathbb{Z}^n$, $A \in \mathbb{Z}^{m\times n}$, $b \in \mathbb{Z}^m$, and

$$\mathcal{X} = \big\{x \in \mathbb{Z}^n_+ :\ Dx \leq e\big\}$$

for some integral matrix D and integral vector e. In SR, we pick a vector $\mu \in \mathbb{R}^m_+$ of *surrogate multipliers*, and solve the following simpler ILP [13,17]:

$$\max\Big\{c^T x :\ (\mu^T A)x \leq \mu^T b,\ x \in \mathcal{X}\Big\}. \tag{2}$$

This gives an upper bound, that we call $U(\mu)$.

Note that computing $U(\mu)$ is itself an ILP, and may even be $\mathcal{NP}$-hard. On the other hand, when $\mathcal{X}$ has a sufficiently simple structure, the ILP in question may be solvable reasonably quickly in practice. For example, if $\mathcal{X}$ is $\mathbb{Z}_+^n$ or $\{0,1\}^n$, then (2) is a knapsack problem, and can be solved in pseudo-polynomial time by dynamic programming [2].

The problem of finding the vector μ that gives the best upper bound is called the *surrogate dual*. It is shown in [13, 17] that the corresponding upper bound is at least as good as the one from LP relaxation. It is also shown that $U(\mu)$ is a quasi-convex function of μ.

For heuristic and exact algorithms for solving the surrogate dual, see, e.g., [3, 4, 22–24, 31].

Note that $U(s\mu) = U(\mu)$ for any positive scalar s. Accordingly, most authors impose the condition that $||\mu||_1 = 1$. We will *not* do this, however, for reasons which will become clear in Section 3.

We will also consider the case in which equations, rather than inequalities, are surrogated. For this situation, we will need a result of Glover & Woolsey [14]. It states that, if $Cx = d$ is a set of m *equations* in n binary variables, one can compute in polynomial time a non-negative integral vector μ, with coefficients bounded by $2^m \prod_{i=1}^m (|d_i| + 1)$, such that the single equation $(\mu^T C)x = \mu^T d$ has the same set of binary solutions.

2.2 Group relaxation

Now consider an ILP written in the slightly different form

$$\max\left\{c^T x : \; Ax \leq b, \; x \in \mathbb{Z}_+^n\right\}, \tag{3}$$

where $c \in \mathbb{Z}^n$, $A \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$. Adding slack variables we obtain

$$\max\left\{c^T x : \; Ax + s = b, \; (x, s) \in \mathbb{Z}_+^{n+m}\right\}. \tag{4}$$

Suppose we solve the continuous relaxation of (4) by the simplex method, yielding a basic optimal solution (x^*, s^*). Exactly m variables will be basic and n variables will be non-basic at (x^*, s^*).

Gomory's GR is obtained from (4) by retaining integrality, but dropping the non-negativity restriction on the basic variables [15, 16]. Provided that (x^*, s^*) is non-degenerate, this is equivalent to dropping all non-binding constraints in the original ILP (3), including all non-binding non-negativity constraints, if any. Thus, the upper bound obtained with GR is at least as good as the one obtained with LP relaxation.

The reason for the name *group relaxation* is that Gomory showed how to express it in group-theoretic terms. He also showed that the GR can be reduced to a shortest-path problem in a graph with D nodes, where D is the determinant of the basis matrix. This has led to several algorithms for solving the GR (see, e.g., [5, 30]). Unfortunately, none of them run in polynomial time.

It is stated in [25] that, for general ILPs, the GR is $\mathcal{NP}$-hard in the strong sense. An explicit proof is given in [10]. Interestingly, however, there are some $\mathcal{NP}$-hard ILPs whose GR can be solved in polynomial time. Indeed, Cornuéjols *et al.* [8] show that this is true for ILPs of the form

$$\max\left\{p^T x : \, x_u + x_v \leq 1 \, \big(\{u,v\} \in E\big), \, x \in \mathbb{Z}_+^n\right\}. \tag{5}$$

where $p \in \mathbb{Z}_+^n$ and E is the edge set of an arbitrary (simple, loopless, undirected) graph on n nodes. Such ILPs are used to model the *independent set problem* (also known as the node packing problem or stable set problem), which is well-known to be strongly $\mathcal{NP}$-hard [21].

3 Surrogating Inequalities

In this section, we consider the computational complexity of the surrogate dual when only inequalities are surrogated. To begin, for any $\theta \in \mathbb{Z}$, let

$$\Lambda(\theta) = \left\{\mu \in \mathbb{R}_+^m : \, U(\mu) \leq \theta\right\}.$$

That is, $\Lambda(\theta)$ is the *lower level set* of $U(\mu)$ for the given θ. Observe that $\Lambda(\theta)$ is the set of points $\mu \in \mathbb{R}_+^m$ satisfying the following linear inequalities:

$$(A\bar{x} - b)^T \mu > 0 \qquad \big(\forall \bar{x} \in \mathcal{X} : \, c^T \bar{x} \geq \theta + 1\big). \tag{6}$$

Thus, $\Lambda(\theta)$ is a convex cone. On the other hand, it is not closed (except in the trivial cases when it is either empty or equal to $\mathbb{R}_+^m$). This makes it rather difficult to work with.

Now, consider the following modified version of the inequalities (6):

$$(A\bar{x} - b)^T \mu \geq 1 \qquad \big(\forall \bar{x} \in \mathcal{X} : \, c^T \bar{x} \geq \theta + 1\big). \tag{7}$$

Observe that, if a vector lies in $\Lambda(\theta)$, we can multiply it by a suitable positive scalar to make it satisfy (7). This leads us to define the following (unbounded) convex set:

$$\tilde{\Lambda}(\theta) = \left\{\mu \in \mathbb{R}_+^m : \, (7) \text{ hold}\right\}.$$

From the above argument, $\Lambda(\theta)$ is empty if and only if $\tilde{\Lambda}(\theta)$ is empty. We also have the following result.

Proposition 1 $\tilde{\Lambda}(\theta)$ *is a polyhedron.*

Proof. Define the polyhedron

$$P^\theta = \left\{x \in \mathbb{R}^n_+ : \ Dx \le e, \, c^T \bar{x} \ge \theta + 1\right\},$$

and let P_I^θ denote the convex hull of the integral points in P. By Meyer's theorem [27], P_I^θ is a polyhedron. Thus, it can be described by a finite set of extreme points and extreme rays. Let $p^1, \dots, p^s$ be the points and $r^1, \dots, r^t$ be the rays. By definition, $\tilde{\Lambda}(\theta)$ is the set of all points in $\mathbb{R}^m_+$ that satisfy the following inequalities:

$$(Ap^k - b)^T \mu \ge 1 \qquad (k = 1, \dots, p) \tag{8}$$

$$(Ar^k)^T \mu \ge 0 \qquad (k = 1, \dots, r). \tag{9}$$

It is therefore the intersection of a finite number of half-spaces. □

The above concepts are made clear by the following example:

Example 1: Consider the following trivial ILP:

$$\max \left\{ x_1 + x_2 : \ 6x_1 + 3x_2 \le 4, \ 3x_1 + 6x_2 \le 4, \ x \in \{0,1\}^2 \right\}.$$

The optimal solution is $(0, 0)$, with profit 0. The optimal solution to the LP relaxation is $(4/9, 4/9)$, giving the upper bound $8/9$.

Suppose we set $\mathcal{X}$ to $\{0,1\}^2$, and surrogate both of the linear constraints, with multipliers μ_1 and μ_2. Suppose also that we set θ to 0. There are three points in $\mathcal{X}$ with profit greater than 0; namely, $(0, 1)$, $(1, 0)$ and $(1, 1)$. The corresponding constraints (6) are $-\mu_1 + 2\mu_2 > 0$, $2\mu_1 - \mu_2 > 0$ and $5\mu_1 + 5\mu_2 > 0$, respectively. The last of these is redundant. Therefore:

$$\begin{aligned} \Lambda(0) &= \left\{\mu \in \mathbb{R}^2_+ : \ -\mu_1 + 2\mu_2 > 0, \, 2\mu_1 - \mu_2 > 0\right\} \\ \tilde{\Lambda}(0) &= \left\{\mu \in \mathbb{R}^2_+ : \ -\mu_1 + 2\mu_2 \ge 1, \, 2\mu_1 - \mu_2 \ge 1\right\}. \end{aligned}$$

One can check that $\tilde{\Lambda}(0)$ has the unique extreme point $(1, 1)$. Thus, SR with this choice of μ yields the upper bound 0. □

We will also need the following lemma.

Lemma 1 *The number of bits needed to represent any inequality of the form (8) or (9) is polynomial in n, size(c), size(D), size(e) and size(θ).*

Proof. This follows from [32], Corollary 17.1c, and the definition of the p^k and r^k. □

Next, consider the following decision problem associated with the surrogate dual:

The Level Set Problem (LSP): *Given some $\theta \in \mathbb{Z}$, is $\Lambda(\theta)$ empty?*

It follows from the above observations that the answer to the LSP is "yes" if and only if the following LP is infeasible:

$$\min \left\{ ||\mu||_1 : \text{ (7) hold, } \mu \in \mathbb{R}^m_+ \right\}. \tag{10}$$

We remark that, if $\mathcal{X}$ has infinite cardinality, then (10) is a *semi-infinite* LP.

Clearly, solving the LSP cannot be harder than solving the surrogate dual. On the other hand, we have the following negative result.

Proposition 2 *Even when only inequalities are surrogated, the LSP is $\mathcal{NP}$-complete.*

Proof. Consider the special case of the ILP (1) in which $m = 1$, $A \geq 0$, $b > 0$ and $\mathcal{X} = \{0,1\}^n$. In this case, the ILP reduces to the 0-1 knapsack problem. Moreover, by setting μ_1 to any positive quantity, we can make the upper bound $U(\mu)$ exact (i.e., equal to the profit of the optimal solution of the knapsack problem). So, the answer to the LSP is "yes" if and only if there exists a solution to the knapsack problem with profit larger than θ. This is $\mathcal{NP}$-hard to check.

To complete the proof, we just have to show that the LSP is in $\mathcal{NP}$. For a given θ, let S be the system of linear inequalities formed by the union of (8) and (9). By definition, the answer to the LSP is "yes" if and only if S is inconsistent. Moreover, if S is inconsistent, then, by Helly's theorem [20], there exists a subset of S of cardinality at most $m+1$ that is also inconsistent. Moreover, by Lemma 1, each of the $m+1$ inequalities involves a polynomially bounded number of bits. Thus, whenever the answer to the LSP is 'yes', there exists a short certificate of that fact. This shows that the LSP is in $\mathcal{NP}$. □

We now return to the surrogate dual itself.

Theorem 1 *Consider once more an ILP of the form (1). Suppose that the following three assumptions hold:*

1. *For any $\theta \in \mathbb{Z}$ and $v \in \mathbb{Q}^n$, one can solve the following ILP in time that is polynomial in n, size(D), size(e), size(θ), size(v) and $||c||_1$:*

$$\min \left\{ v^T x : x \in \mathcal{X},\ c^T x \geq \theta \right\}.$$

2. *The continuous relaxation of the ILP (1) is feasible and bounded.*

3. *One can compute in polynomial time a lower bound L on the optimal profit, whose encoding length is polynomial in n, size(A), size(b), size(c), size(D) and size(e).*

Then one can solve the surrogate dual in time that is polynomial in n, size(A), size(b), size(D), size(e) and $||c||_1$.

Proof. First, suppose that $\theta \in \mathbb{Z}$ is fixed. Consider the *separation problem* for the constraints (7) (i.e., the problem of detecting when a given $\bar{\mu} \in \mathbb{R}^m_+$ violates one of the constraints). The separation problem is equivalent to the ILP

$$\min\left\{(\bar{\mu}^T A)x : \ x \in \mathcal{X}, \ c^T x \geq \theta + 1\right\}.$$

(Indeed, one of the linear constraints is violated if and only if the optimal solution of this ILP, say $\bar{x}$, satisfies $(\bar{\mu}^T A)\bar{x} < 1 + \bar{\mu}^T b$.) Now, under the first assumption, the separation problem can be solved in time that is polynomial in the parameters listed. Then, by Lemma 1 and the polynomial equivalence of separation and optimisation [18], the LP (10) can be solved in time that is polynomial in n, size(A), size(b), size(D), size(e), size(θ) and $||c||_1$. This implies in turn that the LSP can be solved in time that is polynomial in the same parameters.

Now suppose that assumption 2 holds. Using the ellipsoid method [18, 32], we can solve the LP relaxation of the ILP (1) in time that is polynomial in n, size(A), size(b), size(c), size(D) and size(e). This yields an upper bound on the optimal profit, which we denote by U. Moreover, from [32], Theorem 10.3, size(U) is bounded by a polynomial in the same parameters.

Now suppose that assumption 3 also holds. We now have both lower and upper bounds on the optimal value of θ, each of which has polynomial encoding length. Thus, by applying binary search on the value of θ, one can reduce the surrogate dual to a number of LSP instances that is polynomial in the stated parameters. □

This result generalises a result of Boros [4], who proved it for the case in which $\mathcal{X}$ is the hypercube. It means that, if we can solve the surrogate relaxed problem in pseudo-polynomial time, then the surrogate dual cannot be $\mathcal{NP}$-hard in the *strong* sense (unless of course $\mathcal{P} = \mathcal{NP}$).

We remark that computing L is often easy in practice. Indeed, if it is known that all points $\bar{x} \in \mathcal{X}$ satisfy $\ell \leq \bar{x} \leq u$, for some vectors $\ell, u \in \mathbb{Z}^n_+$, then an easily-computable value for L is

$$\sum_{j=1}^{n} \ell_j \max\{0, c_j\} + \sum_{j=1}^{n} u_j \min\{0, c_j\}.$$

4 Surrogating Equations

Now we turn our attention to the case in which equations, rather than inequalities, are surrogated. More precisely, we now assume that the ILP takes the form:

$$\max\left\{c^T x : \ Ax = b, \ x \in \mathcal{X}\right\}.$$

Similarly, we now assume that

$$U(\mu) = \max\left\{c^T x : \left(\mu^T A\right)x = \mu^T b,\ x \in \mathcal{X}\right\}.$$

Note also that, here, it may make sense to allow some of the surrogate multipliers to take negative values.

As in the previous section, we can define lower level sets:

$$\Lambda(\theta) = \left\{\mu \in \mathbb{R}^m : U(\mu) \le \theta\right\}.$$

Note however that, in this case, we have:

$$\Lambda(\theta) = \left\{\mu \in \mathbb{R}^m : \left(\mu^T A\right)\bar{x} \ne \mu^T b \left(\forall \bar{x} \in \mathcal{X} : c^T x > \theta\right)\right\}.$$

The presence of the symbol "$\ne$" suggests that $\Lambda(\theta)$ is not convex in general. This is indeed the case. In fact, $\Lambda(\theta)$ can even be disconnected, as shown by the following example.

Example 2: Consider the following equality-constrained 0-1 LP:

$$\max\left\{x_1 + 5x_2 : x_1 = 0,\ x_1 + x_2 = 0,\ x \in \{0,1\}^2\right\}.$$

Trivially, the optimal solution has profit 0. Now, as before, we set $\mathcal{X}$ to $\{0,1\}^2$ and surrogate both of the linear constraints. Note that the domain of μ is now $\mathbb{R}^2$. One can check that

$$\begin{aligned} U(\mu) &= 6 \quad (\text{if } \mu_1 + 2\mu_2 = 0) \\ &= 5 \quad (\text{if } \mu_2 = 0 \text{ and } \mu_1 \ne 0) \\ &= 1 \quad (\text{if } \mu_1 + \mu_2 = 0 \text{ and } \mu_2 \ne 0) \\ &= 0 \quad (\text{otherwise}). \end{aligned}$$

So the lower level sets $\Lambda(1), \ldots, \Lambda(6)$ are all disconnected. □

Now recall that, in the inequality-constrained case, we reduced the Level Set Problem to the LP (10). The equivalent problem in the equality-constrained case is:

$$\min\left\{||\mu||_1 : \left(A\bar{x}\right)^T \mu \ne b^T \mu \left(\forall \bar{x} \in \mathcal{X} : c^T \bar{x} > \theta\right),\ \mu \in \mathbb{R}^m\right\}. \tag{11}$$

Note that (11) looks much harder to solve than (10). Indeed, we have the following negative result.

Theorem 2 *If only equations are surrogated, then the surrogate dual is $\mathcal{NP}$-hard in the strong sense, and the LSP is $\mathcal{NP}$-complete in the strong sense.*

Proof. Suppose that $\mathcal{X} = \{0,1\}^n$, i.e., we are dealing with a 0-1 LP. By using a slack variable s_j, we can convert the condition $x_j \in \{0,1\}$ to $x_j + s_j = 1$, $x_j, s_j \in \mathbb{Z}_+$. In this way, we can transform a 0-1 LP into an ILP in which all constraints (apart from the non-negativity and integrality constraints) are equations. The result of Glover & Woolsey [14] mentioned in Subsection 2.1 then implies that there exists an integral vector μ, whose encoding length is polynomial in that of the input, such that $U(\mu)$ is equal to the profit of the optimal solution of the original 0-1 LP. Thus, solving the surrogate dual problem is as hard as solving a 0-1 LP, and is therefore strongly $\mathcal{NP}$-complete. Moreover, the answer to the LSP is "yes" if and only if there is a feasible solution to the 0-1 LP with profit greater than θ. For 0-1 LPs in general, checking whether such a feasible solution exists is strongly $\mathcal{NP}$-complete. □

Thus, in the equality-constrained case, one cannot determine the best possible value of $U(\mu)$ in pseudo-polynomial time (unless $\mathcal{P} = \mathcal{NP}$). Bizarrely, however, one can find an optimal vector μ in polynomial time (by applying the procedure in [14]). The explanation of this apparent paradox is that the components of μ may be exponentially large, which makes solving the surrogate relaxed problem as hard as solving the ILP itself.

5 Group Relaxation

In this section, we prove some results concerned with the complexity of GR. We start with two results concerned with knapsack problems (KPs). We remind the reader that the integer KP takes the form

$$\max\left\{p^T x : \, a^T x \le b, \, x \in \mathbb{Z}^n_+\right\}, \tag{12}$$

where $p, a \in \mathbb{Z}^n_+$ and b is a positive integer. The 0-1 KP is similar, except that all variables are constrained to be binary. Both problems are known to be weakly $\mathcal{NP}$-hard [21, 26]. Our results are as follows.

Proposition 3 *The GR is $\mathcal{NP}$-hard for the integer KP.*

Proof. Consider an integer KP instance of the form (12). Let M be a "large" positive integer. (In fact, it suffices to set M to any integer that is larger than $(b+1)\max_{j \in N}\{p_j/a_j\}$.) We construct a different integer KP instance, which we call the *augmented* KP:

$$\begin{array}{rc} \max & p^T x + My \\ \text{s.t.} & a^T x + (b+1)y \le 2b+1 \\ & (x,y) \in \mathbb{Z}^{n+1}_+. \end{array} \tag{13}$$

Note that the LP relaxation of the augmented KP has a unique optimal solution with $x_j^* = 0$ for all j, and $y^* = (2b+1)/(b+1) < 2$. This solution is non-degenerate, since the only binding constraints are the non-negativity constraints on the x variables and the constraint (13). Thus, applying GR to the augmented KP just means dropping non-negativity on y. Since y has a very large profit, it will be set to 1 in the optimal GR solution. So the optimal x for the GR of the augmented KP is identical to the optimal x of the original integer KP. □

Proposition 4 *The GR is $\mathcal{NP}$-hard for the 0-1 KP.*

Proof. Consider again an integer KP instance of the form (12), and let M be as in the previous proof. We construct the following 0-1 KP with $n+2$ variables:

$$\begin{array}{rc} \max & p^T x + 2My + Mz \\ \text{s.t.} & a^T x + (b+1)(y+z) \le 2b+1 \\ & (x, y, z) \in \{0,1\}^{n+2}. \end{array} \tag{14}$$

The LP relaxation of the 0-1 KP has an optimal solution with $x_j^* = 0$ for all j, $y^* = 1$ and $z^* = b/(b+1) < 1$. This solution is non-degenerate, since the only binding constraints are the non-negativity constraints on the x variables, the constraint (14), and the upper bound of 1 on y. Thus, applying GR to the 0-1 KP means dropping the upper bounds of 1 on the x_j, the lower bound of 0 on y, and the lower and upper bounds on z. Since y has a very large profit, it will be set to 1 in the optimal GR solution. This forces z to take the value 0. So the optimal x for the GR of the 0-1 KP is identical to the optimal x of the original integer KP. □

Next, we consider the *set packing problem* (SPP). In the SPP, we are given a positive integer n, a positive integer profit p_j for each $j \in N$, and a collection of m non-empty subsets of N, say $S_1, \ldots, S_m$. The task is to select a subset of N of maximum profit, subject to the constraint that, for $i = 1, \ldots, m$, at most one element of S_i is selected. The SPP is strongly $\mathcal{NP}$-hard [21]. The standard ILP formulation is as follows [29]:

$$\max \left\{ p^T x : \sum_{j \in S_i} x_j \le 1 \ (i = 1, \ldots, m), \ x \in \mathbb{Z}_+^n \right\}. \tag{15}$$

Theorem 3 *The GR is $\mathcal{NP}$-hard in the strong sense, even for set packing problems that are formulated as ILPs of the form (15).*

Proof. Consider an SPP of the form (15), and let M be a "large" positive integer. (Here, it suffices to set M to $2\sum_{j \in N} p_j$.) We construct an augmented SPP with $n+3m$ variables. It takes the form:

$$\begin{array}{lll} \max & p^T x + M \sum_{i=1}^m \left(y_i + y_i'\right) + (M+1) \sum_{i=1}^m z_i & \\ \text{s.t.} & \sum_{j \in S_i} x_j + y_i + y_i' \le 1 & (i = 1, \ldots, m) \quad (16) \\ & y_i + z_i \le 1 & (i = 1, \ldots, m) \quad (17) \\ & y_i' + z_i \le 1 & (i = 1, \ldots, m) \quad (18) \\ & x \in \mathbb{Z}_+^n & \\ & y, y', z \in \mathbb{Z}_+^m. & \end{array}$$

The LP relaxation of the augmented SPP has a feasible solution with $x_j = 0$ for all j, and $y_i = y_i' = z_i = 1/2$ for all i. This LP solution is optimal, as one can verify by setting the dual variables for (16) to $(M-1)/2$, and the dual variables for (17) and (18) to $(M+1)/2$. In fact, it is the unique optimal solution, as one can verify by noting that every x variable has a positive reduced cost.

Now note that, at this LP solution, the binding constraints are (16)–(18), together with the non-negativity constraints on the x variables. Thus, exactly $n+3m$ constraints are binding, and the LP solution is non-degenerate. Applying GR to the augmented SPP therefore means just dropping non-negativity on the y, y' and z variables. Now, note that the remaining constraints imply that $y_i + y_i' + z_i \le 1$ for all i. Since z_i has a larger profit than y_i and y_i', it will be set to 1 in the optimal GR solution. This in turn will force y_i and y_i' to take the value 0. Then, the optimal x of the GR of the augmented SPP is identical to the optimal x of the original SPP. □

We close this section by noting that the complexity of GR for a particular combinatorial optimisation problem depends on the way it is formulated as an ILP. Indeed, Padberg [29] showed that the SPP and the independent set problem are equivalent, i.e., any ILP of the form (15) can be converted into one of the form (5) and vice-versa. Yet, as mentioned in Subsection 2.2, the GR of (5) can be solved in polynomial time.

6 Final Remark

We close the paper by mentioning some interesting open problems concerned with *composite* relaxation (CR), which is a hybrid of Lagrangian and surrogate relaxation [13, 17]. In CR, we choose vectors $\lambda, \mu \in \mathbb{R}_+^m$, and solve the relaxed problem

$$\max \left\{ c^T x + \lambda^T \left(b - Ax\right) : \ (\mu^T A) x \le \mu^T b, \ x \in \mathcal{X} \right\}.$$

In theory, CR can produce better upper bounds than both Lagrangian and surrogate relaxation. Unfortunately, the dual function is not even quasi-convex in λ and μ. In fact, it may have local minima that are not global minima [22]. Thus, the composite dual is even harder to solve than the surrogate dual. We conjecture that, even in the inequality case, the composite dual is strongly $\mathcal{NP}$-hard. As for the Level Set Problem in CR, we do not even know whether it lies in $\mathcal{NP}$ or co-$\mathcal{NP}$.

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